ON A THEOREM RELATING TO THE BINOMIAL COEFFICIENTS

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and

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The President read the following paper, with a NOTE by the late Sir W. R. Hamilton, LL.D.:—

ON A THEOREM RELATING TO THE BINOMIAL COEFFICIENTS.

Towards the end of March, I communicated the following theorem to Sir William Rowan Hamilton:—

Putting

$$\begin{aligned} s_0 &= n_0 + n_3 + n_6 + \&\text{c.}, \\ s_1 &= n_1 + n_4 + n_7 + \&\text{c.}, \\ s_2 &= n_2 + n_5 + n_8 + \&\text{c.}, \end{aligned}$$

where n_0, n_1, &c., are the coefficients of the development

$$(1 + x)^n = n_0x^0 + n_1x^1 + n_2x^2 + \&\text{c.},$$

and n is a positive whole number; we shall find that, of the three quantities s_0, s_1, s_2, two are always equal, and the third differs from them by unity.

I mentioned at the same time that I had arrived at theorems, analogous, but less elegantly expressed, by summing the series formed by taking every *fourth* or *fifth* coefficient, and so on, in the binomial development; and I asked Sir William R. Hamilton whether he remembered to have seen these theorems stated anywhere. I thought it likely that the well-known elementary theorem respecting the equality of the sums of the alternate coefficients in the binomial development would have suggested research in this direction. In a note, written on the day on which he received mine, Sir William stated that my theorem was new to him, and that he had proved it by the help of imaginaries and determinants. The following day he wrote again to me, furnishing me with the following more precise statement of my theorem:—

"Let ν and N be the following (whole) functions of n,

$$\nu = (-1)^n, \quad N = \tfrac{1}{3}(2^n - \nu);$$

then N, N and $N + \nu$ are *always* the value of the *three sums*, if suitably *arranged*; and the *singular sum* is s_0, or s_1, or s_2, according as n, or $n+1$ or $n+2$ is a multiple of 3."

I communicated the following demonstration of my theorem to Sir William, in a letter of the 29th March:—

Using the notation employed above, we know that

$$(n+1)_r = n_r + n_{r-1}, \quad (n+1)_{r-1} = n_{r-1} + n_{r-2},$$

and

$$(n+1)_r - (n+1)_{r-1} = n_r - n_{r-2}.$$

Now, putting

$$\begin{aligned} s_r &= \ldots + n_{r-m} + n_r + n_{r+m} + \ldots, \\ s'_r &= \ldots + (n+1)_{r-m} + (n+1)_r + (n+1)_{r+m} + \ldots, \end{aligned}$$

(m being any positive integer), we have, from equation (1.),

$$s'_r - s'_{r-1} = s_r - s_{r-2};$$

and, in the particular case under consideration, viz. $m = 3$,

$$s'_2 - s'_1 = s_2 - s_0, \quad s'_1 - s'_0 = s_1 - s_2, \quad s'_0 - s'_2 = s_0 - s_1.$$

Thus it appears that the *differences* of the quantities s'_0, s'_1, s'_2, are equal in magnitude, but of opposite signs to those of s_1, s_2, s_0; and if we form these differences for successive values of n, they will arrange themselves in a cycle of six. Thus, if

$$s_2 - s_1 = \Delta_0, \quad s_1 - s_0 = \Delta_2, \quad s_0 - s_2 = \Delta_1,$$

we might form the following Table.

n	Δ_0	Δ_2	Δ_1
1	-1	0	1
2	-1	1	0
3	0	1	-1
4	1	0	-1
5	1	-1	0
6	0	-1	1
7	-1	0	1
8	-1	1	0
...	...	...	...

Combining this result with the well-known theorem—

$$s_0 + s_1 + s_2 = n_0 + n_1 + n_2 + \ldots = 2^n,$$

we arrive at formulae for s_0, s_1, s_2.

A couple of days later, I communicated to Sir William Hamilton my statement and proof of the corresponding theorem respecting the four sums obtained by adding every fourth binomial coefficient.

The theorem is as follows:—

"Writing $\nu = (-1)^i$ where i is any positive integer,
If n is of the form $4i$,

$$\begin{aligned} s_0 &= 2^{n-2} + \nu 2^{\frac{n-2}{2}}, \\ s_1 &= 2^{n-2}, \\ s_2 &= 2^{n-2} - \nu 2^{\frac{n-2}{2}}, \\ s_3 &= 2^{n-2}; \end{aligned}$$

If n is of the form $4i+1$,

$$\begin{aligned} s_0 &= 2^{n-2} + \nu 2^{\frac{n-3}{2}}, \\ s_1 &= 2^{n-2} + \nu 2^{\frac{n-3}{2}}, \\ s_2 &= 2^{n-2} - \nu 2^{\frac{n-3}{2}}, \\ s_3 &= 2^{n-2} - \nu 2^{\frac{n-3}{2}}; \end{aligned}$$

If n is of the form $4i+2$,

$$\begin{aligned} s_0 &= 2^{n-2}, \\ s_1 &= 2^{n-2} + \nu 2^{\frac{n-2}{2}}, \\ s_2 &= 2^{n-2}, \\ s_3 &= 2^{n-2} - \nu 2^{\frac{n-2}{2}}; \end{aligned}$$

If n is of the form $4i+3$,

$$\begin{aligned} s_0 &= 2^{n-2} - \nu 2^{\frac{n-3}{2}}, \\ s_1 &= 2^{n-2} + \nu 2^{\frac{n-3}{2}}, \\ s_2 &= 2^{n-2} + \nu 2^{\frac{n-3}{2}}, \\ s_3 &= 2^{n-2} - \nu 2^{\frac{n-3}{2}}.\text{''} \end{aligned}$$

The proof of this rests upon the equations

$$\begin{aligned} s_3' - s_2' &= s_3 - s_1, \\ s_2' - s_1' &= s_2 - s_0, \\ s_1' - s_0' &= s_1 - s_3, \\ s_0' - s_3' &= s_0 - s_2, \end{aligned}$$

combined with $s_0 + s_1 + s_2 + s_3 = 2^n$.

Though the theorems which I have now stated or indicated are not devoid of interest, I should hardly have brought them under the notice of the Academy if they had not led Sir William R. Hamilton to discuss the more general question treated of in the Note appended to this paper. It is at his suggestion that I have communicated the substance of the letter which I addressed to him on this subject.

I may be allowed to add, that the first theorem stated in this paper was suggested by the investigation of a very simple geometrical problem, and that I have found that it admits of being very curiously illustrated by means of my theory of algebraic triplets.

EXTRACT *from a recent Manuscript Investigation, suggested by a Theorem of* DEAN GRAVES, *which was contained in a letter received by me a week ago.*

1. Let n_r, for any whole value not less than zero of n, and for *any* whole value of r, be defined to be the (always whole) coefficient of the power x^r, in the expansion of $(1+x)^n$ for an arbitrary x; so that we have always $n_0 = 1$, but $n_r = 0$ in each of the cases $r < 0$, $r > n$.

2. Let p be any whole number > 0; and let the sum of all the coefficients n_m, for which $m \equiv r$ (mod. p), the value of n being given, by denoted by the symbol,

$$s^{(p)}_{n,r};$$

which thus represents, when n and p are given, a periodical function of r, in the sense that

$$s^{(p)}_{n,r} = s^{(p)}_{n,r+tp},$$

if t be *any* whole number (positive or negative).

3. A fundamental property of the binomial coefficient n_r is expressed by the equation,

$$(n+1)_r = n_r + n_{r-1};$$

from which follows at once this analogous *equation in differences*,

$$s^{(p)}_{n+1,r} = s^{(p)}_{n,r} + s^{(p)}_{n,r-1};$$

with the p *initial values*,

$$s^{(p)}_{0,0} = 1, \quad s^{(p)}_{0,1} = 0, \quad s^{(p)}_{0,2} = 0, \quad \ldots, \quad s^{(p)}_{0,p-1} = 0.$$

4. Hence may be deduced the *general expression*,

$$s^{(p)}_{n,r} = p^{-1}\Sigma x^{-r}(1+x)^n;$$

in which the summation is to be effected with respect to the p roots x of the binomial equation,

$$x^p - 1 = 0.$$

5. The summand *term*,

$$x^{-r}(1+x)^n,$$

usually involves *imaginaries*, which must however disappear in the *result*; and thus the general expression for the *partial sum*, s, may be reduced to the *real* and *trigonometrical form*,

$$s_{n,r}^{(p)} = p^{-1}\Sigma\left(2\cos\frac{m\pi}{p}\right)^n \cos\frac{m(n-2r)\pi}{p},$$

with the verification that

$$0 = \Sigma\left(2\cos\frac{m\pi}{p}\right)^n \sin\frac{m(n-2r)\pi}{p};$$

each summation being performed with respect to an auxiliary integer m, from $m = 0$ to $m = 1$.

6. Accordingly, *without* using *imaginaries*, it is easy to prove that this expression (5) satisfies all the recent conditions (3), and is therefore a correct expression for the partial sum

$$s_{n,r}^{(p)};$$

while a similar proof of the recent equation $0 =$ &c.

7. But to form *practically*, with the easiest possible *arithmetic*, a *Table of Values* of s, for any given *period*, p, we are led by No. 3 to construct a *Scheme*, such as the following:

	$r = 5$	4	3	2	1	0	Verification
$n = 0$	$s = 1$	0	0	0	0	1	$\Sigma s = 1$
1	1	0	0	0	1	1	2
2	1	0	0	1	2	1	4
3	1	0	1	3	3	1	8
4	1	1	4	6	4	1	16
5	2	5	10	10	5	2	32
6	7	15	20	15	7	7	64