On the uniqueness of solutions of a nonlinear elliptic equation arising in the confinement of a plasma in a Stellarator device.

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Abstract

We obtain the uniqueness of solutions of a nonlocal elliptic problem when the nonlinear terms at the right hand side are assumed to be prescribed. The problem arises in the study of the magnetic confinement of a plasma in a Stellarator device.

1 Introduction.

The main goal of this paper is to study the uniqueness of the solution of a two dimensional free boundary problem modeling the magnetic confinement of a plasma in a Stellarator device [DGP]. The model consists of a second order partial differential equation of elliptic type, obtained from the 3-D Ideal MHD system by Hender and Carreras [HC] by using toroidal averaging arguments and a suitable system of coordinates: the Boozer vacuum flux coordinates. This problem has recently been studied by Díaz [D] who introduced the following formulation in the form of a *free boundary problem*. Let Ω be an open, bounded, regular and connected set contained in $I\!R^2$, and let

$$\lambda > 0, F_v > 0, a, b \in L^\infty(\Omega), b > 0 \text{ a.e. in } \Omega. \tag{1}$$

Given $\gamma \in I\!R_- := \{t \in I\!R : t < 0\}$, the problem is to find

$$u : \Omega \to I\!R \quad \text{and} \quad F : I\!R \to I\!R_+$$

such that $F(s) = F_v$ for any $s \leq 0$ and the following conditions hold:

$$(\mathcal{P}) \begin{cases} -\Delta u &= aF(u) + F(u)F'(u) + \lambda b u_+ \quad \text{in } \Omega \\ u &= \gamma \quad \text{on } \partial\Omega \\ 0 &= \int_{\{u>t\}} \{F(u)F'(u) + \lambda u_+ b\} \quad \forall t \in [\text{essinf}u, \ \text{esssup}u] \end{cases} \tag{2}$$

where, for the sake of simplicity in the exposition, we have replaced the second order symmetric uniformly elliptic operator $\mathcal{L}$ given in ([HC]) by the Laplace operator and we

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have taken the pressure term equal to $\frac{\lambda}{2}u_+^2$. In the sequel we will refer to the family of integral identities stated in $(\mathcal{P})$ as the *Stellarator Condition.*

In order to determine the unknown function F, the above problem was reformulated in Díaz [D] using the notion of *relative rearrangement* (see the definition and properties of this notion in [MT, R]. There, he proved that if (u, F) is a solution of $(\mathcal{P})$ such that $u \in \mathcal{U}$ where

$$\mathcal{U} = \Big\{u \in W^{2,p}(\Omega), \text{ for any } 1 \leq p < \infty : meas\,\{x \in \Omega : \nabla u(x) = 0\} = 0\Big\}$$

then u satisfies the following uncoupled non local equation

$$-\Delta u = a\left[F_v^2 - 2\lambda \int_0^{u_+(x)} \sigma b_{*u}(|u > \sigma|)d\sigma\right]_+^{1/2} + \lambda u_+[b - b_{*u}(|u > u(x)|)] \quad \text{in } \Omega \tag{3}$$

where $|u > t|$ denotes $meas\,\{x \in \Omega : u(x) > t\}$, u_* represents the decreasing rearrangement of u and b_{*u} is the relative rearrangement of b with respect to u (see XXXX for def.). Moreover, if u satisfies (3) then the function $F = F^u$ is given by

$$F^u(t) = \left[F_v^2 - 2\lambda \int_0^{t_+} \sigma b_{*u}(|u > \sigma|)d\sigma\right]_+^{1/2} \quad \text{for any } t \in [\text{essinf}u, \ \text{esssup}u]. \tag{4}$$

The existence of u, solution of $(\mathcal{P})$ in the class of functions $\mathcal{U}$ was proved by Díaz and Rakotoson [DR] under some additional assumptions.

In this article we give a *partial* result to the uniqueness question. Our assertion is: if we fix the function F then, for small enough values of the parameter λ the problem $(\mathcal{P})$ has an unique solution. The interest of this result is twofold:

- On one hand, some numerical experiments are based on fixing a functional form for F. In a first step, the equation of $(\mathcal{P})$ is solved for fixed F. After this, the Stellarator Condition is imposed and then the parameters which define F are modified in each iteration until the algorithm converges. Altough this procedure does not admit a clear mathematical justification, our result would ensure the uniqueness of the solution in each step of the iteration.

- On the other hand, it has a mathematical interest: notice that the equation of $(\mathcal{P})$ involves nonlinear terms which do not need to be convex neither concave functions. Our proof uses some a priori estimates, some properties of the relative rearrangement and the study of a general elliptic problem with the right hand side term depending on the parameter, the space variable and the solution. A suitable weighted eigenvalue problem is considered to solve the problem. The idea of using an auxiliary linear eigenvalue problem is inspired by the technique used in Puel [Pu] to establish the uniqueness of solution of a different free boundary problem arising in the study of the plasma confinement in a Tokamak device.

2 The main result.

To state the uniqueness result we shall need to refer to the weighted eigenvalue linear problem

$$(\mathcal{P}_H^\mu)\begin{cases} -\Delta w &= \mu H(x) w \quad \text{in } \Omega \\ \quad w &= 0 \qquad\quad\;\; \text{on } \partial\Omega \end{cases}$$

as well as to some positive constants which depend on the data of the problem a, b, F_v, on the constant of Poincaré $P(\Omega)$, on the constant of the Sobolev's Imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, denoted by $S(\Omega)$ and on the parameter λ. These constants will be denoted by δ and $C_2(\lambda)$, thus omitting the dependence on the data and on $P(\Omega), S(\Omega)$. Both constants give a restriction on the size of the parameter λ for which uniqueness is ensured. They will be made explicit in Lemma 1 and Remark 3.

Theorem 1 *Let (u, F) with $u \in \mathcal{U}$ be a solution of $(\mathcal{P})$. Suppose that the function FF' is Lipschitz in $I\!\!R$, i.e.,*

$$|F(t)F'(t) - F(\hat{t})F'(\hat{t})| \leq \lambda K|t - \hat{t}| \tag{5}$$

for every $t, \hat{t} \in I\!\!R$ and for some positive constant K. Suppose also that the parameter $\lambda > 0$ is such that

$$\lambda < \delta$$

and

$$\max\{\lambda C_2(\lambda), \lambda\} < \mu_1 \tag{6}$$

where μ_1 is the first eigenvalue of the problem $(\mathcal{P}_H^\mu)$ with H defined by

$$H(x) := \max\left\{|a(x)|\, \|b\|_{L^\infty(\Omega)}, b(x) + K\right\} \ . \tag{7}$$

Then, if (v, F) is another solution of $(\mathcal{P})$ then, necessarily, $v \equiv u$.

Remark 1 *The requirement on the size of λ is physically expected. As the parameter λ represents the ratio between the kinetic pressure and the magnetic pressure, a large λ would give rise to the so called* magnetic islands *(the plasma splits in several dissconected toroidal volumes) or, in mathematical terminology, to the bifurcaion of the solution.*

Remark 2 *A remark on the Lipschitz continuity constant is due. We have set this constant equal to λK, where K is independent on F and λ. The reason to involve λ in this constant is to make explicit, as long as possible, the dependence that the different functions and constants appearing in the discussion have on λ . The dependence of FF' on λ is clear when we use the characterization given in (4): a simple computation yields*

$$F(t)F'(t) = -\lambda t_+ b_{*u}(|u > t|).$$

Now we turn to the statement of a general uniqueness result. The key of the proof (which will be ommited) is the consideration of a suitable weighted eigenvalue problem. A proof is given in [DGP].

Theorem 2 *Consider the problem*

$$\left\{\begin{array}{rcll} \mathcal{L}u(x) & = & f(x,u;\lambda) & \text{in } \Omega \\ u(x) & = & \phi(x) & \text{on } \partial\Omega \end{array}\right. \tag{8}$$

where $\mathcal{L}$ is a symmetric uniformly elliptic operator defined in a bounded domain $\Omega \subset I\!R^N$, λ is a positive constant and the real function f satisfies

$$|f(x,t_1;\lambda) - f(x,t_2;\lambda)| \leq G(\lambda)H(x)\,|t_1 - t_2|$$

for all $t_1, t_2 \in I\!R$, a.e. $x \in \Omega$ and for all $\lambda \geq 0$. Assume that the functions G and H satisfy

(i) G is continuous, $G(0) = 0$, and $G(\lambda) > 0$ if $\lambda > 0$,

(ii) H is an a.e. positive function of $L^r(\Omega)$, with $r > N/2$.

Then, if the parameter λ satisfies

$$G(\lambda) < \mu_1 \tag{9}$$

where μ_1 is the first eigenvalue of

$$\left\{\begin{array}{rcll} \mathcal{L}w(x) & = & \mu H(x)w(x) & \text{in } \Omega \\ w(x) & = & 0 & \text{on } \partial\Omega \end{array}\right.$$

the problem (8) has, as much, an unique non trivial solution in $H^1(\Omega)$.

Proof of the Theorem 1.

Now, we turn to verify that our problem satisfies the necessary conditions to apply Theorem 2. We set

$$\mathcal{L} \equiv -\Delta \qquad \text{and} \qquad f(x,u;\lambda) = a(x)F(u) + F(u)F'(u) + \lambda b(x)u_+.$$

It is straightforward to verify the conditions for the Laplace operator. However, the estimates for f are non trivial. We start again by supposing that there exist two solutions u, v and we define $U = u - v$. We set

$$g_1(x) := \begin{cases} \dfrac{F(u(x)) - F(v(x))}{u(x) - v(x)} & \text{if } u(x) \neq v(x) \\ 0 & \text{if } u(x) = v(x) \end{cases}$$

$$g_2(x) := \begin{cases} \dfrac{F(u(x))F'(u(x)) - F(v(x))F'(v(x))}{u(x) - v(x)} & \text{if } u(x) \neq v(x) \\ 0 & \text{if } u(x) = v(x) \end{cases}$$

$$h(x) := \begin{cases} \dfrac{u_+(x) - v_+(x)}{u(x) - v(x)} & \text{if } u(x) \neq v(x) \\ 0 & \text{if } u(x) = v(x) \end{cases}$$

Then

$$\Phi(x;\lambda) = a(x)g_1(x) + g_2(x) + \lambda b(x)h(x),$$

and our task is to prove that

$$|\Phi(x;\lambda)| \leq G(\lambda)H(x)$$

with G and H satisfying the conditions of Theorem (2). As a direct consequence of the assumption of Lipschitz continuity of the term $F(t)F'(t)$ and as $|h(x)| \leq 1$ we have that

$$|\Phi(x;\lambda)| \leq |a(x)|\,|g_1(x)| + \lambda(K + b(x))$$

To obtain the estimate for $|g_1(x)|$ (or, what is the same, the Lipschitz's constant of F) we shall need some technical results concerning to the estimation of the $\sup_\Omega u$.

Lemma 1 *There exists a constant C_1 depending on F_v, λ, $\|b\|_{L^\infty(\Omega)}$ and the constants $P(\Omega)$ and $S(\Omega)$ such that any solution (u,F) of $(\mathcal{P})$ with $u \in \mathcal{U}$ satisfies*

$$\sup_\Omega u(x) \leq C_1(\lambda) \parallel a \parallel_{L^2(\Omega)} + |\gamma|$$

Where the constant $C_1(\lambda)$ is given by

$$C_1(\lambda) = S(\Omega)F_v\left[2\lambda P^2(\Omega)\,\|b\|_{L^\infty(\Omega)} + 1\right]. \tag{10}$$

Idea of the proof. To obtain the desired estimate we prove that $\|u-\gamma\|_{L^\infty(\Omega)}$ is bounded by a constant times the L^2-norm of $a(x)$. This follows from the continuous injection of $H^2(\Omega)$ in $L^\infty(\Omega)$ and from the estimates that we obtain for the L^2-norms of $u-\gamma$ (by using the Stellarator Condition and the Poincare's and Holder's Inequalities) and for $\Delta(u-\gamma)$ (by using the Agmon-Douglis-Niremberg Theorem).

Now we are ready to estimate the Lipschitz's constant of F:

Lemma 2 *Let $u \in \mathcal{U}$ be any solution of problem $(\mathcal{P})$. Assume that*

$$\sup_\Omega u < \frac{F_v}{\|b\|_\infty^{1/2}} \frac{1}{\lambda^{1/2}} \tag{11}$$

Then the function F^u defined by (4) is strictly increasing in $[0,M]$, where $M := \sup_\Omega u$, and strictly positive and Lipschitz continuous on $(-\infty, M]$. More precisely,

$$|F(t) - F(\hat{t})| \leq \frac{M}{F(M)}\lambda\|b\|_{L^\infty(\Omega)}|t-\hat{t}| \quad \textit{for every} \quad t,\hat{t} \in (-\infty, M]. \tag{12}$$

Remark 3 *After these lemmas we have that, if*

$$\lambda^{1/2}\left[C_1(\lambda)\|a\|_{L^2(\Omega)}+|\gamma|\right]<\frac{F_v}{\|b\|_\infty^{1/2}} \tag{13}$$

then F is strictly positive and Lipschitz continuous. Note that the left hand side of the inequality (13) is continuous and takes the zero value for $\lambda=0$. Therefore, for $\epsilon=\frac{F_v}{\|b\|_\infty^{1/2}}$ there exists a $\delta\equiv\delta(\frac{F_v}{\|b\|_\infty^{1/2}})$ such that if $\lambda<\delta$ then (13) is satisfied.

We also have that

$$|g_1(x)|\le\lambda\|b\|_{L^\infty(\Omega)}\frac{\sup u}{F(\sup u)}\le\lambda C_2(\lambda)\|b\|_{L^\infty(\Omega)}\ .$$

where the constant $C_2(\lambda)$ introduced before of Theorem 1 is given by

$$C_2(\lambda):=\frac{C_1(\lambda)\|a\|_{L^2(\Omega)}+|\gamma|}{F(C_1(\lambda)\|a\|_{L^2(\Omega)}+|\gamma|)}\ . \tag{14}$$

Notice that the positive constant C_2 is increasing in λ and that it is bounded from above, so

$$\lim_{\lambda\to 0}\lambda C_2(\lambda)=0.$$

and thus, it is possible to make this term smaller than any positive constant.

Proof of the lemma 2. We start by recalling that in [D] and [DR] was proved that when (u,F) is a solution of $(\mathcal{P})$ such that $u\in\mathcal{U}$ then the function F is characterized by $F(t):=\sqrt{F(t)^2}$ where

$$F(t)^2:=F_v^2-2\lambda\int_0^{t_+}sb_{*u}(|u>s|)ds\quad\text{for any}\quad t\in(-\infty,M).$$

From (1) we deduce that $F(t)$ is a strictly decreasing function as consequence of the positiveness of the integral. Therefore, if $t\in(-\infty,M]$ then, taking into account the assumption (11) and that $\|b_{*u}\|_{L^\infty(\Omega_*)}\le\|b\|_{L^\infty(\Omega)}$ we conclude that

$$F(t)^2\ge F(\sup u)^2>F\left(\frac{F_v}{(\lambda\|b\|_\infty)^{1/2}}\right)^2=F_v^2-2\lambda\int_0^{F_v/(\lambda\|b\|_\infty)^{1/2}}s_+b_{*u}(|u>s|)ds\ge$$
$$\ge F_v^2-\lambda\|b\|_{L^\infty(\Omega)}\left(\frac{F_v}{(\lambda\|b\|_\infty)^{1/2}}\right)^2=0$$

This proves that $F(t)>0$ if $t\in(-\infty,M]$. Furthermore, if $t,\hat{t}\in(-\infty,M]$ and, for instance, $\hat{t}>t$ then

$$|F(t)-F(\hat{t})|=\frac{F^2(t)-F^2(\hat{t})}{F(t)+F(\hat{t})}=\frac{2\lambda\int_t^{\hat{t}}\sigma_+b_{*u}(|u>\sigma|)d\sigma}{F(t)+F(\hat{t})}\le$$
$$\le\frac{\lambda\|b\|_{L^\infty(\Omega)}(\hat{t}^2-t^2)}{2F(M)}\le\frac{M}{F(M)}\lambda\|b\|_{L^\infty(\Omega)}(\hat{t}-t).\quad\square$$

Continuation of the proof of Theorem 1: From Remark 3 we have that

$$|g_1(x)| \leq \lambda C_2(\lambda)\|b\|_{L^\infty(\Omega)}$$

and then we get

$$|\Phi(x;\lambda)| \leq |a(x)|\,\lambda C_2(\lambda)\|b\|_{L^\infty(\Omega)} + \lambda(K + b(x))$$

hence

$$|\Phi(x;\lambda)| \leq 2\max\{\lambda C_2(\lambda), \lambda\}\max\left\{|a(x)|\,\|b\|_{L^\infty(\Omega)}, K + b(x)\right\}$$

Define now

$$G(\lambda) = 2\max\{\lambda C_2(\lambda), \lambda\} \qquad \text{and} \qquad H(x) = \max\left\{|a(x)|\,\|b\|_{L^\infty(\Omega)}, K + b(x)\right\}$$

from Remark 3 it is clear that $G(\lambda)$ is continuous and $G(0) = 0$. Also, from the assumptions on the data a, b made in the Theorem 1 we have $H \in L^\infty(\Omega)$ so we have the conditions of the Theorem 2 fulfilled.□

Remark 4 *Although in [D, DR] the problem is formulated in a more complicated framework, we prefered to simplify it for the sake of clarity in the exposition. Nevertheless, all the calculus involved in this article can be extended to that case by means of the redefinition of the variables, the functional spaces, the relative rearrangement and the differential operator, and by using suitable generalized versions of Poincare's and Hölder's inequalities and of the Agmon-Douglis-Niremberg's Theorem. Moreover, the pressure term can be taken as any power dominated function.*

References

[D] Díaz, J.I., *Modelos bidimensionales de equilibrio magnetohidrodinámico para Stellarators.* Informe # 1 y # 2, CIEMAT (1992,1993).

[DGP] Díaz, J.I., Galiano, G., Padial, J.F.. *On the uniqueness of solutions of a nonlinear elliptic equation arising in the confinement of a plasma in a Stellarator device.* Preprint.

[DR] Díaz, J.I. and Rakotoson, J.M., *On a nonlocal stationary free boundary problem arising in the confinement of a plasma in a Stellarator geometry.* To appear in Archive for Rational Mechanics and Analysis (1995).

[HC] Hender, T.C. and Carreras, B.A., *Equilibrium calculations for helical axis Stellarators.* Phys, Fluids 27, 2101 (1984).

[MT] Mossino, J. and Temam, R., *Directional derivative of the increasing rearrangement mapping and application to a queer differential equation in Plasma Physics.* Duke Math. J., 48, 1981, pp. 475–495.

[Pa] Padial, J.F. *EDPs no lineales originadas en plasmas de fusión y medios porosos.* Thesis, May 1995, pp. 130

[Pu] Puel, P., *Sur un problème de valeur propre non linearie et de frontière libre.* C.R. Acad. Sci. Paris, **284** 1977, pp. 861-863.

[R] Rakotoson, J.M., *Some properties of the relative rearrangement,* J. Math. Anal. Appl., 135, 1988, pp. 475–495.