A THEOREM OF GLOBAL ASYMPTOTICAL STABILITY

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ABSTRACT: Given a vector field X on the real plane, we study the influence of the curvature of the orbits of $\dot{x} = X^{\perp}(x)$ in the stability of those of the system $\dot{x} = X(x)$. We pay special attention to the case in which this curvature is negative in the whole plane. Under this assumption, we classify the possible critical points and give a criterion for a point to be globally asymptotically stable. In the general case, we also provide expressions for the first three derivatives of the Poincaré map associated to a periodic orbit in terms of geometrical quantities. These results are extracted from the paper [GGG].

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1. INTRODUCTION.

It is known that the divergence of a C^1 vector field $X = (P, Q)$, $\operatorname{div} X$, plays an important role in many problems related to systems of ordinary differential equations on the plane

$$\frac{d\mathrm{x}}{dt} = X(\mathrm{x}) = (P(\mathrm{x}), Q(\mathrm{x})), \qquad \mathrm{x} = (x, y) \in \mathbb{R}^2, \tag{1}$$

such as the classification of critical points, the non-existence of periodic orbits, the stability of limit cycles, the stability of graphics or the global stability of a critical point, for instance.

On the other hand, one may think that the function $\operatorname{div} X$ is not the only one that can control this kind of facts. Roughly speaking, what one intends to control in those problems is the tendency of the trajectories of the flow of (1) to join among them or to separate. However, dealing with the divergence, one does not speculate only about the geometry of the orbits, but also about their parameterization. In some problems, it may happen that the time is not an important factor and the behaviour of the orbits depends only on their geometry. Then, hypotheses on the divergence would turn out to be too strong. Our aim is to get a kind of substitute of the divergence, some operator that can also decide about the behaviour of the flow of (1).

A suitable candidate is found in the literature when one studies the stability of a periodic orbit of (1). In [Ye], the author gives a criterion (due to S. Diliberto, see also [D], [C]) in which the stability is determined by means of the function $K^{\perp}(X)$. The function $K^{\perp}(X)$, that will be called *orthogonal curvature,* associates to a point $\mathrm{x} \in \mathbb{R}^2$, the curvature of the solution of system $(1)^{\perp}$- orthogonal to (1) - at x. Moreover, it seems to be a natural substitute of the divergence because the orbits of (1) are orthogonal to those of $(1)^{\perp}$and so, the shape of these last ones indicates in some sense the tendency of the solutions of (1). Since the curvature is not defined at a critical point of $(1)^{\perp}$, it will be more convenient to deal with the numerator of $K^{\perp}(X)$, which will be denoted by $\tilde{K}^{\perp}(X)$ and will be written as

$$\tilde{K}^{\perp}(X) = Q_y P^2 - (P_y + Q_x) PQ + P_x Q^2,$$

when we consider $X(\mathrm{x}) = (P(\mathrm{x}), Q(\mathrm{x}))$ as in (1). Along this paper, in order to avoid cumbersome notations and where there is no confusion, we will simplify expressions like $K^{\perp}(X)(\mathrm{x})$, $\tilde{K}^{\perp}(X)(x(s), y(s))$ or $K^{\perp}(X)(x, y)$, by the shorter ones $K^{\perp}(\mathrm{x})$, $\tilde{K}^{\perp}(s)$ or $K^{\perp}(x, y)$, respectively.

The concept of orthogonal curvature can be generalized to $\mathbb{R}^2$ endowed with a Riemannian metric g. In this context, a vector field X is denoted also by (X, g)

to remind the present metric and $K_g^{\perp}(X)$ represents again the curvature of the solutions of the orthogonal vector field to X in the metric g.

In the proofs of our results, the geometrical meaning of the orthogonal curvature will be very helpful and some of the proofs will be reached from geometrical arguments rather than analytical (some ideas appeared in [O] and [GLS] are also useful). This is an advantage of this operator compared to the divergence. We mainly focus on systems in which the orthogonal curvature is negative, which we call *path-convergent systems.*

We also discuss the nature of the critical points for path-convergent systems, paying more attention to the case of non-degenerate points. The main results in this direction can be synthesized as:

THEOREM A. *Let E and H represent the number of elliptic and hyperbolic sectors, respectively, of an isolated critical point p of system (1), and U some neighbourhood of p. Then, if $\tilde{K}^{\perp}(x) \neq 0$, for all $x \in U$,*

(a) *$E = 0$ and $H \in \{0, 2\}$. Moreover, there are examples having zero or two hyperbolic sectors.*

(b) *If, in addition, $\det(DX(p)) > 0$, then $H = 0$ and p is locally asymptotically stable (unstable) assuming $\tilde{K}^{\perp}(x) < 0$ ($\tilde{K}^{\perp}(x) > 0$), for all $x \in U$.*

Afterwards, we study the stability of periodic orbits for system (1) and we see that it is controlled by the following result:

THEOREM B. *Let $\gamma(s)$ be the g-arc-length parameterization of a periodic orbit γ of (1) in $\mathbb{R}^2$ endowed with a Riemannian metric g. Then, γ is a hyperbolic stable (unstable) limit cycle, if $\int_0^{l_g} K_g^{\perp}(s)\ ds < (>) 0$, where l_g is the length of $\gamma(s)$ in $(\mathbb{R}^2, g)$.*

For the Euclidean metric, we give also expressions for the second and third derivatives of the return map associated to γ in terms of the curvatures.

Going back to path-convergent systems, we deal with the number of periodic orbits, and we can state that:

THEOREM C. *Let U be an open subset of $\mathbb{R}^2$ with fundamental group $\pi_1(U) = \mathbb{Z} * \overset{r)}{\cdots} * \mathbb{Z}$. Assume that $\tilde{K}^{\perp}(\mathrm{x}) \leq 0$, for all $\mathrm{x} \in U$, and only vanishes on a set of null Lebesgue-measure and that (1) has a finite number of critical points in U. Then, (1) has at most r periodic solutions entirely contained in U.*

In particular, when U is 1-connected, Theorem C reminds the Bendixson-Dulac Criterion for the non-existence of periodic orbits. When r is arbitrary, it is related to some generalizations of the Bendixson-Dulac Criterion, given in [L] and [Y]. Finally, summing up some of the previous results, we give conditions for a local attractor of (1) to be a global attractor. We prove:

THEOREM D. *Suppose that system (1) satisfies:*

(a) $\tilde{K}^{\perp}(\mathrm{x}) \leq 0$ *for all* $\mathrm{x} \in \mathbb{R}^2$*, and only vanishes on a set of null Lebesgue-measure;*
(b) $\det(DX(\mathrm{x})) > 0$ *for all* $\mathrm{x} \in \mathbb{R}^2 \setminus \{(0,0)\}$*;*
(c) *the origin* $(0,0)$ *is a locally asymptotically stable critical point.*

Then, the origin is globally asymptotically stable.

This result is similar to the so-called planar Markus-Yamabe conjecture, solved in the affirmative in 1993 by Gutiérrez and Fessler (see [G], [F]), in which hypothesis (a) reads as: $\operatorname{div} X(\mathrm{x}) < 0$ for all $\mathrm{x} \in \mathbb{R}^2$.

References

[C] Chicone, C., *Bifurcations of nonlinear oscillations and frequency entrainment near resonance*, SIAM J. Math. Anal. **23** (1992), 1577–1608.

[D] Diliberto, S., *On systems of ordinary differential equations*, Contributions to the Theory of Nonlinear Oscillators **1** (1950), 1–38.

[F] Fessler, R., *A solution of the two-dimensional global asymptotic stability Jacobian conjecture* (1994), Preprint. ETH Zürich.

[G] Gutiérrez, C., *A solution to the bidimensional global asymptotic stability conjecture* (1993), In *Conferences given at the Workshop: "Recent results on the global asymptotic Jacobian conjecture"* edited by M. Sabatini. Trento.

[GGG] Garcia, R.A., Gasull, A., Guillamon, A., *Geometrical conditions for the stability of orbits in planar systems*, To appear in *Math. Proc. Cambridge Ph. Soc.*

[GLS] Gasull, A., Llibre, J. and Sotomayor, J., *Global asymptotic stability of differential equations in the plane*, Jour. Differential Equations **91** (1991), 327–335.

[L] Lloyd, N.G., *A note on the number of limit cycles in certain two-dimensional systems*, J. London Math. Soc.(2) **20** (1979), 277–286.

[O] Olech, C., *On the global stability of an autonomous system on the plane*, Contributions to Differential Equations **1** (1963), 389–400.

[Y] K.Yamato, *An effective method of counting the number of limit cycles*, Nagoya Math. J. **76** (1979), 35–114.

[Ye] Ye Yan-qian et al., *Theory of limit cycles*, Transl. of Math. Monographs 66. AMS, Providence, Rhode Island, 1986.