RECENT RESULTS ON TOPOLOGICAL METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMA

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Topological methods of Leray-Schauder type remain a powerful method to study the existence and multiplicity of solutions of various boundary value problems for nonlinear ordinary differential equations. This lecture surveys some results recently obtained in this direction.

The simplest situation for which Leray-Schauder techniques apply occurs when the given problem can be continuously deformed to a linear one and the possible solutions remain a priori bounded during the deformation. A new class of problems to which this technique can be successfully applied is the class of complex valued periodic problems

$$a(t)z'(t) = |z|^q \overline{z}^p + h(t,z),\ z(0) = z(2\pi), \tag{1}$$

where $a \in C^1([0,2\pi], \mathbb{C} \setminus \{0\})$ is 2π-periodic, $q \geq 0$, $p \geq 1$ is an integer, $p+q > 1$, and

$$|h(t,z)| \leq \alpha |z|^{p+q} + \beta,$$

for some $0 \leq \alpha < 1$, $\beta \geq 0$ and all $(t,z) \in [0,2\pi] \times \mathbb{C}$. The corresponding a priori bounds are obtained using L^r-estimates and the topological degree of problem (1) is shown to be equal to 1 or to p according to $p+1$ divides or not the winding number of $a(t)$. This is a joint work with R. Manasevich and F. Zanolin. The difficulties of the problem

$$a(t)z'(t) = |z|^q z^p + h(t,z),\ z(0) = z(2\pi),$$

are also analyzed.

It is more difficult to use the Leray-Schauder method when the set of possible solutions is not a priori bounded. This is the case for the second order equation

$$x''(t) + g(x(t)) = p(t, x(t), x'(t)), \tag{2}$$

under various linear boundary conditions, either when g is superlinear, i.e.

$$\frac{g(x)}{x} \to +\infty \text{ if } |x| \to \infty,$$

or when g has linear growth and satisfies weak nonresonance conditions. Using a refinement of a continuation technique introduced a few years ago by Capietto, Zanolin and the author, we show that when g is superlinear and p has at most linear growth in x and x', equation (2) has, for each sufficiently large n, at least two solutions with $n+1$ zeros satisfying the Dirichlet boundary conditions over $[0,\pi]$. Examples show that this multiplicity result is sharp.

The above results concern boundary value problems over compact intervals. Some of them may be used as intermediate tools to prove, through a limit process, the existence of bounded solutions over $\mathbb{R}$ for some nonlinear ordinary differential equations. We shall describe recent joint results with J.R. Ward related to the extension of Landesman-Lazer type existence conditions for the bounded solutions of some classes of second order ordinary differential equations of the form

$$x'' + cx' + f(t,x) = 0.$$