UPPER BOUNDS FOR THE SPLITTING OF SEPARATRICES

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The Poincaré-Melnikov method provides an asymptotic expression for the distance between the invariant manifolds of a perturbed planar vector field in the following way: consider the system

$$\dot{z} = f(z) + \mu g(z, t, \mu),$$

with $z \in \mathbb{R}^2$ and g being T-periodic with respect its second variable.

Assume that if $\mu = 0$ the system has two hyperbolic saddle points p_1, p_2, not necessarily different, and it has a heteroclinic (homoclinic if $p_1 = p_2$) orbit σ connecting them.

For μ small enough the system has hyperbolic periodic orbits, near p_1, p_2, and they have two dimensional invariant manifolds (in $U \times \mathbb{R}$).

Let p be a point in $\{\sigma(t), t \in \mathbb{R}\}$ and σ parametrized in such a way that $\sigma(0) = p$. Let l be a line through p orthogonal to the direction of the vector field at this point, $f(\sigma(0)) = \dot{\sigma}(0)$. Then the distance between the invariant manifolds in the plane $t = t_0$ along the line l is

$$d(t_0) = \frac{\mu}{\| f(\sigma(0)) \|} M(t_0) + O(\mu^2),$$

where $M(t_0)$ is the so called Melnikov's function. If we consider a rapidly forced perturbation

$$\dot{z} = f(z) + \mu g(z, t/\varepsilon, \varepsilon), \qquad \varepsilon \neq 0,$$

obviously $M(t_0)$ will depend on ε. We prove that, under the hypothesis of analyticity with respect to the space variables and differentiability with respect to the others, the Melnikov function is exponentially small with respect to ε, even in the nonconservative case and we give estimates for the exponential coefficient. In the way of proving this fact we also establish some analytical properties of the heteroclinic orbit σ of the nonperturbed system.

Exponentially small upper bounds were first given by Neishtadt.

Several papers deal with the problem of obtaining and proving asymptotic formulae for the splitting distance when $\mu = \varepsilon^p$ and ε goes to 0, in concrete examples, for different values of p (Lazutkin, Holmes-Marsden-Scheurle, Delshams-T. M. Seara, Gelfreich, Angenent, Treshev, ...)

We provide an example of a two dimensional system for which the Melnikov function is exponentially small, has simple zeros, but the system does not have heteroclinic intersections. However, for conservative systems, we give exponentially small upper bounds for the splitting of separatrices and we provide the exponential coefficient in terms of singularities of the "clinic" orbit of the unperturbed equation. These bounds are valid for some negative values of p depending on the hypotheses on the system. In the most general case $p > -1/2$.

Families of near the identity maps appear as time the period Poincaré maps of the described systems.

We deduce the results on differential equations from the analogous ones for families of near the identity maps:

Let

$$F_\varepsilon(z) = z + \varepsilon^\alpha f(z) + \varepsilon^{\alpha+\beta} g(z,\varepsilon),$$

with $z=(x,y)\in U$, $\alpha,\beta>0$, *such that* F_ε *is area preserving,* f *is analytic,* $f(p)=f(q)=0$ *and* $\det Df(p)<0$, $\det Df(q)<0$, g *is continuous and depends analytically with respect to* z, *the differential equation* $\dot z = f(z)$ *has a heteroclinic orbit* σ *associated to the saddles* p, q, *which is analytic on a complex strip* $\{t\in \mathbb{C}, \mid Im t \mid < \delta_0\}$, F_ε *extends analytically to a neighbourhood of* $\{\sigma(t), \mid Im t \mid < \delta\}$ *for some positive* $\delta \le \delta_0$.

Then, under such hypotheses

(1) if ε *is small,* F_ε *has hyperbolic saddle points* p_ε *and* q_ε *near* p, q,

(2) assuming, in the heteroclinic case, that F_ε *has heteroclinic intersections, given a point* $z\in\sigma\cap\mathbb{R}^2$ *and* $\eta\in(0,\delta)$, *there exists* $N(\eta)>0$ *such that if* ε *is small, the distance between the unstable invariant manifold of* p_ε *and the stable invariant manifold of* q_ε *in a neighbourhood of* z *is bounded by*

$$N(\eta)\exp\left(\frac{-2\pi\eta}{\varepsilon^\alpha}\right).$$

The first proof of this result was based on the use of a uniform Birkhoff normal form of a map near a hyperbolic fixed point. This restricts the application to two dimensional conservative systems.

Now we can substitute this argument by constructing a flow box for the diffeomorphisms near an invariant manifold. For that we use a method introduced by Lazutkin, also used by Gelfreich. This allows us to drop the hypothesis of being conservative and also allows to be able to consider systems of higger dimension. This last part is joint work with X. Roca.