ON THE GEOMETRICAL INTERPRETATION OF SOME RESULTS OBTAINED BY CALCULATION WITH BIQUATERNIONS

By

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By Sir William R. Hamilton.

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Sir William R. Hamilton, LL.D., gave an account of the geometrical interpretation of some results obtained by calculation with biquaternions.

In this communication *bivectors* were employed, and were shown to conduct to interesting conclusions. The conception of such bivectors,

$$\rho + \sqrt{-1}\rho',$$

where ρ and ρ' denote two geometrically real vectors, and $\sqrt{-1}$ is the *old* and ordinary (or commutative) imaginary of common algebra, and generally of *biquaternions* such as

$$q + \sqrt{-1}q',$$

where q and q' are real quaternions, interpretable, geometrically on the author's principles, had occurred to him many years ago; and the remark which he made to the Academy in November, 1844 (see the Proceedings of that date), respecting the representations, in his Calculus, of the geometrically *unreal* tangents to a sphere from an internal point, as having *positive squares*, belonged essentially to this theory of bivectors. In the same year, the more general theory of biquaternions had occurred to him, in connexion with what in his theory presented themselves as the *imaginary roots*, or purely symbolical solutions, of a certain quadratic equation in quaternions. Notices on the subject have since appeared in his subsequent papers, in the Proceedings of the Academy, and in the Philosophical Magazine: and a fuller statement of the theory will be found in his (as yet unpublished) Lectures on Quaternions, of which many sheets have long since been distributed among his friends and others in the University. On the present occasion he has employed *bivectors with null squares*, such as

$$i + hj, \quad \text{or} \quad j + hk,$$

where i, j, k are the *peculiar* symbols of the quaternion calculus, observing the laws communicated by him to the Academy in November, 1843, while h is used as a temporary and abridged symbol for the *old* imaginary $\sqrt{}-1$. In fact the rules of this calculus give

$$(j + hk)^2 = j^2 + h(jk + kj) + h^2k^2 = -1 + 0h + (-1)(-1) = 0,$$

h being a *free* (or commutative) factor in any multiplication, as in algebra, but jk being $= i = -kj$, while

$$h^2 = i^2 = j^2 = k^2 = -1.$$

Thus, at least for any numerical exponent x, we have the simplification,

$$(1 + j + hk)^x = 1 + x(j + hk),$$

which Sir W. R. H. states that he has found useful in a part of a geometrical investigation, respecting the interpretation of certain continued fractions in quaternions, of the form

$$u_x = \left(\frac{b}{a+}\right)^x u_0,$$

already mentioned by him to the Academy on a former occasion, and specially for the case when $\alpha^4 + 4\beta^2 = 0$, in the fraction

$$\rho_x = \left(\frac{\beta}{\alpha+}\right)^x \rho_0,$$

where the vector β is supposed to be perpendicular to α and ρ_0, and therefore also to ρ_x.

By the investigation referred to, he has found, among others, the following results. Let C and D be two given points, and P an assumed point. Perpendicular to DP draw CQ, towards a given hand, and such that the rectangle $CQ\,.\,DP$ may be equal to a given rectangle $CC'DD'$. From Q derive R, as Q has been derived from P, and conceive the process repeated without end. Then, I., the locus of the alternate points $P, R, T, \ldots$ is one circle, and the locus of the other alternate points $Q, S, U, \ldots$ is another circle. II. These two circular loci have the top $C'D'$ of the given rectangle for the common radical axis, of themselves and of the given circle described on CD as diameter. III. The centres of the two alternate loci are harmonic conjugates with respect to the given circle. IV. If from two fixed summits of the two loci chords be drawn to the successive points, and prolonged (if necessary) till they meet the radical axis in other points P', Q', &c.; if also a summit F of the given circle be suitably chosen (on the line of the three centres), then the two lines FP', CQ' will cross in one point on the given circle, the two lines FQ', CR' in another point thereon, and so on for ever: and the same thing holds for the lines DP', FQ', or DQ', FR', &c. Particular forms of these theorems have been published in the Phil. Mag. for this month (February, 1853), but only for the case when the top of the rectangle, or the radical axis, meets the given circle in two *real points*, A, B, in which case the derived points $Q, R, \ldots$ *converge* towards the point B nearer to C. In the contrary case there can be *no convergence*, but there may be *circulation in a period*. For if we then denote by V one of the two common points of the system of common orthogonals, and by W the point of contact of the given circle with a tangent drawn from the middle point between them, the angle $P'VQ'$ or $Q'VR'$ will be constant, and equal to VFW; so that if this latter angle be commensurate with a right angle, the points $P'Q'R' \ldots$, and therefore also the points $PQR \ldots$ will recur in a certain periodical order. These conclusions have been by Sir W. R. Hamilton obtained as results of his quaternion analysis; but he believes that it will not be found difficult to confirm them by a purely geometrical process, founded on the known theory of homographic divisions.*

* NOTE, *added during printing.* Since the foregoing communication was made, the author has seen how to obtain such *geometrical* proofs, or confirmations, of all the foregoing results.