

**ON A THEOREM IN THE
CALCULUS OF DIFFERENCES**

By

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On a Theorem in the Calculus of Differences. By Sir WILLIAM ROWAN HAMILTON.

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It is a curious and may be considered as an important problem in the Calculus of Differences, to assign an expression for the sum of the series

$$X = u_n(x+n)^n - u_{n-1} \cdot \frac{n}{1} \cdot (x+n-1)^n + u_{n-2} \cdot \frac{n(n-1)}{1 \cdot 2} \cdot (x+n-2)^n - \&c.; \quad (1.)$$

which differs from the series for $\Delta^n x^n$ only by its introducing the coefficients u , determined by the conditions that

$$u_i = +1, \ 0, \ \text{or} \ -1, \ \text{according as } x+i > 0, \ = 0, \ \text{or} \ < 0. \quad (2.)$$

These conditions may be expressed by the formula

$$u_i = \frac{2}{\pi} \int_0^\infty \frac{dt}{t} \sin(xt + it); \quad (3.)$$

and if we observe that

$$\begin{aligned} \frac{d}{dt} \sin(at + b) &= a \sin\left(at + b + \frac{\pi}{2}\right), \\ \left(\frac{d}{dt}\right)^n \sin(at + b) &= a^n \sin\left(at + b + \frac{n\pi}{2}\right), \end{aligned}$$

we shall see that the series (1.) may be put under the form

$$X = \frac{2}{\pi} \int_0^\infty \frac{dt}{t} \left(\frac{d}{dt}\right)^n \Delta^n \sin\left(xt - \frac{n\pi}{2}\right); \quad (4.)$$

the characteristic Δ of difference being referred to x . But

$$\begin{aligned} \Delta \sin(2\alpha x + \beta) &= 2 \sin \alpha \sin\left(2\alpha x + \beta + \alpha + \frac{\pi}{2}\right), \\ \Delta^n \sin(2\alpha x + \beta) &= (2 \sin \alpha)^n \sin\left(2\alpha x + \beta + n\alpha + \frac{n\pi}{2}\right); \end{aligned}$$

therefore, changing t , in (4.) to 2α , we find

$$X = \int_0^\infty \frac{d\alpha}{\alpha} \frac{d^n A}{d\alpha^n}, \quad (5.)$$

if we make, for abridgment,

$$A = \frac{2}{\pi} \sin \alpha^n \sin(2x\alpha + n\alpha). \quad (6.)$$

Again, the process of integration by parts gives

$$\int_0^\infty \frac{d\alpha}{\alpha^i} \frac{d^{n-i+1}A}{d\alpha^{n-i+1}} = i \int_0^\infty \frac{d\alpha}{\alpha^{i+1}} \frac{d^{n-i}A}{d\alpha^{n-i}},$$

provided that the function

$$\frac{1}{\alpha^i} \frac{d^{n-i}A}{d\alpha^{n-i}}$$

vanishes both when $\alpha = 0$ and when $\alpha = \infty$, and does not become infinite for any intermediate value of α , conditions which are satisfied here; we have, therefore, finally,

$$X = 1 \cdot 2 \cdot 3 \dots n \int_0^\infty d\alpha \frac{A}{\alpha^{n+1}}. \quad (7.)$$

Hence, if we make

$$P = \frac{X}{1 \cdot 2 \cdot 3 \dots n}, \quad \text{and} \quad c = 2x + n, \quad (8.)$$

we shall have the expression

$$P = \frac{2}{\pi} \int_0^\infty d\alpha \left(\frac{\sin \alpha}{\alpha} \right)^n \frac{\sin c\alpha}{\alpha}, \quad (9.)$$

as a transformation of the formula

$$P = \frac{1}{1 \cdot 2 \cdot 3 \dots n \cdot 2^n} \left\{ (n+c)^n - \frac{n}{1} (n+c-2)^n + \frac{n(n-1)}{1 \cdot 2} (n+c-4)^n - \&c. \right. \\ \left. - (n-c)^n + \frac{n}{1} (n-c-2)^n - \frac{n(n-1)}{1 \cdot 2} (n-c-4)^n + \&c. \right\}; \quad (10.)$$

each partial series being continued only as far as the quantities raised to the n th power are positive. Laplace has arrived at an equivalent transformation, but by a much less simple analysis.