ON THE CONNEXION OF QUATERNIONS WITH CONTINUED FRACTIONS AND QUADRATIC EQUATIONS

By

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[Communicated December 8th, 1851]

The Secretary, in the absence of Sir W. R. Hamilton, read the following remarks on the connexion of Quaternions with continued fractions and quadratic equations.

1. If we write

$$u_x = \frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots\frac{b_x}{a_x} = \frac{N_x}{D_x},$$

it is known (see Sir J. F. W. Herschel's Treatise on Finite Differences) that the numerator and denominator of the resultant fraction satisfy two differential equations in differences, which are of one common form, namely,

$$\begin{aligned} N_{x+1} &= N_x a_{x+1} + N_{x-1} b_{x+1}, \\ D_{x+1} &= D_x a_{x+1} + D_{x-1} b_{x+1}. \end{aligned}$$

And by the nature of the reasoning employed, it will be found that these equations in differences, thus written, hold good for quaternions, as well as for ordinary fractions.

2. Supposing a and b to be two constant quaternions, these equations in differences are satisfied by supposing

$$\begin{aligned} N_x &= Cq_1^x + C'q_2^x, \\ D_x &= Eq_1^x + E'q_2^x, \\ C + C' = 0, &\quad Cq_1 + C'q_2 = b, \\ E + E' = 1, &\quad Eq_1 + E'q_2 = a; \end{aligned}$$

C, C', E, E' being four constant quaternions, determined by the four last conditions, after finding two other and unequal quaternions, q_1 and q_2, which are among the roots of the quadratic equation,

$$q^2 = qa + b.$$

3. By pursuing this track it is found, with little or no difficulty, that

$$2u_x^{-1} + q_1^{-1} + q_2^{-1} = \frac{q_1^x + q_2^x}{q_1^x - q_2^x} \frac{q_1 - q_2}{b};$$

where

$$u_x = \left(\frac{b}{a+}\right)^x 0, \qquad \frac{q_1 - q_2}{b} = q_1^{-1} - q_2^{-1};$$

q_1, q_2, being still supposed to be two unequal roots of the lately written quadratic equation in quaternions,

$$q^2 = qa + b.$$

4. Let the continued fraction in quaternions be

$$u_x = \left(\frac{j}{i+}\right)^x 0;$$

then the quadratic equation becomes

$$q^2 = qi + j:$$

and two unequal roots of it are the following:

$$q_1 = \tfrac{1}{2}(1 + i + j - k),$$
$$q_2 = \tfrac{1}{2}(-1 + i - j - k).$$

Substitution and reduction give hence these two expressions:

$$\left(\frac{j}{i+}\right)^{2n} 0 = \frac{\sin\dfrac{2n\pi}{3}}{i\sin\dfrac{2n\pi}{3} - k\sin\dfrac{(2n-1)\pi}{3}};$$

$$\frac{2 \div \left(\dfrac{j}{i+}\right)^{2n-1} 0}{i - k} = 1 - \frac{\sin\dfrac{(2n-1)\pi}{3}}{\sin\dfrac{2(n-1)\pi}{3} + j\sin\dfrac{2n\pi}{3}};$$

which may easily be verified by assigning particular values to n. No importance is attached by the writer to these particular results: they are merely offered as examples.

5. It may have appeared strange that Sir William R. Hamilton should have spoken of *two* unequal quaternions, as being *among* the roots, or *two of the roots*, of a *quadratic equation* in quaternions. Yet it was one of the earliest results of that calculus, respecting which he made (in November, 1843) his earliest communication to the Academy, that *such* a quadratic equation (if of the above-written form) has generally *six roots*: whereof, however, *two only* are *real quaternions*, while the other four may, by a very natural and analogical extension of received language, be called *imaginary quaternions*. But the theory of such *imaginary*, or *partially* imaginary quaternions, in short, the theory of what Sir William R. Hamilton has ventured to name "*Biquaternions*," in a paper already published, appears to him to deserve to be the subject of a separate communication to the Academy.

[Communicated May 24th, 1852]

6. Sir William R. Hamilton read a supplementary Paper in illustration of his communication of the 8th of December last, on the connexion of Quaternions with continued fractions and quadratic equations.

In this paper he assigned the four Biquaternions which are the *imaginary* roots of the equation

$$q^2 = qi + j;$$

and showed that *these* were as well adapted as the two *real* roots assigned in his former communication, to furnish the real quaternion value of the continued fraction,

$$\left(\frac{j}{i+}\right)^x 0.$$

He also showed that when the continued fraction

$$u_x = \left(\frac{b}{a+}\right)^x 0$$

converges to a *limit*,

$$u = u_\infty = \left(\frac{b}{a+}\right)^\infty 0,$$

the two quaternions a and b being supposed to be given and real, then this limit is equal to *that one of the two real roots of the quadratic equation in quaternions*,

$$u^2 + ua = b,$$

which has the lesser tensor; and gave geometrical illustrations of these results.

The *two real* quaternion roots of the quadratic equation, $q^2 = qi + j$, being, as in the abstract of December, 1851,

$$q_1 = \tfrac{1}{2}(1 + i + j - k), \quad q_2 = \tfrac{1}{2}(-1 + i - j - k),$$

it is now shown that the *four imaginary* roots are

$$q_3 = \frac{i}{2}(1 + \sqrt{-3}) - k, \quad q_4 = \frac{i}{2}(1 - \sqrt{-3}) - k,$$

$$q_5 = \tfrac{1}{2}(i + k) + \tfrac{1}{2}(1 - j)\sqrt{-3}, \quad q_6 = \tfrac{1}{2}(i + k) - \tfrac{1}{2}(1 - j)\sqrt{-3};$$

but that in whatever manner we group them, *two by two*, *even* by taking *one* real and *one* imaginary root, the formula

$$u_x = (1 - v_x)^{-1}(v_x q_1 - q_2), \quad \text{or} \quad \frac{u_x + q_2}{u_x + q_1} = v_x,$$

where $v_x = q_2^x v_0 q_1^{-x}$, $v_0 = \dfrac{u_0 + q_2}{u_0 + q_1}$, and which is at once simpler and more general than the equations previously communicated, conducts still to values of the continued fraction u_x, or $\left(\dfrac{j}{i+}\right)^x 0$, which agree with those formerly found, and may be collected into the following period of six terms,

$$u_0 = 0,\ \ u_1 = k,\ \ u_2 = \tfrac{1}{2}(k - i),\ \ u_3 = k - i,\ \ u_4 = -i,\ \ u_5 = \infty,\ \ u_6 = 0,\ \ u_7 = k,\ \ \&\text{c}.$$

In general it may be remembered that q_1, q_2, are roots of the quadratic equation $q^2 = qa + b$.

As an example of a continued fraction in quaternions which, instead of thus *circulating, converges* to a limit, the general value of

$$u_x = \left(\frac{10j}{5i+}\right)^x c$$

was assigned for any arbitrary quaternion c, by the help of the quadratic equation

$$q^2 = 5qi + 10j;$$

and it was shown that with only one exception, namely, the case when $c = (2k - 4i)$, the limit in question was (for *every other* value of c),

$$u = \left(\frac{10j}{5i+}\right)^\infty c = 2k - i.$$