GENERALIZED FLOQUET REPRESENTATION FOR QUASIPERIODICALLY DRIVEN QUANTUM SYSTEMS

H.R. Jauslin
Laboratoire de Physique - Université de Bourgogne
6,bd. Gabriel, 21000 Dijon - France
jauslin@satie.u-bourgogne.fr

Abstract: We present some results on the relation between the spectral properties of a generalized quasi-energy operator and the stability of quantum systems driven by quasiperiodic time dependent forces.

1. Quasi-energy operator

We consider the time dependent Hamiltonian $H = H_0(x) + V(x;\underline{\theta}(t))$ on a Hilbert space $\mathcal{H}$, where x are the variables of the system and the force $\underline{\theta}(t) = g_t\underline{\theta} \in \Omega$ is described by an invertible flow with an ergodic invariant measure μ. Let $U(t,s;\underline{\theta})$ be the corresponding propagator. The generalized quasi-energy operator (QEO) K is defined [1] on the enlarged space $\mathcal{K} = \mathcal{H} \otimes L_2(\Omega, d\mu)$ by

$$\left[e^{-iKt}\Psi\right](\underline{\theta}) \equiv U(0,-t;\ \underline{\theta})\mathcal{T}_{-t}\Psi(\underline{\theta}) \equiv \mathcal{T}_{-t}U(t,0;\ \underline{\theta})\Psi(\underline{\theta}).$$

where $[\mathcal{T}_{-t}\Psi](\underline{\theta}) = \Psi(g_{-t}\underline{\theta})$. K acts as $[K\Psi](\underline{\theta}) = -i\frac{d}{dt}\Psi(g_t\underline{\theta})\Big|_{t=0} + H(\underline{\theta})\Psi$.

Examples:1) Periodic force: $\Omega = S^1$ is a circle, $g_t\theta = \theta + \omega t$, $d\mu = d\theta$, and $K = -i\frac{\partial}{\partial\theta} + H(\theta)$.

2) Quasi-periodic force with two frequencies ω_1, ω_2: Ω is a torus $S^1_{T_1} \times S^1_{T_2}$, $T_j = 1/\omega_j$, the flow $g_t\ (\theta_1, \theta_2) = (\theta_1 + t, \theta_2 + t)$, $d\mu = d\theta_1 d\theta_2$, and $K = -i\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2} + H(\theta_1, \theta_2)$.

2. Generalized Floquet operator

In the periodic case there is a one to one relation between the spectrum of the QEO and the one of the Floquet operator defined by $U(T,0\ ;\ 0)$. In the quasiperiodic case there is the following natural generalization: Define the generalized Floquet operator [2]
$U_F : \mathcal{K}_1 \longrightarrow \mathcal{K}_1;\qquad \mathcal{K}_1 = \mathcal{H} \otimes L_2(S^1_{T_1}, d\theta_1)\qquad$ as the unitary operator
$U_F(\theta_1) = \mathcal{T}^1_{-T_2}U(T_2, 0; \theta_1, 0) \equiv U(0, -T_2; \theta_1, 0)\mathcal{T}^1_{-T_2}$, where $\left[\mathcal{T}^1_{-T_2}\phi\right](\theta_1) = \phi(\theta_1 - T_2)$.

Theorem 2.1: (i) If $\phi \in \mathcal{K}_1$ is an eigenfunction of $U_F\phi = e^{-i\lambda T_2}\phi$ then the function ψ defined by $\quad \psi(\theta_1, \theta_2) = e^{i\theta_2\lambda}U(0, -\theta_2;\ \theta_1, \theta_2)\phi(\theta_1 - \theta_2) \qquad (2.1)$
belongs to $\mathcal{K}$ and is an eigenfunction of the QEO, with eigenvalue λ:
$e^{-iKt}\psi = e^{-i\lambda t}\psi, \qquad \forall t \in \mathbf{R}.$

(ii) Conversely, if $\psi \in \mathcal{K}$ is an eigenfunction of the QEO, then there is a function $\phi \in \mathcal{K}_1$ such that ψ can be represented by (2.1), and ϕ is an eigenfunction of the generalized Floquet operator .

3. Point spectrum and stability

Theorem 3.1: For initial states $\varphi \in \mathcal{H}$ such that $\varphi \otimes 1$ is in the pure point subspace $\mathcal{K}_{pp} \subset \mathcal{K}$ associated with the QEO the time evolution $\varphi(t)$ is almost-periodic (for almost all $\underline{\theta}$).

Corollary 3.2: If $\varphi \otimes 1 \in \mathcal{K}_{pp}$ then φ has bounded energy for almost all $\underline{\theta}$:

$$\varphi \in \mathcal{H}_{be} = \{\varphi \in \mathcal{H} \mid \lim_{E \to \infty} \sup_{t \geq 0} ||F(H_0 > E)U(t, 0; \theta_1, \theta_2)\varphi||_{\mathcal{H}} = 0\}$$

where $F(H_0 > E) : \mathcal{H} \to \mathcal{H}$ is the projector into the subspace generated by the eigenfunctions of H_0 with energy larger than E.

Remark: As in the case of periodic force, one cannot conclude that the expectation value of the energy is finite, since the decay of the tail might be too slow.

4. Continuous spectrum, RAGE theorem

As oposed to the periodic case [3] [4], continuous spectrum of the QEO does not necessarily imply unbounded growth of the energy of the system. One can construct counterexamples, e.g. for a spin 1/2 in a quasi-periodic magnetic field, for which the energy is trivialy bounded but the spectrum of the QEO is continuous [5]. Here we consider some energy-like operators that play a role analogous to that of the unperturbed energy operator H_0 in the periodic case.

Let $\mathcal{K}_{1,pp}$ and $\mathcal{K}_{1,c}$ be the subspaces of $\mathcal{K}_1$ corresponding respectively to the point and continuous spectrum of the generalized Floquet operator U_F.
(i) Let $\Pi_0(E_0) : \mathcal{H} \to \mathcal{H}$ be the projector into the subspace of $\mathcal{H}$ generated by the eigenfunctions of $H_0\varphi^{(0)} = E\varphi^{(0)}$ with $E < E_0$.
(ii) Let $\Pi_1(E_1)$ be the projector of $L_2(S^1_{T_1}, d\theta_1)$ into the subspace generated by the eigenfunctions of $H_1 \equiv -\partial^2/\partial\theta_1^2$ with energy smaller than E_1.
(iii) $\Pi(E_0, E_1) = \Pi_0(E_0) \otimes \Pi_1(E_1)$ is then a projector in $\mathcal{K}_{1,c}$ into the subspace in which both energies are bounded $H_0 < E_0$, $H_1 < E_1$. Continuous spectrum implies that one or both of the energies H_0, H_1 grows unbounded:

Theorem 4.2: Let $\varphi \in \mathcal{H}$ be such that $\varphi \otimes 1 \in \mathcal{K}_{1,c}$. Then for any fixed energies E_0, E_1, $\sup_{t \geq 0} \left\| [\mathbf{1}_1 - \Pi(E_0, E_1)]\; U(t, 0;\; \theta_1, 0)\varphi \right\|^2_{\mathcal{K}_1} = 1$.

Corollary 4.5: Let $\varphi(t) = U(t, 0; \theta_1, 0)\varphi$. The expectation value $< H_0 + H_1 > (t) \equiv < \varphi(t),\ [H_0 + H_1]\varphi(t) >_{\mathcal{K}_1}$ is unbounded. More precisely, $\forall E > 0$ there is a time t_1 such that either $< H_0 + H_1 > (t_1) > E$ or $\varphi(t_1)$ is not in the domain of $H_0 + H_1$ (i.e. $< H_0 + H_1 > (t_1)$ is infinite).

References

[1]- M. Reed, B. Simon; *Methods of Modern Mathematical Physics*, Vol IV, Thm XIII.76; Academic Press, New York 1979.

[2]- G. Casati, L. Molinari; Progr. Theo. Phys. Suppl. **98** (1989) 287.

[3]- D.L. Shepelyansky; Physica **8D** (1983) 208.

[4]- M. Samuelides, R. Fleckinger, L. Touzillier, J. Bellissard; Europhys. Lett. **1** (1986) 203.

[5]- J.M. Luck, H. Orland, U. Smilansky; J. Stat.Phys. **53** (1988) 551.

[6]- Y. Pomeau, B. Dorizzi, B. Grammaticos; Phys. Rev. Lett. **56** (1986) 681.

[7]- R. Badii, P.F. Meier; Phys. Rev. Lett. **58** (1987) 1045.

[8]- P.W. Miloni, J.R. Ackerhalt, M.E. Goggin; Phys. Rev. **A 35** (1987) 1714.

[9]- M. Combescure; J. Stat.Phys. **62** (1991) 779; and preprint Orsay 1991.

[10]- I. Guarneri; Lett. Nuovo. Cim. **40** (1984) 171.

[11]- C.-A. Pillet; Comm. Math. Phys. **102** (1985) 237, and **105** (1986) 259.

[12]- C.-A. Pillet, in *Stochastic Processes in Classical and Quantum Systems* edited by S. Albeverio, G. Casati and D. Merlini, SpringerVlg., Berlin 1986.

[13]- J. Bellissard, in *Stochastic Processes in Classical and Quantum Systems* edited by S. Albeverio, G. Casati and D. Merlini, Springer 1986.

[14]- H.R. Jauslin, J.L. Lebowitz; *Chaos* **1** (1991).

[15]- L. Bunimovich, H.R. Jauslin, J.L. Lebowitz, A. Pellegrinotti, P. Nielaba; J. Stat.Phys. **62** (1991) 793.

[16]- V. Enss, K. Veselic; Ann. Inst. H. Poincaré **A 39** (1983) 159.

[17]- P. Blekher,H.R. Jauslin, J.L. Lebowitz; Floquet spectrum for two-level systems in quasiperiodic time dependent fields, J. Stat.Phys. **68** (1992) 271.