INTEGRABILITY OF CUBIC SYSTEMS WITH DEGENERATE INFINITY*

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Abstract

In this work we study the integrability of two-dimensional autonomous system in the plane with linear part of center type and non-linear part given by cubic polynomials with degenerate infinity. We give a simple characterisation for all integrable cases in polar coordinates.

1 Introduction

We consider the system

$$\frac{dx}{dt} = -\lambda y + X_2(x,y) + X_3(x,y),\ \frac{dy}{dt} = \lambda x + Y_2(x,y) + Y_3(x,y), \tag{1}$$

where $X_s(x,y)$ and $Y_s(x,y)$, s=2,3, are homogeneous polynomials of degree s, satisfying $xY_3(x,y) - yX_3(x,y) \equiv 0$ whenever $X_3^2 + Y_3^2 \not\equiv 0$. Such systems are a particular class of the cubic systems with degenerate infinity. This name is due to the fact that, in Poincaré compactification of (1), all the infinity is filled up of critical points, see Sotomayor [12]. This problem has been studied by Chen Guang-Qing and Liang Zhao-Jun [4] and recently by Gasull and Prohens [6], in this work he gives an affine classification of quadratic systems with degenerate infinity.

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Yasmin [14] calculates the Liapounov quantities for the origin of the system (1) in order to find its maximum order, and hence the number of limit cycles that can be produced from it by bifurcation. Christopher [5] gives local first integrals or integrating factors for system (1).

Besides, these systems are a subfamily of systems with linear part of center type and nonlinear part given by polynomials of cubic degree (The Cubic Center-Focus Problem). The integrable cases for some cubic systems have been studied by Chavarriga [2], Giacomini and Ndiaye [7], Lloyd and Pearson [8], Schlomiuk [10] and Zoladek [14]. The aim of this paper is to find the integrable cases of system (1) (see Theorem 1). Poincaré [9] developed an important technique for the general solution of these problems. It consists on finding a formal power series of the form $H(x,y) = \sum_{n=2}^{\infty} H_n(x,y)$, where $H_2(x,y) = \frac{(x^2+y^2)}{2}$, and $H_n(x,y)$ are homogeneous polynomials of degree n, so that

$$\frac{dH}{dt} = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k,$$

where $V_{2k},\ k = 2, 3, \cdots$, are real numbers called *Lyapunov constants*.

The vanishing of all Lyapunov constants is a necessary condition for the integrability of the system (1); in this case the series $H(x,\ y)$ would be a first integral of the system if it converges, this last point is an open question today. On the other hand, in general it is not possible to express this first integral (if it exists) by means of elementary functions.

Lyapunov constants are polynomials in the variables given by the coefficients of the polynomials $X_{\rm s}(x,\ y)$ and $Y_{\rm s}(x,\ y)$, s=2,3. Thus the ideal generated by $V_{\rm 2k}$ has a finite number of generators by Hilbert's Theorem. We denote by $M(s)$ the minimum number of generators of this ideal. About the function $M(s)$, Bautin [1] proved $M(2)=3$ and it has been proved by Shi Songling [11] that under certain hypotheses on the Lyapunov constants, the number of small amplitude limit cycles around the origin is at least $M(s)$. The vanishing of these generators is a sufficient condition for the vanishing of all Lyapunov constans and for the integrability of the system.

In Section 2 we give some results which are necessary for the proof of Theorem 1. In Section 3 we prove Theorem 1. This theorem characterizes the integrable cases by means of polar coordinates. Finally, we give an appendix on the computation of the Lyapunov constants.

2 Some Preliminary Results

In the study of this problem we have used polar coordinates. In Lemma 1 we give the expression of system (1) in polar coordinates. In Proposition 1 we give the evaluation of the Poincaré series in these coordinates. In Lemmas 2 and 3 we give different forms that the system (2.1) can take, eliminating unessential parameters.

LEMMA 1. *In polar coordinates $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ we can write system (1) as*

$$\begin{aligned} \tfrac{dr}{dt} &= P_2(\varphi)r^2 + P_3(\varphi)r^3, \\ \tfrac{d\varphi}{dt} &= \lambda + Q_2(\varphi)r, \end{aligned} \tag{2}$$

where $P_2(\varphi)$, $Q_2(\varphi)$ and $P_3(\varphi)$, are trigonometric polynomials of the form

$$\begin{aligned} P_2(\varphi) &= a_3\cos 3\varphi + a_1\cos\varphi + a_2\sin\varphi, \\ Q_2(\varphi) &= -a_3\sin 3\varphi + b_1\cos\varphi + b_2\sin\varphi, \\ P_3(\varphi) &= c_0 + c_1\cos 2\varphi + c_2\sin 2\varphi. \end{aligned} \tag{3}$$

Proof. Differentiating $x = r\cos\phi$, $y = r\sin\phi$, with respect to t gives

$$\frac{dr}{dt} = \frac{dx}{dt}\cos\phi + \frac{dy}{dt}\sin\phi,\ \ \frac{d\phi}{dt} = \frac{\frac{dy}{dt}\cos\phi - \frac{dx}{dt}\sin\phi}{r},$$

and from (1) we obtain

$$\frac{dr}{dt} = [-\lambda r\sin\phi + \sum_{s=2}^{3} X_s(r\cos\phi, r\sin\phi)]\cos\phi + [\lambda r\cos\phi + \sum_{s=2}^{3} Y_s(r\cos\phi, r\sin\phi)]\sin\phi,$$

$$\frac{d\phi}{dt} = \frac{[\lambda r\cos\phi + \sum_{s=2}^{3} Y_s(r\cos\phi, r\sin\phi)]\cos\phi - [-\lambda r\sin\phi + \sum_{s=2}^{3} X_s(r\cos\phi, r\sin\phi)]\sin\phi}{r}.$$

By homogeneity of X_s and Y_s and grouping terms, we obtain

$$\frac{dr}{dt} = \sum_{s=2}^{3}(X_s(\cos\phi, \sin\phi)\cos\phi + Y_s(\cos\phi, \sin\phi)\sin\phi)r^s,$$

$$\frac{d\phi}{dt} = \lambda + \sum_{s=2}^{3}\left(Y_s(\cos\phi, \sin\phi)\cos\phi - X_s(\cos\phi, \sin\phi)\sin\phi\right)r^{s-1}.$$

Let $X_2(x, y) = A_{20}x^2 + A_{11}xy + A_{02}y^2$, $Y_2(x, y) = B_{20}x^2 + B_{11}xy + B_{02}y^2$, $X_3(x, y) = xL(x, y)$ and $Y_3(x, y) = yL(x, y)$, where $L(x, y) = C_{20}x^2 + C_{11}xy + C_{02}y^2$, then we have

$$P_2(\phi) = X_2\cos\phi + Y_2\sin\phi = A_{20}\cos^3\phi + (A_{11}+B_{20})\cos^2\phi\sin\phi + (A_{02}+B_{11})\cos\phi\sin^2\phi + B_{02}\sin^3\phi$$
$$= \frac{(A_{20}-B_{11}-A_{02})\cos 3\phi + (A_{11}-B_{02}+B_{20})\sin 3\phi + (A_{02}+3A_{20}+B_{11})\cos\phi + (A_{11}+3B_{02}+B_{20})\sin\phi}{4}.$$

$$Q_2(\phi) = Y_2\cos\phi - X_2\sin\phi = B_{20}\cos^3\phi + (B_{11}-A_{20})\cos^2\phi\sin\phi + (B_{02}-A_{11})\cos\phi\sin^2\phi - A_{02}\sin^3\phi$$
$$= \frac{(A_{11}-B_{02}+B_{20})\cos 3\phi + (B_{11}+A_{02}-A_{20})\sin 3\phi + (B_{02}+3B_{20}-A_{11})\cos\phi + (B_{11}-3A_{02}-A_{20})\sin\phi}{4},$$

$$P_3(\phi) = X_3\cos\phi + Y_3\sin\phi = C_{20}\cos^2\phi + C_{11}\cos\phi\sin\phi + C_{02}\sin^2\phi = \frac{(C_{20}-C_{02})\cos 2\phi + C_{11}\sin 2\phi + (C_{02}+C_{20})}{2},$$

$$Y_3\cos\phi - X_3\sin\phi = 0.$$

So we can write

$$\begin{aligned} P_2(\phi) = &\ \alpha_3\cos 3\phi + \alpha_4\sin 3\phi + \alpha_1\cos\phi + \alpha_2\sin\phi, \\ Q_2(\phi) = &\ \alpha_4\cos 3\phi - \alpha_3\sin 3\phi + \beta_1\cos\phi + \beta_2\sin\phi, \\ P_3(\phi) = &\ \gamma_1\cos 2\phi + \gamma_2\sin 2\phi + \gamma_0. \end{aligned}$$

Finally, if we make $\phi = \varphi + \varphi_0$, being $\tan 3\varphi_0 = \frac{\alpha_4}{\alpha_3}$, in the preceding expression we obtain the values of $P_2(\varphi), Q_2(\varphi)$ and $P_3(\varphi)$, which completes the proof.$\square$

NOTE. Let $\lambda = 0$. In this case, the integration of system (2) reduces to a Bernoulli differential equation: $\frac{dr}{d\varphi} = \frac{P_2(\varphi)r + P_3(\varphi)r^2}{Q_2(\varphi)}$. If $\lambda \neq 0$ we can take $\lambda = 1$ without lost of generality, making a linear scalate of time $t_1 = \lambda t$. So we will take $\lambda = 1$ from now on.

Proposition 1. *In polar coordinates the Poincaré series for the system (1) is* $H(r,\varphi) = \sum_{n=2}^{\infty} H_n(\varphi) r^n$ *where* $H_1(\varphi) = 0$, $H_2(\varphi) = \frac{1}{2}$ *and* $H_n(\varphi)$, $n = 3, 4, \cdots$, *are homogeneous trigonometric polynomials of degree n, satisfying the differential equations*

$$\frac{dH_{n+1}}{d\varphi} + nH_nP_2(\varphi) + \frac{dH_n}{d\varphi}Q_2(\varphi) + (n-1)H_{n-1}P_3(\varphi) = \begin{cases} 0 & if\ n\ is\ odd, \\ V_{2n} & if\ n\ is\ even, \end{cases} \quad n = 2,\ 3, \cdots, \tag{4}$$

where V_{2n}, $n = 2,\ 3, \cdots$, *are the Lyapunov constants.*

Proposition 1 is proved in [2].

Lemma 2. *We can write system (2) in the form*

$$\begin{aligned} \frac{dz}{d\tau} = &\ (a_3\cos 3\varphi + \bar{a}_1\cos\varphi + \bar{a}_2\sin\varphi)\, z^2 + \bar{c}_0 z^3, \\ \frac{d\varphi}{d\tau} = &\ 1 + \left(-a_3\sin 3\varphi + \bar{b}_1\cos\varphi + \bar{b}_2\sin\varphi\right) z, \end{aligned} \tag{5}$$

where $z = \frac{r}{1-u(\varphi)r}$, *being* $u(\varphi)$ *a trigonometric polynomial of first degree and* $\frac{dt}{d\tau} = 1 + u(\varphi)z$.

Proof. Let $u(\varphi) = k_1 \cos\varphi + k_2 \sin\varphi$, where k_1 and k_2 are real numbers. If we make the change $z = \frac{r}{1-u(\varphi)r}$, system (2) takes the form

$$\begin{array}{rcl} \frac{dz}{dt} & = & \frac{(P_2+u')z^2+(P_2+u'+uu'+Q_2u'+P_3)z^3}{1+uz}, \\ \frac{d\varphi}{dt} & = & \frac{1+(Q_2+u)z}{1+uz}. \end{array} \tag{6}$$

where $' = \frac{d}{d\varphi}$. Setting

$$a_3 = a_3,\ a_1 = a_1,\ a_2 = a_2,\ b_1 = a_2 + m_1,\ b_2 = 2 - a_1 + m_2,$$

$$c_1 = m_1 + \frac{m_1 m_2}{4} + \frac{(a_3-1)m_1}{2} + m_3,$$

$$c_2 = -2 + \frac{m_2^2 - m_1^2 + m_4 + 4(1-a_3)m_2 - 4(5+3a_3)(-1+a_3)}{8},$$

$$k_1 = -\frac{m_1}{2} + y,\ k_2 = -\frac{m_2}{2} + z - 1 - a_3,$$

and using the same change in the expression (3) we have

$$P_2 + u' = a_3 \cos 3\varphi + (a_1 - \frac{m_2}{2} + z - 1 - a_3)\cos\varphi + (a_2 + \frac{m_1}{2} - y)\sin\varphi,$$

$$Q_2 + u = -a_3 \sin 3\varphi + (b_1 - \frac{m_1}{2} + y)\cos\varphi + (b_2 - \frac{m_2}{2} + z - 1 - a_3)\sin\varphi,$$

$$\begin{array}{c} P_2 u + Q_2 u' + uu' + P_3 = (m_3 + yz)\cos 2\varphi + \frac{-m_4+4y^2+16a_3z-4z^2}{8}\sin 2\varphi \\ +\frac{2c_0-2a_2-2a_2a_3-a_1m_1-a_3m_1-a_2m_2-2y+2a_1y-m_2y+2a_2^2z+m_1z}{2}. \end{array} \tag{7}$$

If $m_3 \neq 0$, by substituting $y = -\frac{m_3}{z}$ into (7), the coefficient of $\cos 2\varphi$ is zero and the coefficient of $\sin 2\varphi$ becomes $\frac{-4m_3^2+m_4z^2-16a_3z^3+4z^4}{8z^2}$. Let $M(z)$ the numerator of preceding expression. It is obvius that $M(0) = -4m_3^2 < 0$ and $M(\infty) > 0$. Hence, by Bolzano's Theorem, exits at least a real root.

Let $m_3 = 0$. If $m_4 \leq 0$ we take y=0; in this case the coefficient of $\sin 2\varphi$ is $-m_4+16a_3z-4z^2$. On the other hand, if $m_4 > 0$ we take z=0 and the coefficient of $\sin 2\varphi$ is $-m_4+4y^2$. In both cases the coefficient of $\cos 2\varphi$ is zero and the coefficient of $\sin 2\varphi$ is a square polynomial in z, whose discriminant is positive and therefore it a real root exits. Finally, the expresion (7) is a constant.

Notice that the change of variable $z = \frac{r}{1-ru(\varphi)}$ is an analytical change for r small enough $\left(r < \sqrt{k_1^2 + k_2^2}\right)$. $\square$

Lemma 3. *We can write system (2) in the form*

$$
\begin{aligned}
\frac{dz}{d\tau} &= \left(a_3\cos 3\varphi + A_1\cos\varphi + A_2\sin\varphi\right)z^2 + \left(C_1\sin 2\varphi + C_2\cos 2\varphi + C_0\right)z^3,\\
\frac{d\varphi}{d\tau} &= 1 + \left(-a_3\sin 3\varphi\right)z.
\end{aligned}
\tag{8}
$$

where $z = \frac{r}{1-u(\varphi)r}$ *and* $\frac{dt}{d\tau} = 1 + u(\varphi)z$, *being* $u(\varphi) = -b_1\cos\varphi - b_2\sin\varphi$.

Proof. It is obvius from the expresion (6).□

3 The Main Result

Theorem 1. *For all integrable cases of system (2) there is a local analitycal change of coordinates at the neighborhood of the origin plus an scalate of the time variable t such that the cubic system with degenerate infinity has one of the following expressions in polar coordinates:*

i) $\frac{d\rho}{dT} = \left(R_3\sin 3\omega + R_1\sin\omega\right)\rho^2 + R_2\sin 2\omega\rho^3,\ \frac{d\omega}{dT} = 1 + R_3\cos 3\omega\rho;$

ii) $\frac{d\rho}{dT} = \left(R_3\cos 3\omega + R_1\cos(\omega+\omega_1)\right)\rho^2,\ \frac{d\omega}{dT} = 1 - \left(R_3\sin 3\omega + 3R_1\sin(\omega+\omega_1)\right)\rho;$

iii) $\frac{d\rho}{dT} = R_3\left(\cos 3\omega \pm 3\cos(\omega+\omega_1)\right)\rho^2,\ \frac{d\omega}{dT} = 1 + R_3\left(-\sin 3\omega \pm \sin(\omega+\omega_1)\right)\rho;$

iv) $\frac{d\rho}{dT} = \left(R_3\cos 3\omega + R_1\cos(\omega+\omega_1)\right)\rho^2,\ \frac{d\omega}{dT} = 1 + \left(-R_3\sin 3\omega + R_1\sin(\omega+\omega_1)\right)\rho;$

Proof of Theorem. The first not zero Lyapunov constant for system (8) is $V_2 = C_0$. So the first condition to have integrability is $C_0 = 0$.

Let $A_2(3A_1 + 4a_3) \neq 0$. From V_8 we obtain C_2 as a funtion of the remaining parameters, that is

$$
C_2 = -\frac{9A_1^2A_2a_3 - 3A_2^3a_3 + 6A_1^2C_1 - 6A_2^2C_1 - 16A_1a_3C_1}{4A_2\left(3A_1 + 4a_3\right)}. \tag{9}
$$

By substituting this value in V_8, V_{10}, and V_{12} results $V_k = \Delta_1\Delta_2\Delta_3\ W_k,\ k = 8, 10, 12$, where $\Delta_1 = A_2^2 - 3A_1^2$, $\Delta_2 = a_3$, and $\Delta_3 = 3A_1A_2 + 12A_2a_3 + 16C_1$.

We also assume that $-9A_1^2 - 9A_2^2 + 64a_3^2 \neq 0$. In this assumtion, we can find C_1 from W_8 as a function of the remaining parameters, that is

$$
C_1 = -\frac{A_2\left(36A_1^3 + 36A_1A_2^2 + 39A_1^2a_3 + 39A_2^2a_3 - 186A1a_3^2 - 184a_3^3\right)}{128a_3^2 - 18A_1^2 - 18A_2^2}. \tag{2}
$$

If we introduce this value in W_{10} and W_{12} we obtain $W_k = \Delta_4U_k\ k = 10, 12$, where $\Delta_4 = 4a_3^2 - A_1^2 - A_2^2$. Let $A_2 = \sqrt{z - A_1^2}$. U_{10} and U_{12} take the form

$$U_{10} = \frac{7\left(336A_1^3a_3^3 - 1024a_3^6 - 36A_1^3a_3z - 252A_1a_3^3z + 1456a_3^4z + 27A_1a_3z^2 - 384a_3^2z^2 + 27z^3\right)}{64(3A_1 + 4a_3)(-64a_3^2 + 9z)^2},$$

$$\begin{aligned} U_{12} = & \frac{7\Big(80400A_1^3a_3^5 - 13312a_3^8 - 207972A_1^3a_3^3z - 60300A_1a_3^5z + 568048a_3^6z + 22572A_1^3a_3z^2}{1152(3A_1^2+4a_3)(-64A_3^2+9z)^2} \\ + & \frac{155979A_1a_3^3z^2 - 768348a_3^4z^2 - 16929A_1a_3z^3 + 201618a_3^2z^3 - 14256z^4\Big)}{1152(3A_1+4a_3)(-64a_3^2+9z)^2}. \end{aligned}$$

Eliminating A_1 and z from the system give by $U_{10} = 0$ and $U_{12} = 0$, we obtain

$$(a)\ z = a_3^2,\ A_1 = \tfrac{a_3}{2};\quad (b)\ z = a_3^2,\ A_1 = -a_3;\quad (c)\ z = 4a_3^2,\ A_1 = a_3;$$
$$(d)\ z = 4a_3^2,\ A_1 = -2a_3.$$

It is easy to see that cases (a) and (c) verifying $\Delta_1 = 0$ and cases (b) and (d) implies $A_2 = 0$.

Finally, from the vanishing of $V_4,\ V_6,\ V_8,\ V_{10}$ and V_{12} we arrive at the following cases:

A. Let $A_2(3A_1 + 4a_3) \neq 0$. In this assumption we have:

Case 1. Let $C_0 = 0$ and $\Delta_1 = A_2^2 - 3A_1^2 = 0$, that is $A_2 = \pm\sqrt{3}A_1$. From (9) we have $C_2 = \pm\frac{C_1}{\sqrt{3}}$. If we make in system (8) $z = \rho,\ \tau = T,\ \varphi = \omega - \frac{\pi}{6},\ a_3 = R_3,\ A_1 = \frac{R_1}{2}$ and $C_1 = \frac{\sqrt{3}}{2}R_2$ then system (8) becomes system **(i)**.

Case 2. Let $C_0 = 0$ and $\Delta_2 = a_3 = 0$. From (9) we have $C_2 = \frac{(A_2^2 - A_1^2)C_1}{2A_1A_2}$; Let $A_1 \neq 0$, if we make in system (8) $z = \rho,\ \tau = T,\ \varphi = \omega - \varphi_1,\ A_1 = R_1\cos\varphi_1,\ A_2 = -R_1\sin\varphi_1$ and $C_1 = -R_2\sin 2\varphi_1$ then we obtain a particular case of system **(i)** ($R_3 = 0$). If $A_1 = 0$ implies $C_1 = 0$, in this case we make $z = \rho,\ \tau = T,\ \varphi = \omega,\ A_2 = R_1$ and $C_2 = R_2$.

Case 3. Let $C_0 = 0$ and $\Delta_3 = 3A_1A_2 + 12A_2a_3 + 16C_1 = 0$. From (9) we have $C_2 = \frac{3(A_1^2 - A_2^2 - 8A_1a_3)}{32}$. If we make the change of variable in system (8) $\rho = \frac{z}{1 - v(\varphi)z}$ and $\frac{d\tau}{dT} = 1 + v(\varphi)\rho$, being $v(\varphi) = \frac{3}{4}(A_2\cos\varphi - A_1\sin\varphi)$, we obtain system **(ii)** setting $\varphi = \omega - \varphi_1,\ a_3 = R_3,\ A_1 = 4R_1\cos\varphi_1,\ A_2 = -4R_1\sin\varphi_1$.

If $-9A_1^2 - 9A_2^2 + 64a_3^2 = 0$ implies a_3=0 and therefore $A_1 = A_2 = 0$ which will be studied forward.

Case 4. Let $-9A_1^2 - 9A_2^2 + 64a_3^2 \neq 0,\ C_0 = 0$ and $\Delta_4 = 4a_3^2 - A_1^2 - A_2^2 = 0$. If we make $A_1 = 2a_3\cos\varphi_1$ and $A_2 = -2a_3\sin\varphi_1$, from (9) and (10), we have $C_1 = -a_3^2(1 + 3\cos\varphi_1)\sin\varphi_1$ and $C_2 = \frac{a_3^2}{2}(2\cos\varphi_1 - 3\cos 2\varphi_1)$. If we make the change of variable in system (8) $\rho = \frac{z}{1 - v(\varphi)z}$ and $\frac{d\tau}{dT} = 1 + v(\varphi)\rho$, being $v(\varphi) = a_3\sin(\varphi + \varphi_1)$, we obtain system **(iii)** setting $\varphi = \omega - \varphi_1,\ a_3 = R_3$.

B. Let $A_2(3A_1 + 4a_3) = 0$. In this assumption we have:

Case 5. $C_0 = 0,\ A_2 = 0$. From the vanishing of V_6 and V_8 we obtain the following subcases

$$(5.a)\ C_1 = 0;\quad (5.b)\ A_1 = 0;\quad (5.c)\ a_3 = \frac{3}{8}A_1\ and\ C_2 = -\frac{83}{256}A_1^2.$$

(5.a) If we make in system (8) $z = \rho,\ \tau = T,\ \varphi = \omega - \frac{\pi}{2},\ a_3 = R_3,\ A_1 = -R_1\ and\ C_2 = -R_2$ we obtain system **(i)**.

(5.b) Applying Lemma 2 to system (8) we obtain $\bar{a}_1 = k_2,\ \bar{a}_2 = -k_1,\ \bar{b}_1 = k_1,\ \bar{b}_2 = k_2$ and it is easy to see that $\bar{c}_0 = 0$. Finally, if we make $z = \rho,\ \tau = T,\ \varphi = \omega - \varphi_1,\ a_3 = R_3,\ k_1 = R_1 \sin\varphi_1$ and $k_2 = R_1 \cos\varphi_1$ then we obtain system **(iv)**.

(5.c) In this case the values of V_{10} and V_{12} are

$$V_{10} = \frac{A_1^2 C_1(207515A_1^4 - 65536C_1^2)}{3932160},\ V_{12} = \frac{A_1^4 C_1(-2279552275A_1^4 + 631832576C_1^2)}{1509949440}$$

therefore we do not find anything new, except the integrable subcases mentioned before.

Case 6. $C_0 = 0$ and $3A_1 + 4a_3 = 0$. From the vanishing of V_6 and V_8 we obtain

$$\begin{gathered}(6.a)\ a_3 = 0,\ A_2 = 0;\quad (6.b)\ a_3 = \pm\tfrac{\sqrt{3}}{4}A_2,\\ C_2 = \mp\tfrac{C_1}{\sqrt{3}};(6.c)\ a_3 = \pm\tfrac{\sqrt{3}}{4}A_2,\ C_2 = \tfrac{83A_2^2 \pm 96\sqrt{3}C_1}{96};\\ (6.d) 2C_1 + A_2 a_3 = 0,\ 96C_2 - 112a_3^2 + 9A_2^2 = 0;\\ (6.e)\ 2C_1 + A_2 a_3 = 0,\ 18C_2 + 49a_3^2 - 18A_2^2 = 0.\end{gathered}$$

Subcases **(6.a)**, **(6.b)** and **(6.d)** are particular cases of **(5.b)**, **(5.a)** and **Case 3** respectively.

(6.c) In this subcase the values of V_{10} and V_{12} are

$$V_{10} = \frac{A_2^2(\pm 83A_2^2 + 128\sqrt{3}C_1)(-5839A_2^4 \pm 1992\sqrt{3}A_2^2 C_1 + 4608C_1^2)}{3317760\sqrt{3}}$$

$$V_{12} = \frac{A_2^4(\pm 83A_2^2 + 128\sqrt{3}C_1)(65009429A_2^4 \mp 19204872\sqrt{3}A_2^2 C_1 - 44425728C_1^2)}{955514880\sqrt{3}}$$

From the simultaneous vanishing of V_{10} and V_{12} we obtain $C_2 = \mp\frac{C_1}{\sqrt{3}}$ which is a particular case of **Case 1**

(6.e) In this subcase the values of V_{10} and V_{12} are

$$V_{10} = \frac{245A_2 a_3(9A_2^2 - 20a_3^2)(9A_2^2 + 4a_3^2)(-9A_2^2 + 32a_3^2)}{419904}$$

$$V_{12} = \frac{245A_2a_3(9A_2^2-32a_3^2)(9A_2^2-20a_3^2)(7128A_2^4+14625A_2^2a_3^2+5488a_3^4)}{11337408}$$

From the simultaneous vanishing of V_{10} and V_{12} we obtain $A_2 = \pm\frac{\sqrt{20}}{3}a_3$ which corresponds to a particular case of **Case 4** or $A_2 = \pm\frac{4\sqrt{2}}{3}a_3$ which corresponds to a particular case of **Case 3**. This completes the proof.□

FINAL REMARKS.

(a) The integration of system **(i)** of Theorem 1 can be done by reducing equation $\frac{d\rho}{d\omega} = \frac{(R_3\sin 3\omega+R_1\sin\omega)\rho^2+R_2\sin 2\omega\rho^3}{1+R_3\cos 3\omega\rho}$ to a linear differential equation of first degree $\frac{d\varrho}{dw} = \frac{2\left((R_1-R_3+2R_2w)\varrho+4R_3w^2\right)}{-1+(R_1+2R_3)w+2R_2w^2}$, making the change of variables $w = \rho\cos\omega$ and $\varrho = \rho^2$. On the other hand, by appliying Lemma 2, there is a local analitycal change of coordinates in complex variable at neighborhood of the origin plus an scalate of the time variable t such that system **(i)** reduce to a quadratic system of the same form with complex parameters, (i.e R_2=0).

(b) The first integral of systems **(ii)**, **(iii)** and **(iv)** can be found in [2].

(c) With reference to the number of small amplitude limit cycles around the origin that appear in system (1) we note that the third integrable case (system **(iii)**) the number of relations of parameters is four. So, the number of small amplitude limit cycles is at least four. Notice from this work that the number of independent Lyapunov constants is five.

Appendix.

Lyapunov constants V_6, V_8, V_{10} and V_{12}:

$$24V_6 = 9A_1^2A_2a_3 - 3A_2^3a_3 + 6A_1^2C_1 - 6A_2^2C_1 - 16A_1a_3C_1 + 12A_1A_2C_2 + 16A_2a_3C_2.$$

$$\begin{gathered}576V_8 = -2754^4A_2a_3 - 1836^2A_2^3a_3 + 918A_2^5a_3 + 531^2A_2a_3^3 - 177A_2^3a_3^3 - \\ 1800^4C_1 + 1800A_2^4C_1 + 4626^3a_3C_1 + 4626A_2^2a_3C_1 + 72^2a_3^2C_1 - 72A_2^2a_3^2C_1 + \\ 160a_3^3C_1 + 192A_2a_3C_1^2 - 3600^3A_2C_2 - 3600A_2^3C_2 - 4626^2A_2a_3C_2 - \\ 4626A_2^3a_3C_2 + 144A_2a_3^2C_2 - 160A_2a_3^3C_2 - 384a_3C_1C_2 - 192A_2a_3C_2^2.\end{gathered}$$

$$\begin{gathered}17280V_10 = 1279395A_1^6A_2a_3 + 2132325A_1^4A_2^3a_3 + 426465A_1^2A_2^5a_3 - 426465A_2^7a_3 + \\ 115830A_1^5A_2a_3^2 - 386100A_1^3A_2^3a_3^2 + 115830A_1A_2^5a_3^2 - 451008A_1^4A_2a_3^3 - 300672A_1^2A_2^3a_3^3 + \\ 150336A_2^5a_3^3 + 20772A_1^2A_2a_3^5 - 6924A_2^3a_3^5 + 831870A_1^6C_1 + 831870A_1^4A_2^2C_1 - \\ 831870A_1^2A_2^4C_1 - 831870A_2^6C_1 - 2039148A_1^5a_3C_1 - 4143096A_1^3A_2^2a_3C_1 - 2017548A_1A_2^4a_3C_1 - \\ 340992A_1^4a_3^2C_1 + 541836A_1^2A_2^2a_3^2C_1 + 160380A_2^4a_3^2C_1 + 140580A_1^3a_3^3C_1 + 140580A_1A_2^2a_3^3C_1 + \\ 9504A_1^2a_3^4C_1 - 9504A_2^2a_3^4C_1 - 4480A_1a_3^5C_1 + 263520A_1^3A_2C_1^2 - 263520A_1A_2^3C_1^2 - \\ 878328A_1^2A_2a_3C_1^2 - 52200A_2^3a_3C_1^2 + 124608A_1A_2a_3^2C_1^2 + 384A_2a_3^3C_1^2 - 18000A_1^2C_1^3 + \\ 18000A_2^2C_1^3 + 47232A_1a_3C_1^3 + 1663740A_1^5A_2C_2 + 3327480A_1^3A_2^3C_2 + 1663740A_1A_2^5C_2 + \\ 2071548A_1^4A_2a_3C_2 + 4035096A_1^2A_2^3a_3C_2 + 2049948A_2^5a_3C_2 - 140148A_1^3A_2a_3^2C_2 - 862596A_1A_2^3a_3^2C_2 - \\ 140580A_1^2A_2a_3^3C_2 - 140580A_2^3a_3^3C_2 + 19008A_1A_2a_3^4C_2 + 4480A_2a_3^5C_2 - 131760A_1^4C_1C_2 + \\ 790560A_1^2A_2^2C_1C_2 - 131760A_2^4C_1C_2 + 517464A_1^3a_3C_1C_2 + 517464A_1A_2^2a_3C_1C_2 + 124608A_1^2a_3^2C_1C_2 - \\ 124608A_2^2a_3^2C_1C_2 - 768A_1a_3^3C_1C_2 - 36000A_1A_2C_1^2C_2 - 47232A_2a_3C_1^2C_2 - 263520A_1^3A_2C_2^2 + \\ 263520A_1A_2^3C_2^2 - 360864A_1^2A_2a_3C_2^2 + 465264A_2^3a_3C_2^2 - 124608A_1A_2a_3^2C_2^2 - 384A_2a_3^3C_2^2 - \\ 18000A_1^2C_1C_2^2 + 18000A_2^2C_1C_2^2 + 47232A_1a_3C_1C_2^2 - 36000A_1A_2C_2^3 - 47232A_2a_3C_2^3.\end{gathered}$$

$$
\begin{gathered}
1244160V_1 2 = -1936661400A_1^8A_2a_3 - 5164430400A_1^6A_2^3a_3 - 3873322800A_1^4A_2^5a_3 + \\
645553800A_2^9a_3 - 483961230A_1^7A_2a_3^2 + 1129242870A_1^5A_2^3a_3^2 + 1129242870A_1^3A_2^5a_3^2 - \\
483961230A_1A_2^7a_3^2 + 890630235A_1^6A_2a_3^3 + 1484383725A_1^4A_2^3a_3^3 + 296876745A_1^2A_2^5a_3^3 - \\
296876745A_2^7a_3^3 + 67915665A_1^5A_2a_3^4 - 226385550A_1^3A_2^3a_3^4 + 67915665A_1A_2^5a_3^4 - \\
90319941A_1^4A_2a_3^5 - 60213294A_1^2A_2^3a_3^5 + 30106647A_2^5a_3^5 + 1967904A_1^2A_2a_3^7 - \\
655968A_2^3a_3^7 - 1257933240A_1^8C_1 - 2515866480A_1^6A_2^2C_1 + 2515866480A_1^2A_2^6C_1 + \\
1257933240A_2^8C_1 + 2874768408A_1^7a_3C_1 + 9298319184A_1^5A_2^2a_3C_1 + 9073647864A_1^3A_2^4a_3C_1 + \\
2650097088A_1A_2^6a_3C_1 + 1163777706A_1^6a_3^2C_1 - 1118882160A_1^4A_2^2a_3^2C_1 - 2685550950A_1^2A_2^4a_3^2C_1 - \\
402891084A_2^6a_3^2C_1 - 500665212A_1^5a_3^3C_1 - 947367900A_1^3A_2^2a_3^3C_1 - 518652720A_1A_2^4a_3^3C_1 - \\
44013564A_1^4a_3^4C_1 + 22881960A_1^2A_2^2a_3^4C_1 + 36386244A_2^4a_3^4C_1 + 23666544A_1^3a_3^5C_1 + \\
23666544A_1A_2^2a_3^5C_1 + 476928A_1^2a_3^6C_1 - 476928A_2^2a_3^6C_1 + 573440A_1a_3^7C_1 - \\
812903040A_1^5A_2C_1^2 + 812903040A_1A_2^5C_1^2 + 2467133100A_1^4A_2a_3C_1^2 + 2480516352A_1^2A_2^3a_3C_1^2 + \\
13383252A_2^5a_3C_1^2 - 322955856A_1^3A_2a_3^2C_1^2 - 448810416A_1A_2^3a_3^2C_1^2 - 11020536A_1^2A_2a_3^3C_1^2 - \\
4949208A_2^3a_3^3C_1^2 + 8098560A_1A_2a_3^4C_1^2 + 434688A_2a_3^5C_1^2 + 34894800A_1^4C_1^3 - \\
34894800A_2^4C_1^3 - 94726512A_1^3a_3C_1^3 - 33919920A_1A_2^2a_3C_1^3 + 10586304A_1^2a_3^2C_1^3 + \\
2746944A_2^2a_3^2C_1^3 + 4826880A_1a_3^3C_1^3 - 2446848A_2a_3C_1^4 - 2515866480A_1^7A_2C_2 - \\
7547599440A_1^5A_2^3C_2 - 7547599440A_1^3A_2^5C_2 - 2515866480A_1A_2^7C_2 - 3211775388A_1^6A_2a_3C_2 - \\
8511969564A_1^4A_2^3a_3C_2 - 8287298244A_1^2A_2^5a_3C_2 - 2987104068A_2^7a_3C_2 + 44895546A_1^5A_2a_3^2C_2 + \\
3133337580A_1^3A_2^3a_3^2C_2 + 3088442034A_1A_2^5a_3^2C_2 + 473683950A_1^4A_2a_3^3C_2 + 1037305440A_1^2A_2^3a_3^3C_2 + \\
491671458A_2^5a_3^3C_2 - 65145168A_1^3A_2a_3^4C_2 - 95654448A_1A_2^3a_3^4C_2 - 23666544A_1^2A_2a_3^5C_2 - \\
23666544A_2^3a_3^5C_2 + 953856A_1A_2a_3^6C_2 - 573440A_2a_3^7C_2 + 406451520A_1^6C_1C_2 - \\
2032257600A_1^4A_2^2C_1C_2 - 2032257600A_1^2A_2^4C_1C_2 + 406451520A_2^6C_1C_2 - 1253641428A_1^5a_3C_1C_2 - \\
2507282856A_1^3A_2^2a_3C_1C_2 - 1253641428A_1A_2^4a_3C_1C_2 - 417346776A_1^4a_3^2C_1C_2 + 188781840A_1^2A_2^2a_3^2C_1C_2 + \\
354419496A_2^4a_3^2C_1C_2 + 12934080A_1^3a_3^3C_1C_2 + 12934080A_1A_2^2a_3^3C_1C_2 + 8098560A_1^2a_3^4C_1C_2 - \\
8098560A_2^2a_3^4C_1C_2 - 869376A_1a_3^5C_1C_2 + 69789600A_1^3A_2C_1^2C_2 + 69789600A_1A_2^3C_1^2C_2 - \\
57289968A_1^2A_2a_3C_1^2C_2 + 125129808A_2^3a_3C_1^2C_2 + 7839360A_1A_2a_3^2C_1^2C_2 - 4826880A_2a_3^3C_1^2C_2 + \\
4893696A_1a_3C_1^3C_2 + 812903040A_1^5A_2C_2^2 - 812903040A_1A_2^5C_2^2 + 1213491672A_1^4A_2a_3C_2^2 - \\
26766504A_1^2A_2^3a_3C_2^2 - 1240258176A_2^5a_3C_2^2 + 322955856A_1^3A_2a_3^2C_2^2 + 448810416A_1A_2^3a_3^2C_2^2 + \\
1913544A_1^2A_2a_3^3C_2^2 + 7984872A_2^3a_3^3C_2^2 - 8098560A_1A_2a_3^4C_2^2 - 434688A_2a_3^5C_2^2 + \\
34894800A_1^4C_1C_2^2 - 34894800A_2^4C_1C_2^2 - 33919920A_1^3a_3C_1C_2^2 - 216339696A_1A_2^2a_3C_1C_2^2 - \\
16080192A_1^2a_3^2C_1C_2^2 - 23919552A_2^2a_3^2C_1C_2^2 + 4826880A_1a_3^3C_1C_2^2 + 69789600A_1^3A_2C_2^3 + \\
69789600A_1A_2^3C_2^3 + 125129808A_1^2A_2a_3C_2^3 + 64323216A_2^3a_3C_2^3 + 7839360A_1A_2a_3^2C_2^3 - \\
4826880A_2a_3^3C_2^3 + 4893696A_1a_3C_1C_2^3 + 2446848A_2a_3C_2^4.
\end{gathered}
$$

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