Simulations in Early-Universe Theory

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Lattice '05, Dublin

LG: leptogenesis

EW & QCD transitions

BG: baryogenesis

NS: nucleosynthesis

EQ: equal energy in ‘radiation’ and ‘matter’ (non-rel.)

DEC: photon decoupling (↔ CMB)

This talk: EW and earlier, field theory

Plan:

1. Inflation, preheating, defects, baryogenesis, thermalization

2. Classical approximation

3. Φ-derivable approximations

4. Outlook

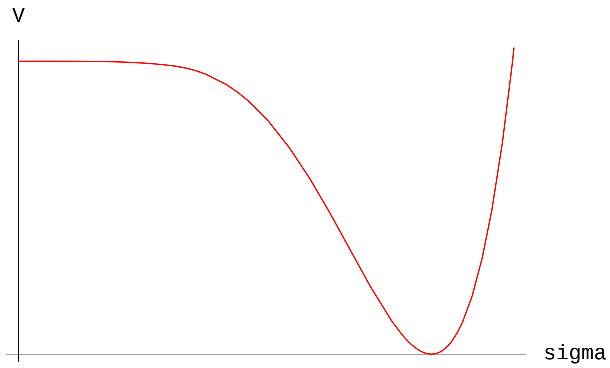
examples, not a review

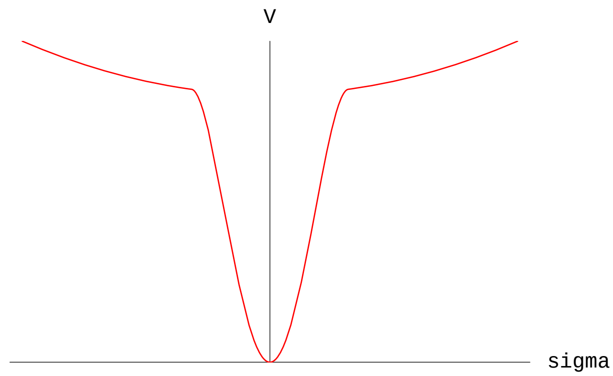
Inflation

inflaton field σ, FLRW scale factor a (flat), Hubble rate H, energy density ρ

$$
\begin{aligned}
0 &= \ddot{\sigma} + 3H\dot{\sigma} - \frac{\nabla^2\sigma}{a^2} + \frac{\partial V(\sigma,\cdots)}{\partial\sigma} \\
H \equiv \frac{\dot{a}}{a} &= \sqrt{\frac{\rho}{3m_{\mathrm{P}}^2}} \\
\rho &= \frac{\dot{\sigma}^2}{2} + \frac{(\nabla\sigma)^2}{2a^2} + V(\sigma,\cdots) + \cdots
\end{aligned}
$$

potential energy dominates, strong damping, ‘slow roll’, exponential expansion





'small-field' potential 'large-field' potential

$\sigma \uparrow$ small-field models, 'new inflation'
$\sigma \downarrow$ large-field models, 'chaotic inflation'

e.g. large-field potential (hybrid inflation):

$$V(\sigma, \varphi) = V_0 + \frac{1}{2}\mu_\sigma^2 \sigma^2 + g^2\sigma^2\varphi^\dagger\varphi - \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$$

($\mu_\sigma \ll \mu$, when $g^2\sigma^2 - \mu^2 < 0$, $\varphi^\dagger\varphi \to (\mu^2 - g^2\sigma^2)/\lambda$)

Preheating

after inflation: universe is cold*

σ-energy transferred to other d.o.f.

$$V(\sigma, \cdots) = V(\sigma) + (g^2\sigma^2 - \mu^2)\varphi^\dagger\varphi + \cdots$$

*different in 'warm inflation' Berera & Ramos time-dependent effective φ-mass

$$\mu_{\text{eff}}^2 = -\mu^2 + g^2\sigma^2$$

- $\langle\sigma\rangle$ oscillates $\rightarrow$ φ resonates

- $\mu_{\text{eff}}^2 < 0 \rightarrow$ tachyonic instability

σ-modes can also resonate and σ is also tachyonic during $\partial^2 V/\partial\sigma^2 < 0$

$\Rightarrow$ large occupation numbers in σ and φ in limited momentum range: preheating

Zoo of defects

domain walls, strings, textures, monopoles, Q-balls, I-balls, oscillons, half knots, sphalerons, Chern-Simons numbers, . . .

Baryogenesis

explain baryon to photon ratio

$$\frac{n_B}{n_\gamma} \simeq 6.5\,10^{-10}$$

from $n_B = 0$ after inflation and B, C & CP violation during particle physics processes out of equilibrium

very many scenarios proposed

here:

Cold ElectroWeak Baryogenesis

anomaly in divergence of baryon current

$$\partial_\mu j_B^\mu = 3\,q, \qquad\qquad q = \frac{1}{16\pi^2}\,\mathrm{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

topological-charge density

tachyonic EW transition with CP violation $\Rightarrow$ baryon asymmetry

$$B(t) = 3\int_0^t dt' \int d^3x\,\langle q(\mathbf{x},t')\rangle$$

*García-Bellido, Grigoriev, Kusenko, Shaposhnikov, PRD 60 (1999) 123504; Krauss, Trodden, PRL 83 (1999) 1502

Thermalization

redistribution of energies in d.o.f.

how long does it take?
(in the expanding universe)

(re?)heating temperature $T_{\rm rh}$ marks beginning of radiation-dominated universe

need $T_{\rm reh} \gtrsim 1$ GeV for nucleo-synthesis

Classical Approximation

real time

$$\langle O(t) \rangle = \mathrm{Tr}\rho\, O(t) = \mathrm{Tr}\rho\, U^\dagger(t)\, O\, U(t)$$

difficult in field theory

classical approximation

- reasonable for large occupation numbers

- depends on initial conditions

coherent-state* or Wigner represenation

$$\langle O(\phi,\pi)\rangle = \int D\phi\, D\pi\, \rho_c(\phi_c,\pi_c)\, O(\phi_c,\pi_c)$$

classical approximation**

1. draw initial configuration from $\rho_c(\phi,\pi)$
2. solve classical e.o.m. for ϕ and π
3. evaluate $O(\phi,\pi)$
4. average over initial configurations

*Sallé, JS, Vink, PRD 64 (2001) 025016

**perturbative analysis: Aarts, JS, PLB 393 (1997) 395; NPB 511 (1998) 451 ; this meeting: Yavin occupation numbers

$$a_{\mathbf{k}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(\omega_{\mathbf{k}}\phi_{\mathbf{k}} + i\pi_{\mathbf{k}}), \qquad n_{\mathbf{k}} = \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle$$

generalize to time-dependent (quasi)particle energies $\omega_{\mathbf{k}}$ and numbers $n_{\mathbf{k}}$:

$$\langle \phi_{\mathbf{k}}\phi^\dagger_{\mathbf{k}}\rangle \equiv \left(n_{\mathbf{k}} + \frac{1}{2}\right)\frac{1}{\omega_{\mathbf{k}}}, \qquad \langle \pi_{\mathbf{k}}\pi^\dagger_{\mathbf{k}}\rangle \equiv \left(n_{\mathbf{k}} + \frac{1}{2}\right)\omega_{\mathbf{k}}$$

$$\langle \phi_{\mathbf{k}}\pi^\dagger_{\mathbf{k}}\rangle \equiv \tilde{n}_{\mathbf{k}} + \frac{i}{2}$$

tachyonic transition at $t = t_c$, $\mu_{\text{eff}}(t_c) = 0$

gaussian approximation

$$\ddot{\phi}_{\mathbf{k}} + (\mu_{\text{eff}}^2 + k^2)\phi_{\mathbf{k}} = 0$$

$$\mu_{\text{eff}}^2 = -M^3(t - t_c), \qquad M^3 = -2g^2\,\sigma_c\dot{\sigma}_c$$

quench:

$$\begin{aligned}\mu_{\text{eff}}^2 &= +\mu^2,\ t < t_c,\\ &= -\mu^2,\ t > t_c\end{aligned}$$

reproduce quantum $\langle \cdots \rangle$ with classical distribution

$$\exp\left[-\frac{1}{2}\sum_{\mathbf{k}}{}' \left(\frac{|\xi_{\mathbf{k}}^{+}|^2}{n_k + 1/2 + \tilde{n}_k} + \frac{|\xi_{\mathbf{k}}^{-}|^2}{n_k + 1/2 - \tilde{n}_k}\right)\right]$$

where

$$\xi_{\mathbf{k}}^{\pm} = \frac{1}{\sqrt{2\omega_k}}\,(\omega_k\,\phi_{\mathbf{k}} \pm \pi_{\mathbf{k}})$$

n_k and $\tilde{n}_k$ grow faster than exponential* and $n_k + 1/2 - \tilde{n}_k \to 0$, for $k \lesssim k_{\text{max}}$

$$\begin{aligned} k_{\text{max}} &= \sqrt{M^3(t - t_c)} \\ &= \mu, \quad \text{for the quench} \end{aligned}$$

$\sum{}'$: $k \lesssim k_{\text{max}}$

*for the quench: $n_k \sim \exp[\sqrt{\mu^2 - k^2}(t - t_c)]$

initial distribution for classical approximation:

gaussian ensemble

at time $t_i > t_c$ when $n_k, \tilde{n}_k \gg 1$ and non-linear terms in e.o.m. still small,

or ‘just the half’ at $t_i = t_c$

$k_{\text{max}} \ll$ cutoff, or $k_{\text{max}} =$ cutoff milestones in preheating:

're-scattering' effects after parametric-resonance* and tachyonic** preheating lead to efficient mixing

*Khlebnikov, Tkachev, PRL 77 (1996) 219

**Felder, García-Bellido, Greene, Kofman, Linde, Tkachev, PRL 87 (2001) 011601

Example*: 'new inflation' with Coleman-Weinberg potential**

$$V(\sigma) = \frac{1}{4}\lambda\sigma^4 \left(\ln\frac{|\sigma|}{v} - \frac{1}{4}\right) + \frac{1}{16}\lambda v^4$$

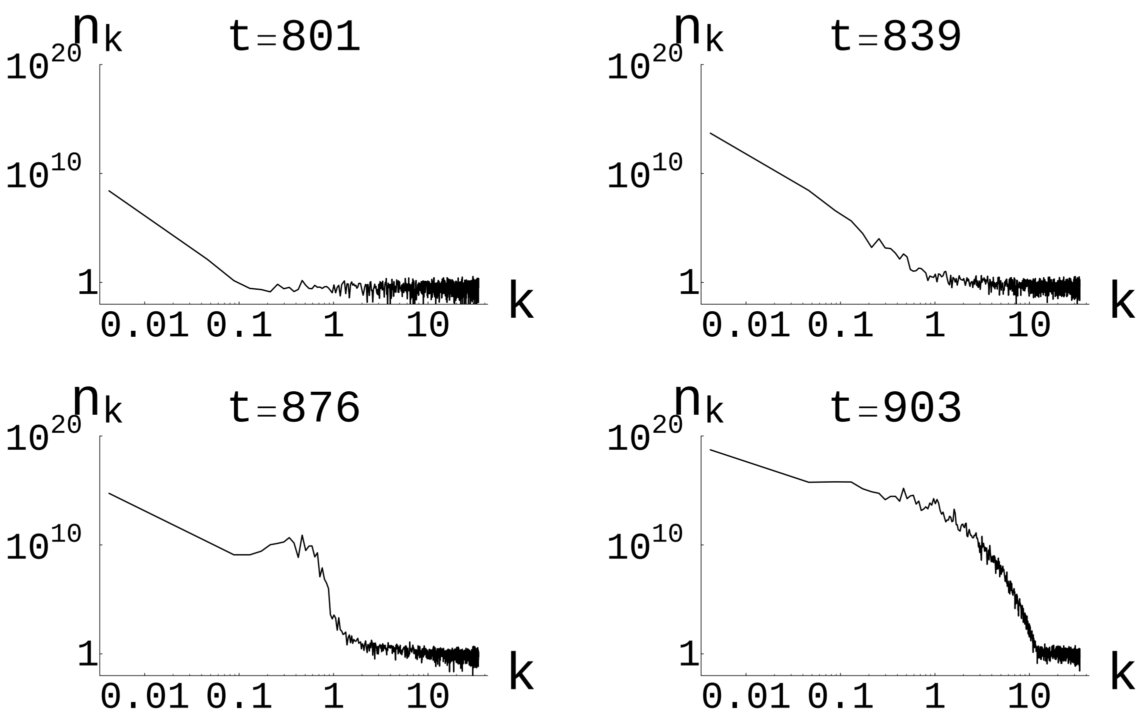
$\lambda = 10^{-12}$, $v = 10^{-3}\, m_{\mathrm{P}}$, with expansion

initial $\langle\sigma\rangle \approx 0$, it rolls into minimum of V and oscillates, energy decays into σ-particles

tachyonic & parametric-resonance preheating

*Desroche, Felder, Kratochvil, Linde, PRD71 (2005) 103516

**questionable in real time!

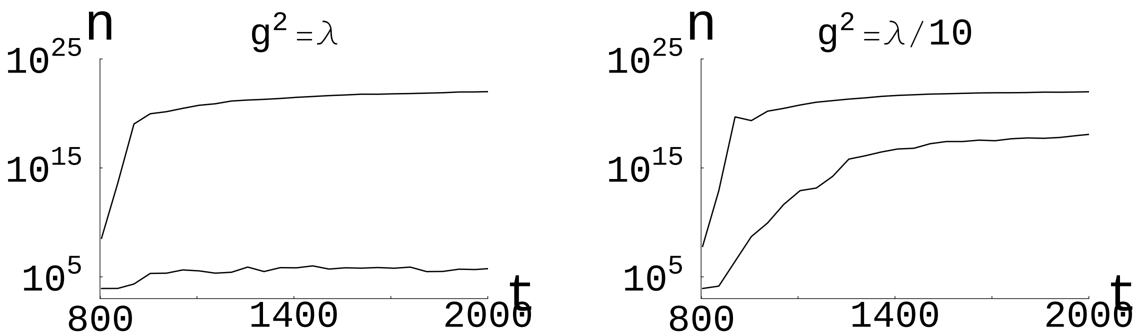


occupation numbers n_k, resonance visible at $t = 876$

t and k in λv-units

with ϕ, $V = V(\sigma) + \frac{1}{2} g^2 \sigma^2 \phi^2$ (non-tachyonic for ϕ)

number densities n_σ, n_ϕ



inefficient resonance, $n_\phi \ll n_\sigma$

$\sigma \to \phi\phi$ perturbatively,

$T_{\rm rh} \approx \sqrt{m_{\rm P}\,\Gamma_{\sigma\to\phi\phi}}$ ($\approx 10^7$ GeV for $g^2 \lesssim \lambda$)

Example*: large-field inflaton-'Higgs' SU(2) model

$$V(\sigma,\varphi) = \frac{1}{2}\mu_\sigma^2\sigma^2 + g^2\sigma^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi - \frac{1}{2}v^2)^2$$

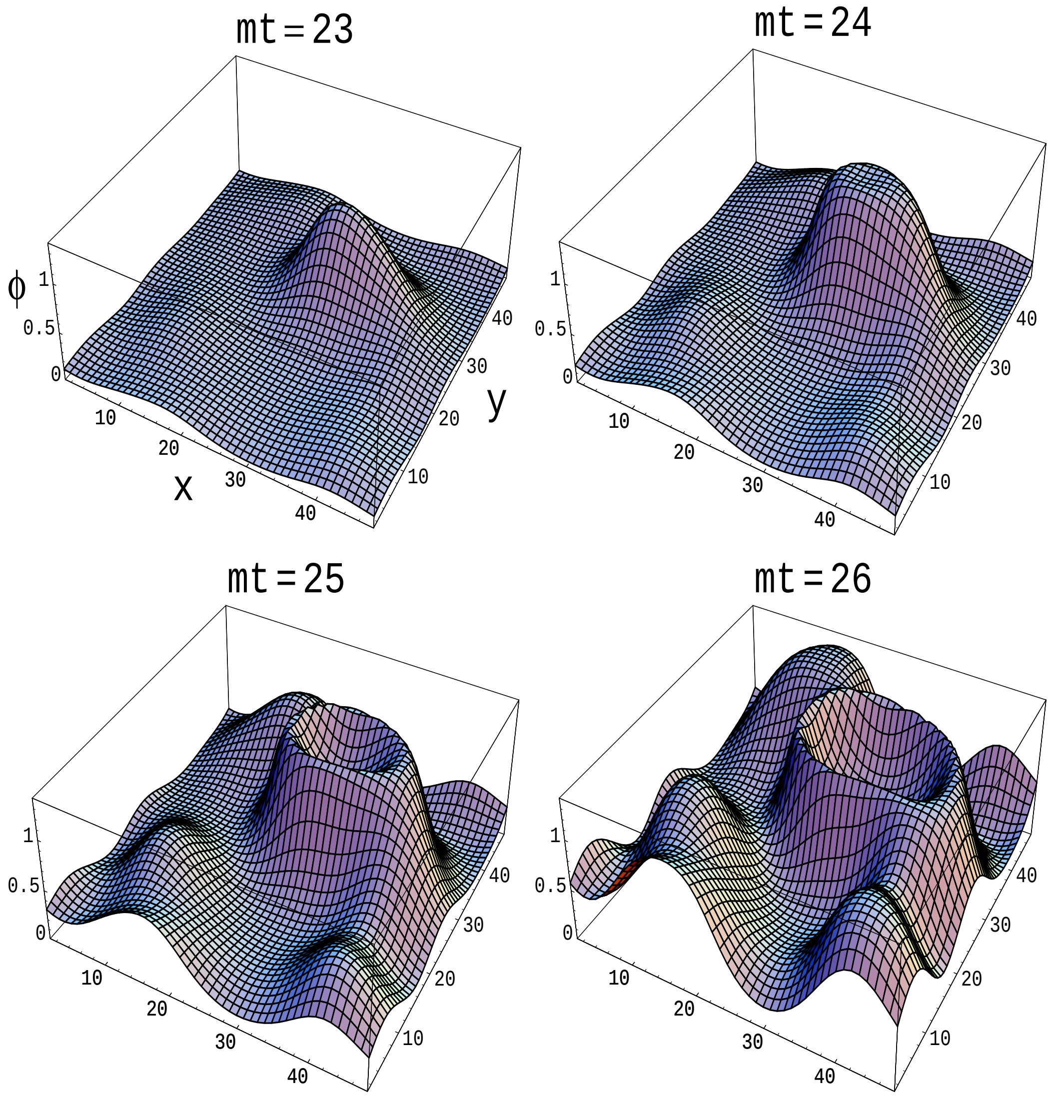
fit k-dependence of initial distribution at t_i to gaussian, $k_{\rm max} \ll$ cutoff, no expansion

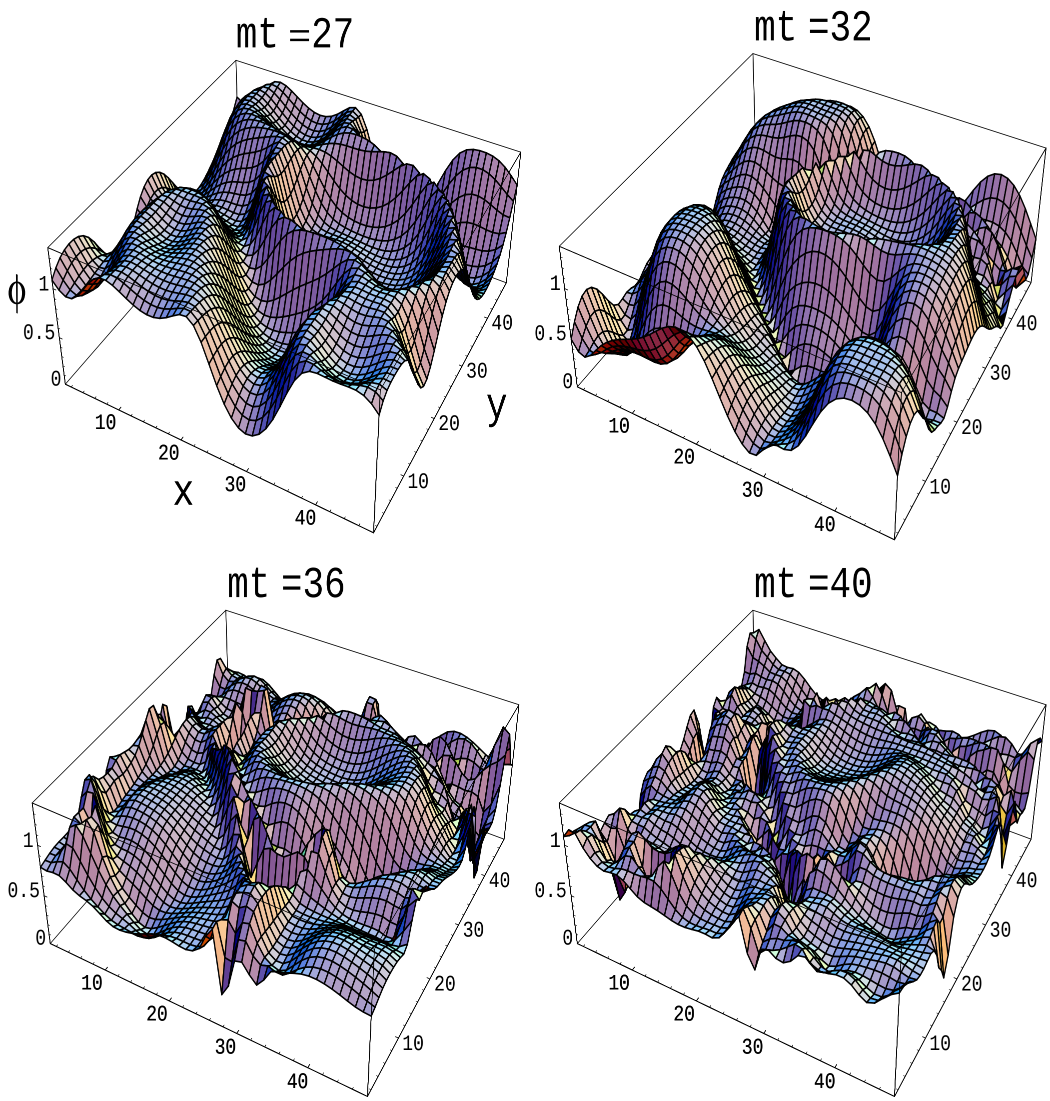
next slide:
'mixing' through scattering of waves

$\lambda = 0.11/4$, $g^2 = 2\lambda$, $M = 0.18\, m$, $m = \sqrt{\lambda}\, v$

$\sigma \to \chi$, $t_c \to 0$, no expansion,

*García-Bellido, García-Pérez, González-Arroyo, PRD67 (2003) 103501


mt = 23
mt = 24
mt = 25
mt = 26
φ
x
y
0
0.5
1
10
20
30
40



mt =27
mt =32
mt =36
mt =40
ϕ
x
y
0
0.5
1
10
20
30
40

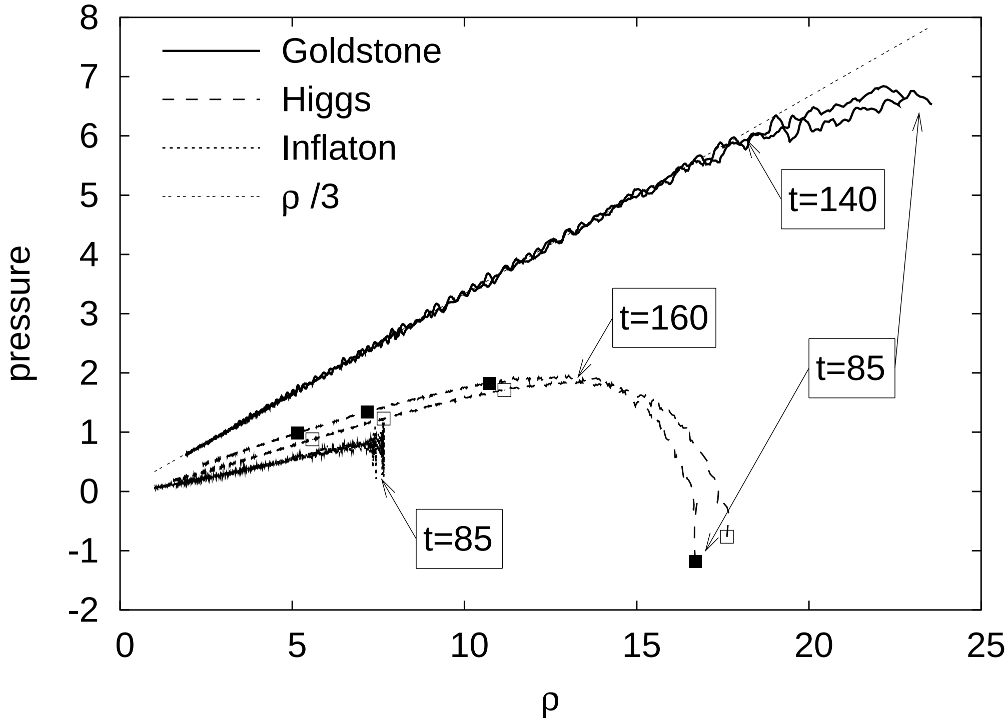
Example*: equation of state (p/ρ) in large-field inflaton-'Higgs' U(1) model

$$V = \frac{1}{2}\mu_\sigma^2\sigma^2 + g^2\sigma^2\varphi^\dagger\varphi + \lambda\left(\varphi^\dagger\varphi - \frac{1}{2}v^2\right)^2$$

$\varphi = r\,e^{i\alpha}/\sqrt{2}$, pressures:

$$\begin{aligned} p_{\mathsf{H}} &= \frac{1}{2}\dot{r}^2 - \frac{1}{6}(\nabla r/a)^2 + \frac{1}{4}\lambda(r^2 - v^2)^2 \\ p_{\mathsf{G}} &= \frac{1}{2}r^2\dot{\alpha}^2 - \frac{1}{6}(\nabla\alpha/a)^2 \\ p_\sigma &= \frac{1}{2}\dot{\sigma}^2 - \frac{1}{6}(\nabla\sigma/a)^2 + \frac{1}{2}(\mu_\sigma^2 + g^2r^2)\sigma^2 \end{aligned}$$

*Borsányi, Patkos, Sexty, PRD 68 (2003) 063512



$g^2 = 10^{-2}$, $\lambda = g^2/2$, $\mu_\sigma = 4\,10^{11}$, $m_{\mathrm{H}} = 9\,10^{14}$ GeV, with expansion

Example*: strings and hot spots

$V(\sigma,\varphi) = \frac{1}{2}\mu_\sigma^2\sigma^2 + g^2\sigma^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi - v^2)^2$

$g^2 = 10^{-4}$, $\lambda = 10^{-2}$, $m \equiv \sqrt{2\lambda}\, v = 3\, 10^{15}$ GeV
constant expansion rate $H = 3.6\, 10^{-3}\, m$

$\mu_{\rm eff}^2 = g^2\sigma^2 - m^2 = 0$ at $\sigma = \sigma_c$

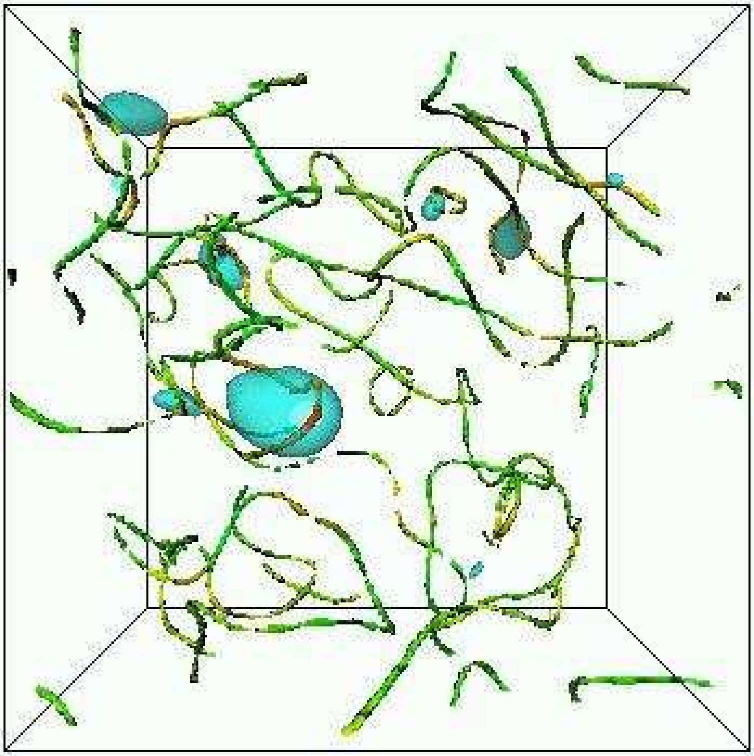
all modes ($k_{\rm max}$ = cutoff) initialized with 'just the half'

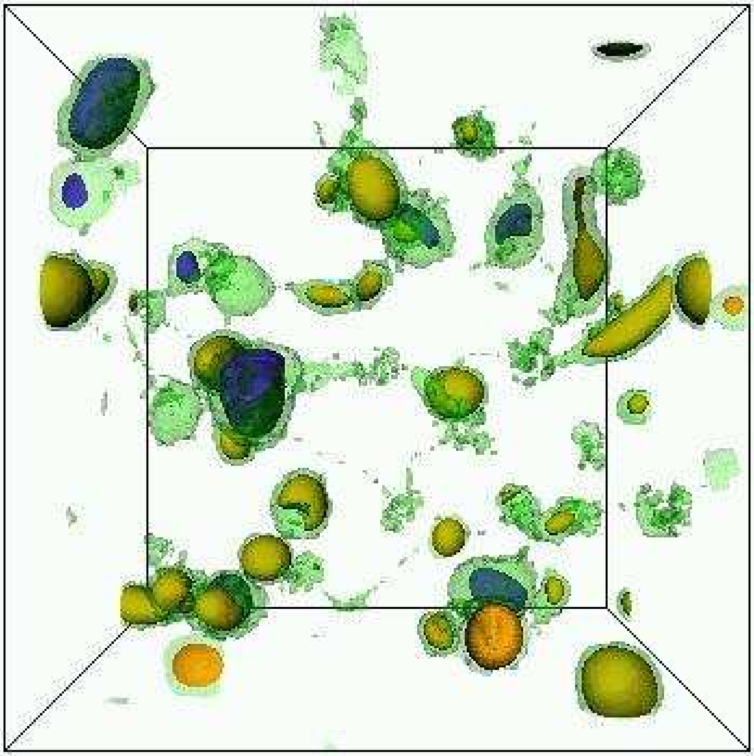
next two slides:

global strings decay, overshoots to $\sigma < -\sigma_c \Rightarrow$ hot spots,

good mixing

*Copeland, Pascoli, Rajantie, PRD65 (2002) 103517


transparant surface: $\sigma = -\sigma_c$ at time $tm = 130$; strings: $|\varphi|^2 = v^2/10$ at time for which spatial average $\overline{\sigma} = 0$ for the first time


transparant green surface: $|\varphi|^2 = v^2/10$; blue: $\sigma = \sigma_c$; yellow: $\sigma = -\sigma_c$; time $tm = 270$

Example*:

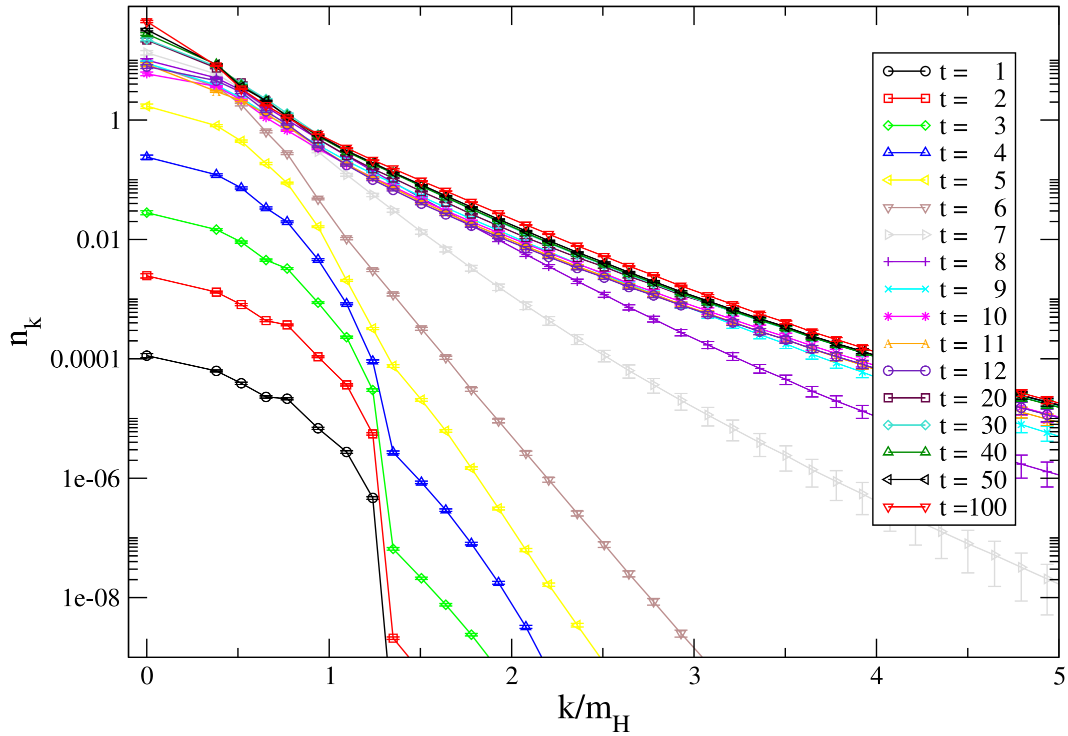
tachyonic electroweak quench

SU(2)-Higgs model

$$-\mathcal{L} = \frac{1}{2g^2}\,\mathrm{tr} F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger D^\mu\varphi - \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$$

'just the half', $k_{\mathrm{max}} = \mu$,
$\mathbf{A}_{\mathrm{init}} = 0$, $A_0 = 0$, Gauss' law

*Skullerud, JS, Tranberg, JHEP 0308 (2003) 045



Coulomb-gauge W-particle numbers as a function of momentum, time in units of m_H^{-1}

approx. gauge-independence of transverse n_k^W (Coulomb- and unitary-gauge) after $t = 50\, m_H^{-1}$ not much happens in $50 \lesssim t m_H < 100$

temperature $T \approx 0.4\, m_{\mathrm{H}}$

chemical potential $\mu_{\mathrm{ch}} \approx 0.7(0.9) m_{\mathrm{H}}$ for W(H)

longitudinal modes settle at slower rate

- suggests continuing with inclusion of lighter d.o.f. of Standard Model, e.g. through Boltzmann eqns. & quantum scattering rates

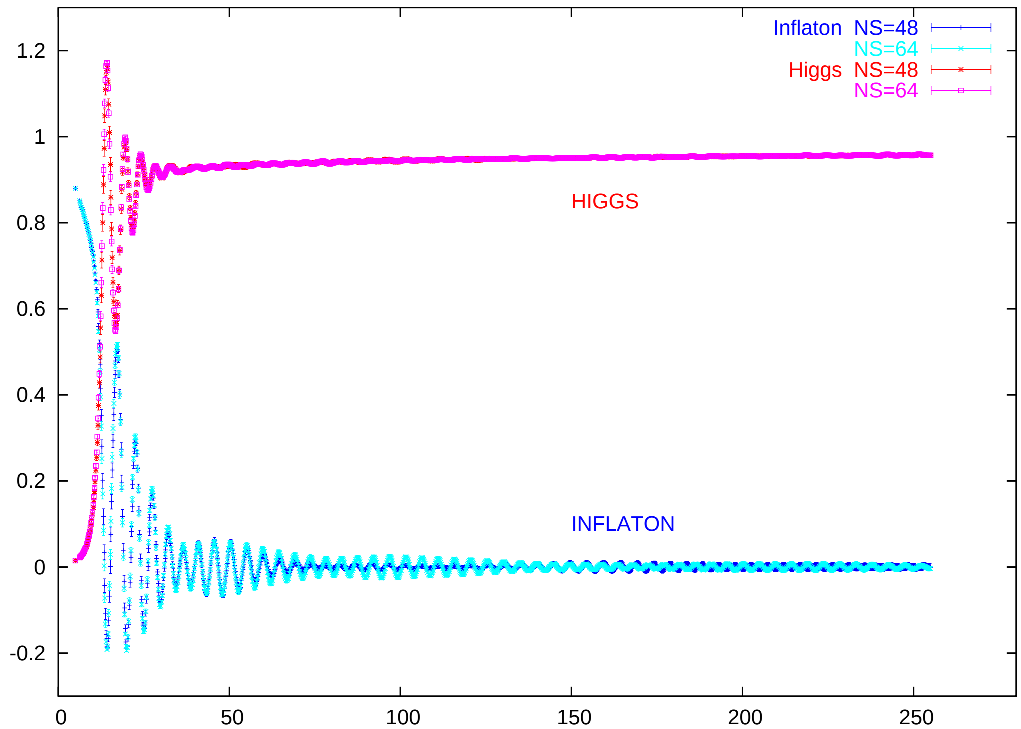
Example: kinetic turbulence and self-similar scaling* in electromagnetic field after tachyonic electroweak transition**

inflaton + SU(2)×U(1) model

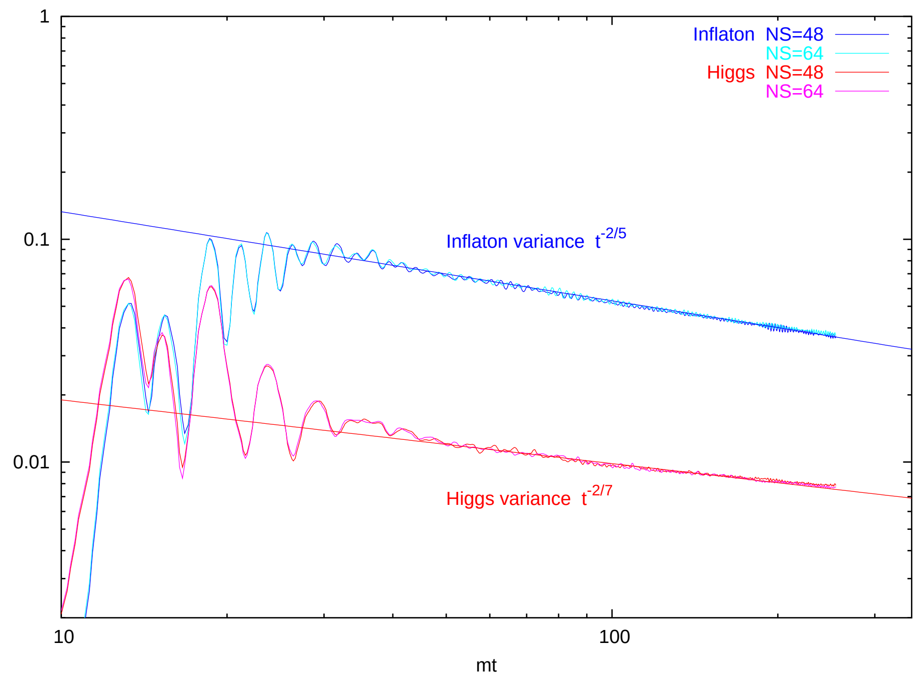
m_W = 6.15 GeV, m_{H}/m_W = 4.65, Mt_i = 5, g/g' as in SM

*Micha, Tkachev, PRD 70 (2004) 043538

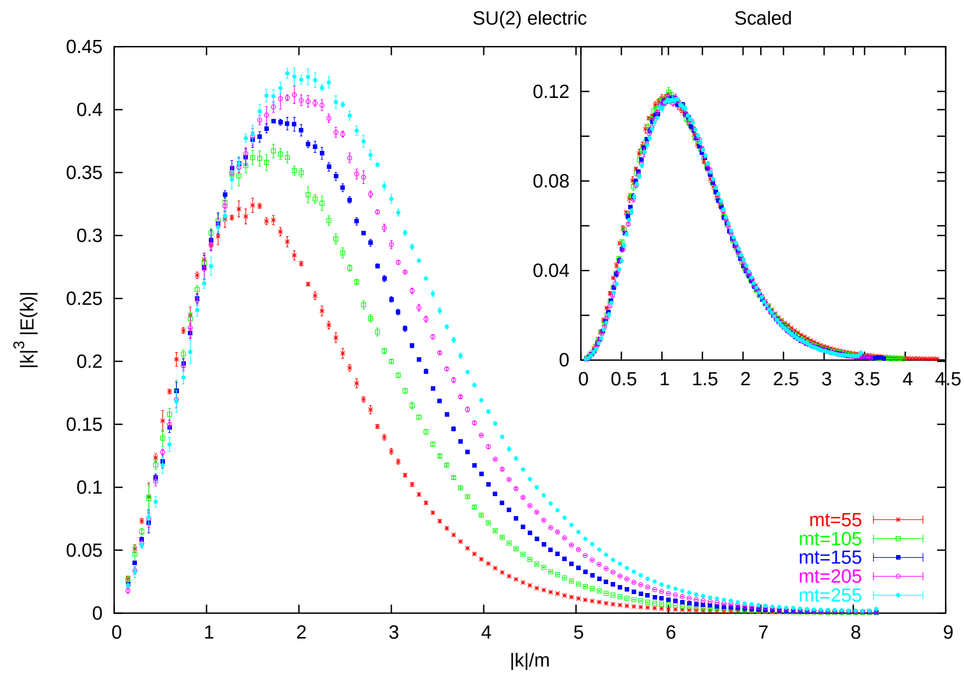
**Diaz-Gil, García-Bellido, García-Pérez, González-Arroyo, poster this meeting



Higgs $\langle \varphi^\dagger \varphi \rangle$ and inflaton $\langle \sigma \rangle$ vs time



turbulent scaling of Higgs $\langle \varphi\varphi^\dagger \rangle$ and inflaton $\langle \sigma\sigma \rangle \propto t^{-\nu}$, $\nu = -2/(2m-1)$, $m = 3$ (σ), $m = 4$ (H)



self-similar spectrum: $n(k,t) = t^{-q}\, n_0(kt^{-p})$, $q = (7/2)p$, $p = 1/(2m-1)$

Example*: cold electroweak baryogenesis

SU(2) Higgs model with effective CP violation

$$\mathcal{L} = \mathcal{L}_{\text{SU(2)H}} - \frac{k}{m_W^2}\varphi^\dagger\varphi\, q, \quad q = \frac{1}{16\pi^2}\text{tr}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

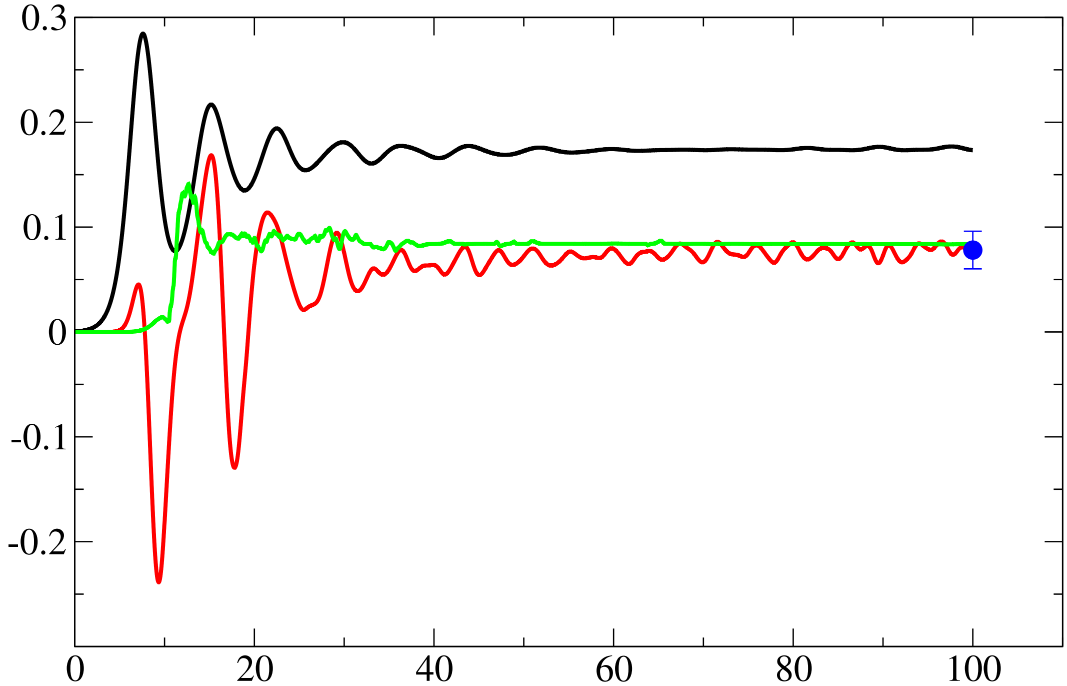
quench, 'just the half', $k_{\text{max}} = \mu = m_{\text{H}}/\sqrt{2} = 0.25/a_s$

$q = \partial_\mu j^\mu_{\text{CS}}$, $N_{\text{CS}} = \int d^3x\, j^0_{\text{CS}}$, Chern–Simons number

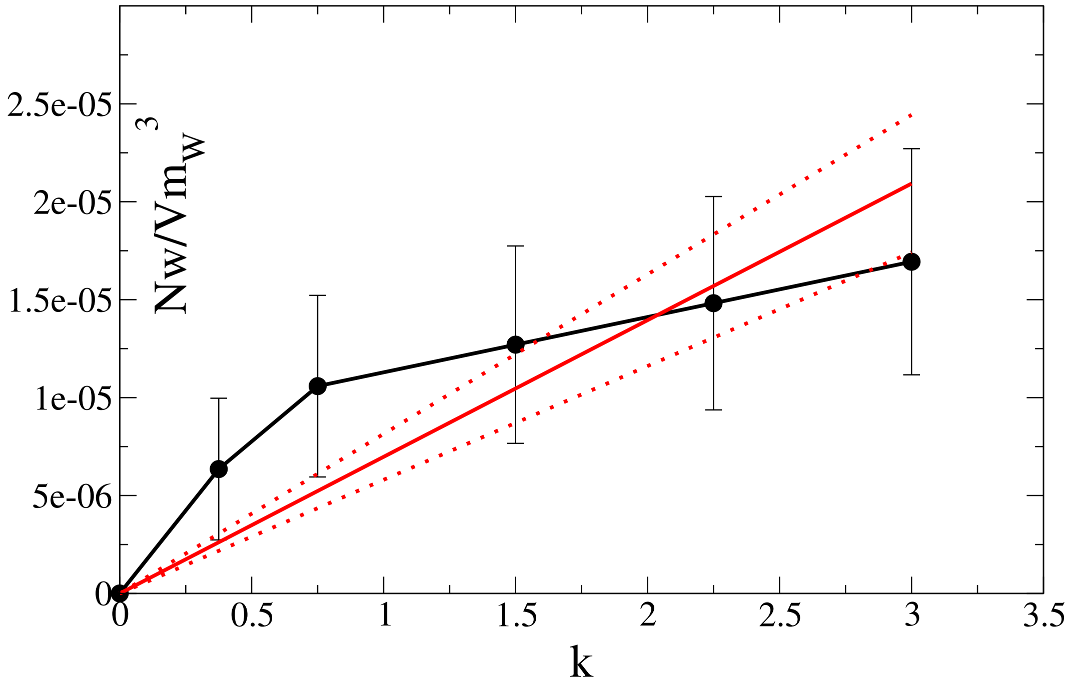
$$B(t) = 3\langle N_{\text{CS}}(t) - N_{\text{CS}}(0)\rangle$$

N_{w}: winding number in the Higgs field
$N_{\text{CS}} \approx N_{\text{w}}$, expected at low temperature

*Tranberg, JS, JHEP 0311 (2003) 016; Tranberg, this meeting



$\langle \varphi^\dagger \varphi \rangle$(black), $\langle N_{\mathrm{CS}} \rangle$(red), $\langle N_{\mathrm{w}} \rangle$(green) Higgs winding number, versus time; $k = 3$, $m_{\mathrm{H}} = \sqrt{2}\, m_W$



dependence of n_{CS} on k, $m_{\text{H}} = 2m_W$

resulting in

$$\begin{aligned} \frac{n_B}{n_\gamma} &= (4 \pm 1)10^{-5}\, k,\; m_{\mathsf{H}} = \sqrt{2}\, m_W \\ &= -(4 \pm 1)10^{-5}\, k,\; m_{\mathsf{H}} = 2\, m_W \end{aligned}$$

Thermalization

classical: Rayleigh-Jeans effects at large times

- need quantum description

approach equilibrium through scattering and damping

nonperturbative, e.g.

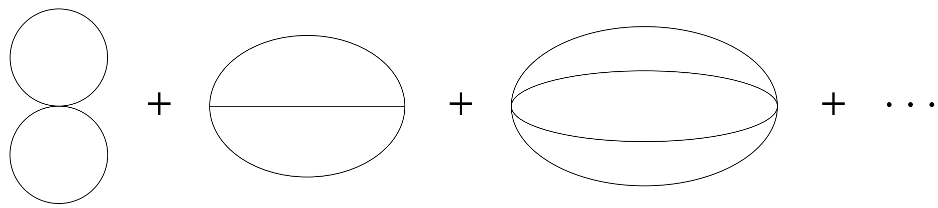
$$e^{-\gamma t}\cos(\omega t), \quad \gamma = c\lambda^2, \quad e^{-\gamma t} = 1 - c\lambda^2 t + \cdots$$

need summing infinite series of diagrams e.g. use Dyson-Schwinger hiearchy

Phi-derivable*

$$\Gamma(\phi, G) = S(\phi) - \frac{i}{2}\mathrm{Tr}\ln G + \frac{i}{2}\mathrm{Tr}\frac{\delta^2 S(\phi)}{\delta\phi\delta\phi} G + \Phi(\phi, G)$$

$i\,\Phi =$



sum of all 2PI diagrams

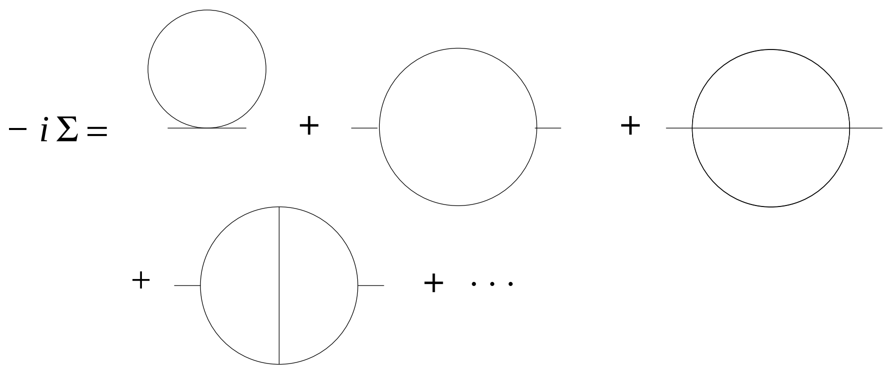
*revived simulation-wise by Berges, Cox, PLB 517 (2001) 369 equations of motion

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)} = 0, \qquad \frac{\delta\Gamma(\phi,G)}{\delta G(x,y)} = 0$$

typical form

$$\begin{aligned}\left[\partial^2 - \mu^2 - 3\lambda\phi^2 - 3\lambda G(x,x)\right] G^>(x,y) \; &= \\ 2\int_0^{x^0} dz^0 \int d^3z \, \mathrm{Im}[\Sigma^>(x,z;\phi,G)] G^>(z,y) & \\ -\, 2\int_0^{y^0} dz^0 \int d^3z \, \Sigma^>(x,z;\phi,G) \, \mathrm{Im}[G^>(z,y)] &\end{aligned}$$

selfenergy $\Sigma(x,y) = 2i\frac{\delta\Phi}{\delta G(x,y)}$



Φ-derivable:

- 'thermodynamically consistent'

- global symmetries $\rightarrow$ WTIs

- renormalizable*

*Van Hees and Knoll, PRD 65 (2002) 025010 & 105005; PRD 66 (2002) 045008

Blaizot, Iancu, Reinosa, PLB 568 (2003) 160; NPA 736 (2004) 149

Cooper, Mihaila, Dawson, PRD 70 (2004) 105008; hep-ph/0502040

Berges, Borsányi, Reinosa, Serreau, hep-ph/0409123;hep-ph/0503240

- truncate Φ by order in coupling- or $1/N$-expansion

- put Γ on space-time lattice ($a_t \ll a_s$), or use spatial lattice and solve e.g. with Runge-Kutta

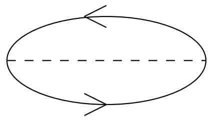
- initial conditions in terms of ϕ and G

  results directly for average $\langle \cdots \rangle$

- numerically challenging 'memory kernels' Σ

Example*: chiral quark-meson model

$$-\mathcal{L} = \bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi^a\partial^\mu\pi^a) + \frac{1}{2}\mu^2(\sigma^2 + \pi^a\pi^a) + g\bar{\psi}(\sigma + i\gamma_5\tau^a\pi^a)\psi$$

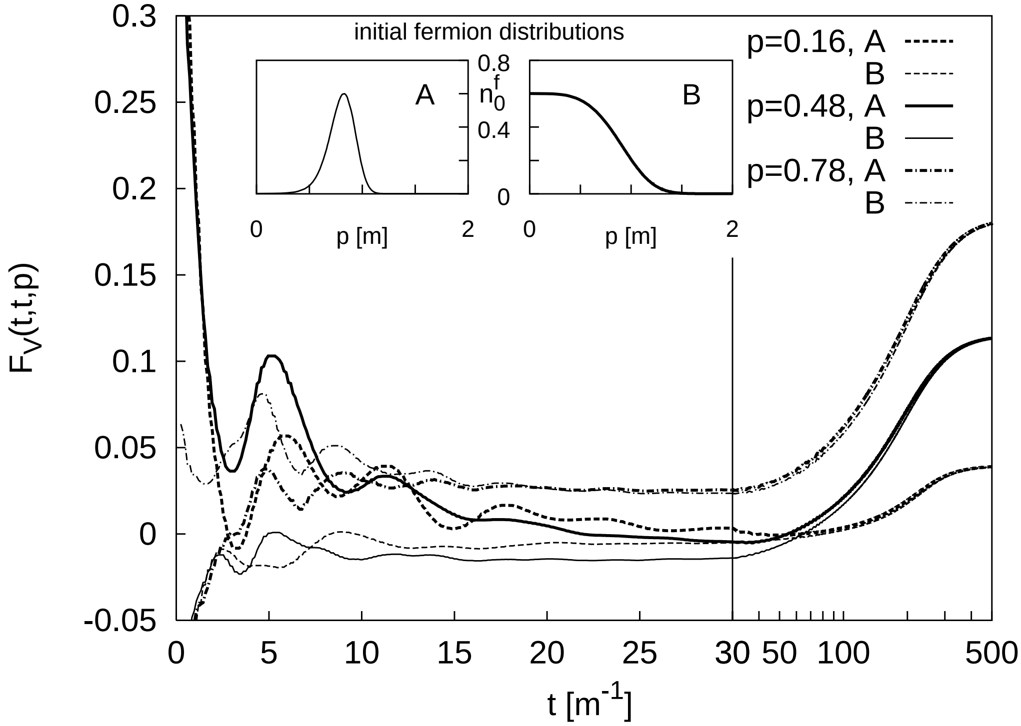
$i\,\Phi =$ 

spectral function ρ, statistical function F
$G(x,y) \sim F(x,y) - i\rho(x,y)/2$

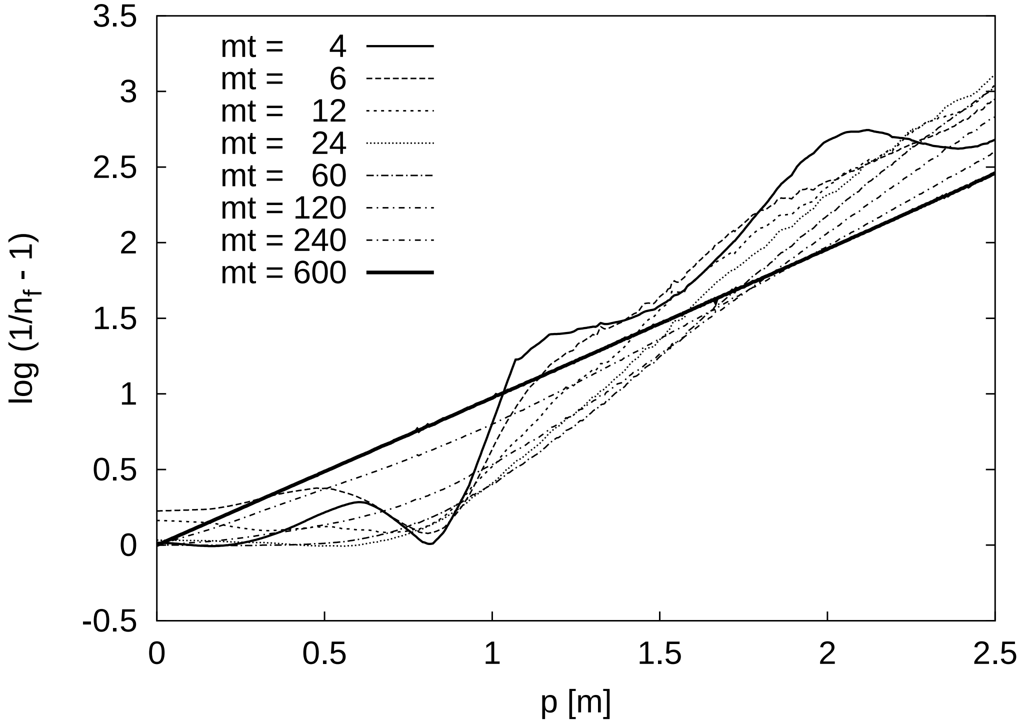
e.o.m. discretized in terms of components, e.g.

$$\rho = \rho_S + \rho_P\, i\gamma_5 + \rho_V^\mu\gamma_\mu + \rho_A^\mu\gamma_\mu\gamma_5 + \rho_T^{\mu\nu}\sigma_{\mu\nu}/2$$

*Berges, Borsányi, Serreau, NPB 660 (2003) 51



memory loss $1/\gamma_F^{\text{damp}} \approx 15\, m^{-1}$, $m =$ thermal mass



thermalization: $\ln(-1 + 1/n_p) \to p/T$, Fermi-Dirac
$1/\gamma_F^{\text{therm}} \approx 95\, m^{-1}$, $g = 1$

thermalization, relatively fast

no fermion doubling seen*

* differs from Aarts &JS, NPB555 (1999) 355; PRD61 (2000) 025002, who used lattice fermion techniques in real time

Example*: electroweak quench

$-\mathcal{L} = (1/2)\partial_\mu\phi_a\partial^\mu\phi_a - (\mu^2/2)\phi_a\phi_a + (\lambda/4)(\phi_a\phi_a)^2$

$a = 1, \ldots, N$, $\lambda \propto 1/N$, approximate Φ to Next-to-Leading-Order in $1/N$ expansion**, $N = 4$

next two slides:

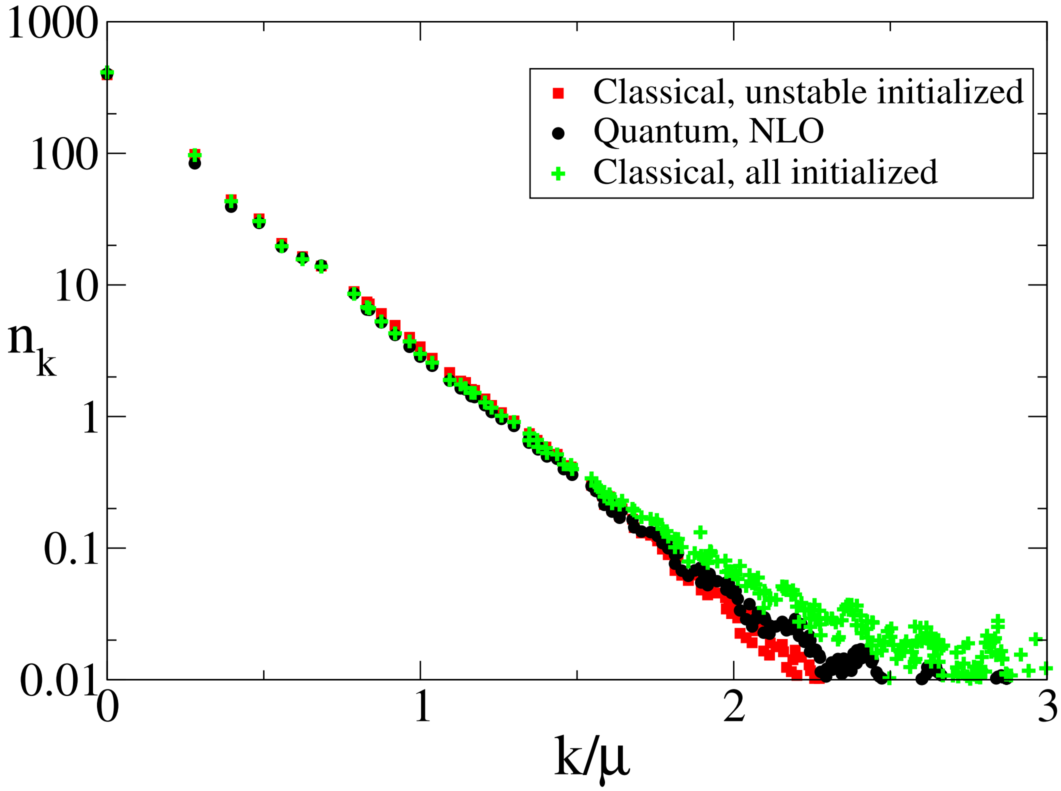
comparison of n_k in NLO with two 'just the half' classical approximations:

(a) only unstable modes initialized ($k_{\text{max}} = \mu = 0.7/a_s$)

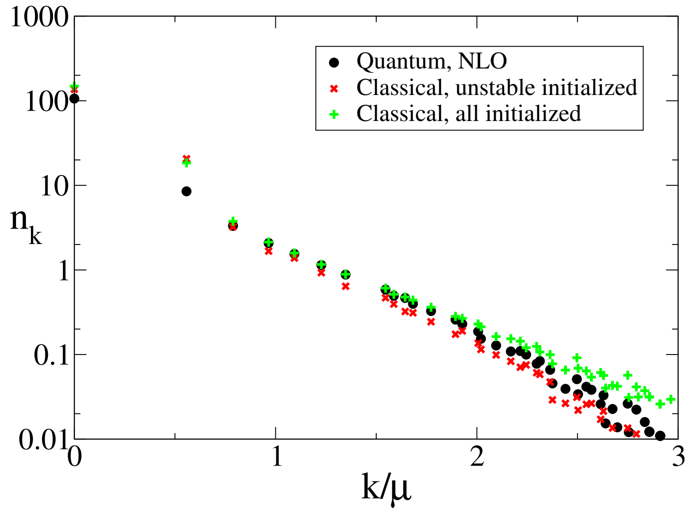
(b) all modes initialized ($k_{\text{max}} =$ cutoff)

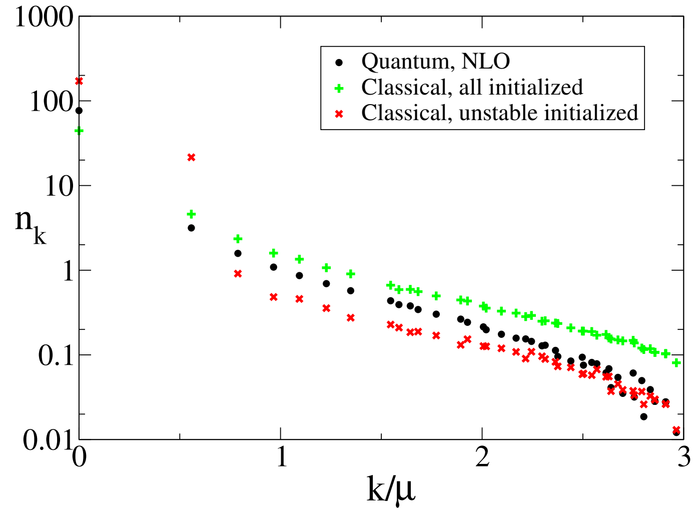
*Arrizabalaga, JS, Tranberg, JHEP 0410 (2004) 017

**Berges, NPA 699 (2002) 847



particle numbers at time $t\mu = 14$; $\lambda = 1/24$, $L\mu = 22.4$





particle numbers at time $t\mu = 100$ (Left) and 700 (Right);
$\lambda = 1/4$ ($m_{\mathrm{H}} = 174$ GeV), $L\mu = 11.2$

furthermore

Kibble-Zurek vs Hindmarsch-Rajantie* scaling in formation of topological defects in gauge theories

. . .

*Hindmarsch, Rajantie, PRL 85 (2000) 4660

Outlook

- lots to do in the Big Bang

- quantum fields out of equilibrium: approximations unavoidable

- classical approximation can be well motivated and is very useful up to intermediate times

  in tachyonic preheating, topological-like terms in e.o.m. and -observables appear feasible with $k_{\mathrm{max}} \ll 1/a_s$

  need more experience with non-abelian gauge fields*

*Moore, JHEP 0111 (2001) 017

- Φ-derivable approximations good for later times

  three-point interactions (non-zero range) are efficient for thermalization (broken symmetry phase, Yukawa, gauge fields), but difficult to incorporate in Φ beyond two loops

  Nambu-Goldstone boson mass is still an issue

  gauge fields not simulated yet (?)

  dependence on gauge fixing parameter admissable* as part of controlled approximation?

  fermion doubling?

  need to gauge fix: a blessing for the chiral case?

  real-time approach** to finite density?

*Arrizabalaga, JS, PRD 66 (2002) 065014
**Sallé, JS, Vink, NP Proc.Suppl. 94 (2001) 427

Example*: development of winding and Chern-Simons number densities in tachyonic EW transition

same parameters as before

next two slides:

n_{W}, $tm_{\mathrm{H}} = 1, \ldots, 15$

n_{CS}, $tm_{\mathrm{H}} = 7, \ldots, 15$

*Van der Meulen, Sexty, JS, Tranberg, to be pub.