On the Phænomena presented by Light in its Passage along the Axes of Biaxal Crystals. By the Rev. HUMPHREY LLOYD, *A.M. M.R.I.A. Fellow of Trinity College, and Professor of Natural and Experimental Philosophy in the University of Dublin**.

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It is well known that when a ray of light is incident upon certain crystals, such as Iceland spar and quartz, it is in general divided into two pencils, one of which is refracted according to the known law of sines, while the direction of the other is determined by a new and extraordinary law first assigned by Huyghens.

These laws were long supposed to apply to all doubly-refracting substances; and it was not until the subject was taken up by Fresnel, that the problem of double refraction was solved in all its generality. Setting out from the hypothesis, that the elasticity of the vibrating medium within the crystal is unequal in three rectangular directions, Fresnel has shown that the surface of the wave is neither a sphere nor a spheroid, as in the Huyghenian law, but a surface of the 4th order, consisting of two sheets, whose points of contact with the tangent planes determine the directions of the two rays. From this construction it follows that neither of the rays, in general, obeys the law of Snellius, or that of Huyghens, but that they are both refracted according to a new and more complicated law. Such crystals have two optic axes, and are said to be *biaxal*. When the elasticity of the medium is equal to two of the three directions, the equation of the surface of the wave is resolvable into two quadratic factors, which give the equations of the sphere and spheroid of the Huyghenian theory. The two optic axes in this case coincide in one; and the law of Huyghens is thus deduced from the general solution, and proved to belong to the case of *uniaxal* crystals. Finally, when the elasticity is equal in all the three directions, the surface of the wave becomes a sphere; and the refraction is single, and takes place according to the ordinary law of the sines.

There are two remarkable cases, however, in this elegant and profound theory, which its author seems to have overlooked, if not to have misapprehended. In a communication made, some months since, to the Royal Irish Academy, Professor Hamilton has supplied these omissions in the theory of Fresnel, and has thus been led to results in the highest degree novel and remarkable.

To understand these conclusions, it will be necessary to examine for a moment the form of the wave. Its equation, referred to polar coordinates, is

$$\begin{aligned}&(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma)r^4\\&-[a^2(b^2+c^2)\cos^2\alpha + b^2(a^2+c^2)\cos^2\beta + c^2(a^2+b^2)\cos^2\gamma]r^2\\&+a^2b^2c^2 = 0\end{aligned}$$

* Communicated by the Author.

in which α, β, γ, denote the angles made by the radius vector with the three axes of coordinates. If now we make $\cos\gamma = 0$ in this equation, so as to obtain the section of the surface made by the plane of $x\,y$, the result is reducible to the form,

$$(r^2 - c^2)[(a^2\cos^2\alpha + b^2\sin^2\alpha)r^2 - a^2b^2] = 0;$$

so that the surface of the wave intersects the plane of $x\,y$ in a circle and ellipse, whose equations are

$$r = c, \quad (a^2\cos^2\alpha + b^2\sin^2\alpha)r^2 = a^2b^2.$$

Now if c, the radius of the circle, be intermediate between a and b, the semiaxes of the ellipse, the two curves will intersect in four points, or *cusps*; and the angle which the radius vector drawn to the cusp makes with the axis of x, is found by eliminating r between the two equations, by which means we obtain

$$\sin\alpha = \pm\frac{1}{c}\sqrt{\frac{c^2 - b^2}{a^2 - b^2}}.$$

At each of the points thus determined, there will be two tangents to the plane section, and therefore two tangent planes to the surface; and consequently a single ray, proceeding within the crystal to one of these points, will at emergence be divided into two, whose directions are determined by those of the tangent planes.

Such seems to have been Fresnel's conception of this case. Professor Hamilton has shown, however, that there is a cusp at each of these points, not only in this particular section, but in every section of the wave-surface passing through the line whose direction has just been determined; of that there are, in fact, *four conoïdal cusps* on the general wave-surface at the points of intersection of the circle and ellipse. So that there must be an infinite number of tangent planes at each of these points, and consequently, a *single ray*, proceeding from a point within the crystal in any of the above-mentioned directions, ought to be divided into an *infinite number of emergent rays*, forming a cone of the 4th order.

It is evident, further, that the circle and ellipse which thus intersect must have four common tangents. Fresnel has shown that the planes passing through these tangents, and parallel to the 3rd or mean axis, are parallel to the circular sections of a curved surface which he calls the surface of elasticity; and he seems to have concluded that these planes touched the wave-surface only in the two points just mentioned; and consequently that a single ray, proceeding from a point without a biaxal crystal, and refracted in the direction of the optic axis, would necessarily be divided into two, determined by the points of contact. Professor Hamilton, however, has shown that the four planes in question touch the wave-surface, not in two points only, but in an *infinite number of points*, constituting each a small *circle of contact*, whose plane is parallel to one of the two circular sections of the surface of elasticity; and that, consequently, a *single ray* of common light, proceeding from an external point, and refracted in the required direction, ought, if the theory be true, to be divided *within* the crystal into an *infinite number of rays*, constituting a *conical surface*.

Here then are two singular and unexpected consequences of the undulatory theory, not only unsupported by any phænomena hitherto noticed, but even opposed to all the analogies derived from experience. If confirmed by experiment, they would furnish a new and almost convincing proof of the truth of that theory; and if disproved, on the other hand, it is evident that the theory must be abandoned or modified.

Being naturally anxious to submit the theory of waves to this delicate test, and to ascertain how far these new theoretical conclusions were in accordance with actual phænomena, Professor Hamilton requested me to undertake a series of experiments with that view. I accordingly applied myself to this experimental problem with all the attention which the subject so well deserved, and have fortunately succeeded in verifying the first-mentioned species of conical refraction. I hope before long to be able to make similar researches on the second*.

The mineral I employed in these experiments was *arragonite*, which I selected partly on account of the magnitude of the cone which theory indicated in this instance, and partly because the three elasticities in this mineral have been determined, apparently with great care, by Professor Rudberg, and therefore the results of theory could be applied to it at once without further examination. The specimen I used was one of considerable size and purity, procured for me by Mr. Dollond, and cut with its parallel faces perpendicular to the line bisecting the optic axes. If we suppose a ray of common light to pass in both directions out of such a crystal, along the line connecting the two cusps in the wave, it is evident that it must emerge similarly at both surfaces: consequently the ray which passes along this line, and forms a diverging cone of rays at emergence at the second surface of the crystal, must arise from a converging cone incident upon the first surface. Having therefore nearly ascertained the direction of the optic axis by means of the rings, I placed a lens of short focus at the distance of its own focal length from the first surface, and in such a position that the central rays of the pencil might after refraction pass along the axis. Then looking through the crystal at the light of a lamp placed at a considerable distance, I observed, in the expected direction, a point more luminous than the space immediately about it, and surrounded by something like a stellar radiation. Fearing that this appearance might have arisen from some imperfection in the crystal, I examined it with polarized light, and was happy to find the system of rings in the same direction. This was afterwards confirmed by numerous observations on different parts of the crystal.

This result is of some interest in itself, independently of its connexion with theory. It has been hitherto supposed that the only means of determining experimentally the direction of the optic axes, in substances of weak double refraction, was by observation of the rings which appear around them, when the incident and emergent light is polarized. Here, however, it is seen that common or unpolarized light undergoes such modifications in the neighbourhood of the optic axes of biaxal crystals, that the apparent direction of the axes may be at once determined, and with the simplest contrivance.

But to examine the emergent cone it was necessary to exclude the light which passed through the crystal in other directions. For this purpose a plate of thin metal, having a minute aperture, was placed on the surface of the crystal next the eye; and the position of the aperture so adjusted that the line connecting it with the luminous point on the first surface might be, as nearly as possible, in the direction of the optic axis. The exact adjustment to this direction was made by subsequent trial. The phænomenon which presented itself,

* Since we received this paper, we have been informed by the author that he has now *observed* phænomena corresponding to the *second* species of conical refraction, and of which an account will be given in our next Number.—Edit.

on looking through the aperture, when this adjustment was complete, was in the highest degree curious. There appeared a luminous circle with a small dark space round the centre, and in this dark space (which was also nearly circular) were two bright points divided by a narrow and well-defined dark line. When the aperture in the plate was slightly shifted, the appearances rapidly changed. In the first stage of its change the central dark space became greatly enlarged, and a double cone appeared within it. The circle was reduced to about a quadrant, and was separated by a dark interval from the cone just mentioned. The remote cone then disappeared, and the circular arch diminished; and, as the obliquity of the line to the axis was further increased, these two luminous portions merged gradually into the two pencils into which a single ray is divided in the other parts of the crystal.

The same experiments were repeated by bringing the flame of the lamp close to the first surface of the crystal. In this case the lens was removed, and the incident cone of rays formed by covering the surface of incidence with a thin metallic plate perforated with a minute aperture. The results were perfectly similar to those obtained in the former case.

But to apply a yet more palpable test to this theory, I substituted a narrow linear aperture for the point, in the plate next the lamp; and fixed it so that the plane passing through the line of the first plate and the point on the second, should be the plane of the optic axes. In this case, according to the received theory, all the rays transmitted through the two apertures should be refracted doubly in the plane of the optic axes, so that no part of the line should appear enlarged in breadth on looking through the second aperture; whereas, according to Professor Hamilton's beautiful deduction from the same theory, the ray proceeding in the direction of an optic axis should be refracted in every plane passing through that line. In accordance with this conclusion I found, on looking through the second aperture, that the luminous line was undilated, except in the direction of one of the optic axes; and that in the neighbourhood of this direction its boundaries ceased to be rectilinear, and it swelled out into an oval curve.

The experiment seems to go far in affording a general verification of the principle. I was anxious, however, to observe the emergent cone more directly. After some trials I effected this with the suns' light, and received the rays, emerging from the aperture in the second plate, on a screeen of roughened glass. I was thus enabled to observe the phaenomenon at various distances, and with all the advantages of enlargement. The light was sufficiently bright, and the appearance distinct, when the plane section of the cone of rays on the screen was even two inches in diameter.

Examining the emergent cone with a tourmaline plate, I was surprised to observe that one radius only of the section of the cone vanished, in a given position of the axis of the tourmaline; and that the ray which disappeared ranged through 360°, as the tourmaline plate was turned through 180°. From this it appeared that all the rays of the cone are polarized in different planes.

On examining this curious phænomenon more attentively, I discovered the remarkable law,—that "the angle between the planes of polarization of any two rays of the cone is half the angle contained by the planes passing through the rays themselves and its axis." This law accounts for the disappearance of one radius only of the section of the cone, the opposite radius being in fact polarized in a plane at right angles to the plane of polarization of the first. The law itself can be easily shown to be a necessary consequence of the general theory applied to this particular case; it is, however, but approximately true, and holds only on the assumption that the biaxal energy of the crystal is small,—an assumption justified by the phænomena of all crystals hitherto examined.

The general phænomena being observed, it remained to take measurements, and to compare them with the results of theory. For this purpose I determined the magnitude of a section of the cone, at a considerable distance from the crystal, by observing, with the assistance of a small telescope, the points at which the aperture ceased to be visible by means of the transmitted light. The distance being then accurately measured, the angle of the cone could be obtained from a table of tangents. This angle was thus found to amount to $6^\circ\,14'$ in the plane of the optic axes, and to $5^\circ\,46'$ in the perpendicular plane,—the mean being exactly 6°. I then placed the flame of a wax taper at the centre of this section, and removing the plate from the second surface of the crystal, placed a mark at a considerable distance on the line of the reflected ray. Then placing a Hadley's sextant with its centre in the place of the crystal, I measured the angular distance between the flame and the mark. This angle was found to by $31^\circ\,56'$, and consequently the angle of emergence corresponding to the axis of the cone was $15^\circ\,58'$.

Now assuming the three indices for arragonite to be $1 \cdot 5326$, $1 \cdot 6863$, $1 \cdot 6908$, which are the indices for the mean ray E, as determined by Professor Rudberg*, Professor Hamilton has shown that the direction of the emergent rays in the plane of the optic axes will be given by the formulæ

$$\sin R_{\text{o}} = 1 \cdot 6863 \,.\, \sin I$$
$$\sin R_{\text{e}} = 1 \cdot 68708 \,.\, \sin(I - 1^\circ\,44'\,48'')$$

in which I is the internal angle of incidence, or the angle which the cusp ray makes with the normal to the surface of emergence; and R_{o}, R_{e} are the corresponding angles of refraction in air. But in the present instance the normal to the surface of emergence bisects the angle of the optic axes, and therefore $I = 9^\circ\,56'\,27''$. Consequently $R_{\text{o}} = 16^\circ\,55'\,27''$, and $R_{\text{e}} = 13^\circ\,54'\,49''$. Now the difference of these angles, or $3^\circ\,0'\,38''$, may be called the angle of the cone; and half the sum, or $15^\circ\,25'\,8''$, is the mean angle of emergence. The angle $15^\circ\,58'$, found above, differs from this by $33'$ only; but the observed angle of the cone is about double of that given by theory.

I also measured the angle of the cone by receiving it on a screen of roughened glass at different distances, and tracing the outline of the section on the screen: the diameter of this section and the distance being then measured, the angle was determined. Three measurements taken in this manner gave for the magnitude of that angle respectively $6^\circ\,24'$, $5^\circ\,56'$, $6^\circ\,22'$, the mean of which, $6^\circ\,14'$, agrees very nearly with that determined by the former method.

Conceiving that the difference between experiment and theory arose chiefly from the rays which were inclined to the optic axis all round at small angles, and which were transmitted at the second surface in consequence of the sensible magnitude of the aperture, I determined to try the effects of apertures of various forms and dimensions.

When the aperture was at all considerable, two concentric circles were seen to surround the optic axis, the interior circle having about double the brightness of the annulus which surrounded it. The light of the interior circle was unpolarized, while that of the surrounding annulus was polarized according to the law already mentioned. When the aperture was di-

* See the Lond. and Edinb. Phil. Mag. and Journ., vol. i. p. 140–141.—Edit.

minished, the inner circle contracted in diameter, the breadth of the outer annulus remaining nearly the same; until the former was finally reduced to a point in the centre of the exterior circle. When the aperture was still further diminished, a dark space sprung up in the centre, which enlarged as the aperture decreased; until finally, with a very minute aperture, I succeeded in rendering this space about $\frac{3}{4}$ of the whole, or of reducing the breadth of the luminous annulus to about $\frac{1}{8}$ of its exterior diameter.

With this diminished aperture I examined the appearance produced by a line of light on the first surface parallel to the plane of the optic axes. The swelling curves, which it has been already remarked, surrounded the optic axis in this case, were reduced to a breadth corresponding to that of the annulus in the former experiment, and were separated by a considerable dark interval. When the plane passing through the two apertures deviated a little from the plane of the optic axes, the phænomena underwent many beautiful changes, the curves all assuming in all cases the form of the conchoid, whose pole was the projection of the optic axis, and asymptote the line on the first surface.

Finally, when the apertures on the two surfaces were transposed, I found that no change was made in the resulting phænomena; and that they seemed, in fact, to be in all respects similarly related to the surfaces of incidence and emergence.

It is easy to render an account of these various appearances. When the aperture on the second surface is considerable, the rays proceeding to its circumference from a point on the first surface will be sensibly inclined to the optic axis, which we shall suppose to be in the line connecting the point with the centre of the aperture. Consequently the interior, as well as the exterior rays, into which each of them is divided, will be inclined *outwards*; and it is obvious that there will be a central bright space, every point of which is illuminated by one interior and one exterior ray. This space therefore will have double the brightness of the surrounding space, each point of which is illuminated by one ray only; and as the rays which combine to form it are polarized in planes at right angles to one another, the resulting light will be unpolarized.

When the aperture is diminished, the inclination of these interior rays to one another decreases; until finally they become parallel, and the central bright space is reduced to a point. When the aperture is still further diminished, the interior rays become inclined *inwards*, and cross; and it is obvious that beyond the point of junction there will be a dark space illumined by no ray whatever. As there is no meeting of rays oppositely polarized in this case, the whole of the light will be polarized, and according to the law already explained. Finally, when the aperture is still further diminished, the interior rays at one side approach to parallelism with the exterior at the other; and the central dark space enlarges, and approaches to equality with the outer and limiting cone. Thus the annulus of light is indefinitely diminished in breadth, and the cone approaches to a mathematical surface.

It will be easily seen that the angle of the true cone is, nearly, half the sum of the angles of the exterior and interior limits of the observed conical annulus; and that when a bright space appears in the centre, as is the case when the aperture is considerable, the true angle is half the difference of the angles of the interior and exterior cones. When the whole cone is of uniform brightness, and the central dark space reduced to a point, the observed cone is just double of that sought.

Now this last was very nearly the case in the experiments from which the measures already mentioned were taken; and consequently the corrected angle, being in this case half the observed, coincides very nearly with that deduced from theory.

As there must be an equal cone of rays incident upon the first surface of the crystal, I took other measurements with a view to determine its magnitude. For this purpose I placed a rough micrometer, consisting of two moveable metallic plates, immediately before the lens, and closed the plates, until on looking through the aperture on the second surface I could see them just touching the opposite sides of the circular images. The same thing was done for the interior circle of the annulus, and the focal length of the lens accurately measured. In this manner the extreme dimensions of the conical annulus were ascertained, and the true angle calculated. The mean of three such measurements gave $3^\circ\,47'$ for the corrected angle of the cone.

It has been observed that the theoretical angle of the cone has been computed from the three indices of refraction as determined by M. Rudberg. Now a very small error in the determination of these indices, or a very minute difference between their values in different specimens of the same mineral, would make a considerable change in the angle. On the other hand, the effects of diffraction must in some degree modify the experimental results. And hence, though the measures were not taken with all the means to insure accuracy of which they are susceptible, it will be seen that their correspondence with theory is as close as could be reasonably expected.