SURGERY AND GEOMETRIC TOPOLOGY

ANDREW RANICKI AND MASAYUKI YAMASAKI, EDITORS

Proceedings of a conference held at Josai University
Sakado, September 1996

Contents

Foreword

As the second of a conference series in mathematics, Josai University sponsored the Conference on Surgery and Geometric Topology during the week of September 17 – 21, 1996. The scientific program consisted of 13 lectures, listed below, and there was an excursion to Takaragawa, Gunma on the 21st.

This volume collects papers by participants, as well as some of the abstracts prepared by the lecturers for the conference. The articles are also available electronically on WWW from:

`http://math.josai.ac.jp/~yamasaki/conference.html`

at least for several years.

We would like to thank Josai University, and Grant-in-Aid for Scientific Research (A)(1) of the Ministry of Education, Science, Sports and Culture of Japan, for their generous financial support. Thanks are also due to the university staff for their various support, to the lecturers and to the participants of the conference.

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(Edinburgh, Scotland)
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(Sakado, Japan)
December, 1996

Program of the Conference

Surgery and Geometric Topology

Sakado, Saitama

September 17, Tuesday

10:00 – 11:20 **Andrew Ranicki**
Chain Duality.

13:20 – 14:40 **Erik Pedersen**
Simplicial and Continuous Control

15:00 – 16:20 **Toshiyuki Akita**
Cohomology and Euler Characteristics of Coxeter Groups

September 18, Wednesday

10:00 – 11:20 **Frank Quinn**
Problems with Surgery and Handlebodies in Low Dimensions

13:20 – 14:40 **John Roe**
Surgery and Operator Algebras

15:00 – 16:20 **Tsuyoshi Kato**
The Asymptotic Method in the Novikov Conjecture

16:30 – 17:20 **Eiji Ogasa**
On the Intersection of Spheres in a Sphere

September 19, Thursday

10:00 – 11:20 **Bruce Hughes**
Stratified Spaces and Approximate Fibrations

13:20 – 14:40 **Francis Connolly**
An End Theorem for Manifold Stratified Spaces

15:00 – 16:20 **Tatsuhiko Yagasaki**
Infinite-dimensional Manifold Triples of Homeomorphism Groups

September 20, Friday

10:00 – 11:20 **Yongjin Song**
The Braid Structure of Mapping Class Groups

13:20 – 14:40 **Masaharu Morimoto**
On Fixed Point Data of Smooth Actions on Spheres

15:00 – 16:00 **Masayuki Yamasaki**
Epsilon Control and Perl

List of Participants

(The numbers correspond to the keys to the photograph on page xi.)

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Surgery and Geometric Topology

Contents

COHOMOLOGY AND EULER CHARACTERISTICS OF COXETER GROUPS

TOSHIYUKI AKITA

1. Introduction

Coxeter groups are familiar objects in many branches of mathematics. The connections with semisimple Lie theory have been a major motivation for the study of Coxeter groups. (Crystallographic) Coxeter groups are involved in Kac-Moody Lie algebras, which generalize the entire theory of semisimple Lie algebras. Coxeter groups of finite order are known to be finite reflection groups, which appear in invariant theory. Coxeter groups also arise as the transformation groups generated by reflections on manifolds (in a suitable sense). Finally, Coxeter groups are classical objects in combinatorial group theory.

In this paper, we discuss the cohomology and the Euler characteristics of (finitely generated) Coxeter groups. Our emphasis is on the rôle of the parabolic subgroups of finite order in both the Euler characteristics and the cohomology of Coxeter groups.

The Euler characteristic is defined for groups satisfying a suitable cohomological finiteness condition. The definition is motivated by topology, but it has applications to group theory as well. The study of Euler characteristics of Coxeter groups was initiated by J.-P. Serre [22], who obtained the formulae for the Euler characteristics of Coxeter groups, as well as the relation between the Euler characteristics and the Poincaré series of Coxeter groups. The formulae for the Euler characteristics of Coxeter groups were simplified by I. M. Chiswell [7]. From his result, one knows that the Euler characteristics of Coxeter groups can be computed in terms of the orders of parabolic subgroups of finite order.

On the other hand, for a Coxeter group W, the family of parabolic subgroups of finite order forms a finite simplicial complex $\mathcal{F}(W)$. In general, given a simplicial complex K, the Euler characteristics of Coxeter groups W with $\mathcal{F}(W) = K$ are bounded, but are not unique. However, it follows from the result of M. W. Davis that $e(W) = 0$ if $\mathcal{F}(W)$ is a generalized homology $2n$-sphere (Theorem 4). Inspired by this result, the author investigated the relation between the Euler characteristics of Coxeter groups W and the simplicial complexes $\mathcal{F}(W)$, and obtained the following results:

1. If $\mathcal{F}(W)$ is a PL-triangulation of some closed $2n$-manifold M, then
$$e(W) = 1 - \frac{\chi(M)}{2}.$$
2. If $\mathcal{F}(W)$ is a connected graph, then $e(W) \geq \gamma(\mathcal{F}(W))$, where $\gamma(-)$ denotes the genus of the graph.

See Theorem 5 and 7. Conversely, given a PL-triangulation K of a closed $2n$-manifold M, we obtain an equation for the number of i-simplices of K $(0 \leq i \leq 2n)$ by considering a Coxeter group W with $K = \mathcal{F}(W)$ (Theorem 6 and its corollary).

The family of parabolic subgroups of finite order is also important in understanding the cohomology of a Coxeter group W. For instance, let k be a commutative ring with unity, ρ a ring homomorphism

$$\rho : H^*(W, k) \to \prod_{W_F} H^*(W_F, k).$$

induced by restriction maps, where W_F ranges all the parabolic subgroups of finite order. Then $u \in \ker \rho$ is nilpotent and cannot be detected by any finite subgroup of W. And we can say more about the homomorphism ρ.

We remark that, according to the results of D. Quillen [19] and K. S. Brown [5], the family of elementary abelian p-subgroups also plays an important rôle. However, it is p-local. The rôle of the parabolic subgroups of finite order is *not* p-local, a phenomenon in which I am very interested.

Notation 1. For a finite set X, the cardinality of X is denoted by $|X|$. In particular, for a finite group G, the order of G is denoted by $|G|$.

2. Definitions and Examples

In this section, we give the definition and elementary examples of Coxeter groups.

Definition 2.1. Let S be a finite set. Let $m : S \times S \to \mathbb{N} \cup \{\infty\}$ be a map satisfying the following three conditions:

1. $m(s,t) = m(t,s)$ for all $s, t \in S$,
2. $m(s,s) = 1$ for all $s \in S$,
3. $2 \leq m(s,t) \leq \infty$ for all distinct $s, t \in S$.

The group W defined by the set of generators S and the fundamental relation $(s \cdot t)^{m(s,t)} = 1$ $(m(s,t) \neq \infty)$ is called a *Coxeter group*. Some authors permit S to be an infinite set.

Remark 1. We frequently write (W, S) or (W, S, m) instead of W to emphasize S and m. The pair (W, S) is sometimes called a *Coxeter system* in the literature.

Remark 2. Each generator $s \in S$ is an element of order 2 in W. Hence W is generated by involutions.

Example 2.1. Let (W, S) be a Coxeter group with $S = \{s, t\}$. If $m(s,t) < \infty$, then W is isomorphic to $D_{2m(s,t)}$, the dihedral group of order $2m(s,t)$. If $m(s,t) = \infty$, then W is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, the free product of two copies of the cyclic group of order 2.

Example 2.2. A *finite reflection group* is a finite subgroup of the orthogonal group $O(n)$ (for some n) generated by orthogonal reflections in the Euclidean space. A finite reflection group is known to be a Coxeter group, i.e., it admits a presentation of Coxeter groups. Conversely, any Coxeter group of finite order can be realized as a finite reflection group. Hence one can identify Coxeter groups of finite order with finite reflection groups in this way.

For example, an elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^n$ and a symmetric group Σ_n can be regarded as Coxeter groups. Finite reflection groups are completely classified and their list is short. By using the list, it is easy to determine if a given Coxeter group is of finite order. See [4], [15] for details.

Example 2.3. Coxeter groups are closed under free products and direct products.

Example 2.4 (Full triangular group)**.** Let p, q, r be integers greater than 1. The group $T^*(p,q,r)$ defined by the presentation

$$T^*(p,q,r) =< s_1, s_2, s_3 | s_i^2 = (s_1s_2)^p = (s_2s_3)^q = (s_3s_1)^r = 1 >$$

is called the *full triangular group*. It is obvious from the presentation that $T^*(p,q,r)$ is a Coxeter group. The group $T^*(p,q,r)$ is known to be of finite order if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

The *triangular group*

$$T(p,q,r) =< u, v | u^p = v^q = (uv)^r = 1 >$$

is a subgroup of $T^*(p,q,r)$ of index 2 (via $u = s_1s_2$ and $v = s_2s_3$).

The full triangular group $T^*(p,q,r)$ can be realized as a planar discontinuous group acting on a sphere S^2 (if $1/p + 1/q + 1/r > 1$), on the Euclidean plane E^2 (if $1/p + 1/q + 1/r = 1$), or on the hyperbolic plane H^2 (if $1/p + 1/q + 1/r < 1$). The orbit space of the action of $T^*(p,q,r)$ on S^2, E^2, or H^2 is homeomorphic to a disk D^2.

Example 2.5. Given integers p, q, r greater than 1, let $O(p,q,r)$ be the orbifold defined as follows. (See [21] for the notion of orbifolds.) The underlying space of O is a standard 2-simplex Δ^2. Vertices v_0, v_1, and v_2 of Δ^2 are corner reflection points of order $2p$, $2q$, and $2r$. The points in the interior of edges are reflection points, while the points in the interior of the whole Δ^2 are manifold points.

The orbifold $O(p,q,r)$ is uniformable (i.e., it has a manifold cover). Indeed, the orbifold $O(p,q,r)$ comes from the orbit space of the action of the full triangular group $T^*(p,q,r)$ on one of S^2, E^2, or H^2 mentioned in the bottom of Example 2.4. The orbifold fundamental group $\pi_1^{orb}(O(p,q,r))$ is isomorphic to $T^*(p,q,r)$.

Let $O'(p,q,r)$ be the orbifold, whose underlying space is a 2-sphere S^2, with three cone points of order p, q, and r. Then there is a double orbifold covering

$$O'(p,q,r) \to O(p,q,r).$$

The orbifold fundamental group $\pi_1^{orb}(O'(p,q,r))$ is isomorphic to the triangular group $T(p,q,r)$. See [17] and [21] for the details.

Example 2.6. Example 2.4 and 2.5 are special cases of reflection orbifolds and groups generated by reflections on a manifold, both of which are closely related to Coxeter groups. See [8] and [16] for the general theory.

3. Parabolic Subgroups

Let (W, S, m) be a Coxeter group. For a subset $T \subset S$, define W_T to be the subgroup of W generated by the elements of T (i.e. $W_T =< T >\subseteq W$). In particular, $W_\emptyset = \{1\}$ and $W_S = W$. W_T is called a *parabolic subgroup* (or special subgroup) of W. The subgroup W_T is known to be a Coxeter group. Indeed, $(W_T, T, m|T \times T)$ is a Coxeter group. It is obvious from the definition that the number of parabolic subgroups of a Coxeter group is finite.

Example 3.1. Parabolic subgroups of the full triangular group $T^*(p,q,r)$ consist of 8 subgroups. Namely,

1. The trivial subgroup $\{1\}$.
2. Three copies of a cyclic group of order 2 (generated by single element).
3. Dihedral groups of order $2p$, $2q$, and $2r$ (generated by two distinct elements).
4. $T^*(p,q,r)$ itself.

The following observation asserts that the parabolic subgroups of finite order are maximal among the subgroups of finite order in a Coxeter group.

Proposition 1 ([9, Lemma 1.3]). *Let W be a Coxeter group and H its finite subgroup. Then there is a parabolic subgroup W_F of finite order and an element $w \in W$ such that $H \subset wWw^{-1}$.*

4. Euler characteristics

In this section, we introduce the Euler characteristics of groups. First we introduce the class of groups for which the Euler characteristic is defined.

Notation 2. Let Γ be a group. Then $\mathbb{Z}\Gamma$ is the integral group ring of Γ. We regard $\mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module with trivial Γ-action.

Definition 4.1. A group Γ is said to be of *type FL* if $\mathbb{Z}$ admits a free resolution (over $\mathbb{Z}\Gamma$) of finite type. In other words, there is an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

of finite length such that each F_i is a finitely generated free $\mathbb{Z}\Gamma$-module.

Remark 3. If Γ is a group of type FL, then $\operatorname{cd}\Gamma < \infty$ and hence Γ is torsion-free.

Definition 4.2. A group Γ is said to be of *type VFL* if some subgroup of finite index is of type FL.

Now we define the Euler characteristic of a group. Let Γ be a group of type FL, and let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

b a free resolution of finite length. The *Euler characteristic* $e(\Gamma)$ of Γ is defined by

$$e(\Gamma) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}\Gamma} F_i.$$

Let Γ be a group of type VFL. Then its Euler characteristic $e(\Gamma)$ is defined by

$$e(\Gamma) = \frac{e(\Gamma')}{(\Gamma : \Gamma')} \in \mathbb{Q},$$

where Γ' is a subgroup of finite index which is of type FL. The rational number $e(\Gamma)$ is independent of the choice of a subgroup Γ', and we have

Proposition 2. *Let Γ be a group and Γ' a subgroup of finite index. Then Γ is of type VFL if and only if Γ' is of type VFL. If Γ is of type VFL, then*

$$e(\Gamma') = (\Gamma : \Gamma') \cdot e(\Gamma).$$

We give some examples of groups of type VFL and their Euler characteristics.

Example 4.1. Any finite group Γ is of type VFL. Its Euler characteristic is given by

$$e(\Gamma) = \frac{1}{|\Gamma|}.$$

(Take Γ' to be a trivial group $\{1\}$.)

Example 4.2. Let K be a finite aspherical polyhedron. Then its fundamental group $\pi = \pi_1(K)$ is of type FL, and

$$e(\pi) = \chi(K),$$

where $\chi(K)$ is the Euler characteristic of K. The fact that the Euler characteristic of a finite aspherical polyhedron depends only on its fundamental group is the motivation of the definition of Euler characteristics of groups.

For instance, the circle S^1 is aspherical and $\pi_1(S^1) \cong \mathbb{Z}$, hence

$$e(\mathbb{Z}) = \chi(S^1) = 0.$$

Let Σ_g be a closed orientable surface of genus $g > 0$. Then Σ_g is aspherical, proving

$$e(\pi_1(\Sigma_g)) = \chi(\Sigma_g) = 2 - 2g.$$

Example 4.3. If Γ_1, Γ_2 are groups of type VFL, then their free product $\Gamma_1 * \Gamma_2$ and their direct product $\Gamma_1 \times \Gamma_2$ are of type VFL, and

$$e(\Gamma_1 * \Gamma_2) = e(\Gamma_1) + e(\Gamma_2) - 1,$$
$$e(\Gamma_1 \times \Gamma_2) = e(\Gamma_1) \cdot e(\Gamma_2).$$

As a consequence, a free group F_n and a free abelian group $\mathbb{Z}^n$ are of type VFL (in fact type FL), and we have

$$e(F_n) = 1 - n,$$
$$e(\mathbb{Z}^n) = 0,$$

where F_n is the free group of rank n.

Example 4.4. The group $SL(2, \mathbb{Z})$ has a subgroup of index 24 which is isomorphic to the free group of rank 3. Hence $SL(2, \mathbb{Z})$ is of type VFL. Using Example 4.2 and 4.3, one can compute the Euler characteristic of $SL(2, \mathbb{Z})$ as

$$e(SL(2, \mathbb{Z})) = \frac{e(F_3)}{24} = -\frac{1}{12}.$$

Example 4.5. The Euler characteristics of groups are closely related to the Euler characteristics of orbifolds. (See [24] or [21] for the definition of the orbifold Euler characteristics.) Namely, let O be an orbifold such that

1. O has a finite manifold covering $M \to O$ for which M has the homotopy type of a finite complex.
2. The universal cover of O is contractible.

Then the orbifold fundamental group $\pi = \pi_1^{orb}(O)$ of O is of type VFL and one has

$$e(\pi) = \chi^{orb}(O),$$

where $\chi^{orb}(O)$ is the orbifold Euler characteristic of O.

Example 4.6. Let Γ be a full triangular group $T^*(p,q,r)$ of infinite order. Then, as in Example 2.4, Γ is isomorphic to the orbifold fundamental group of the orbifold $O(p,q,r)$. The orbifold $O(p,q,r)$ satisfies the conditions 1 and 2 in Example 4.5. Hence the Euler characteristic of Γ is identified with the orbifold Euler characteristic of $O(p,q,r)$. Using this, one has

$$e(\Gamma) = \frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right).$$

Finally, we mention two properties of Euler characteristics of groups. Let G be a group of type VFL.

Theorem 1 (Gottlieb-Stallings [12], [23])**.** *If $e(G) \neq 0$, then the center of G is a finite subgroup.*

Theorem 2 (Brown [5])**.** *Let p be a prime. If p^n divides the denominator of $e(G)$, then G has a subgroup of order p^n.*

In view of Example 4.1, Theorem 2 is a generalization of (a part of) Sylow's theorem.

5. Euler characteristics of Coxeter groups (I)

J.-P. Serre [22] proved that Coxeter groups are of finite homological type. In fact he proved that Coxeter groups satisfy a much stronger finiteness condition than finite homological type, called type WFL. He also provided the formulae for the Euler characteristics of Coxeter groups.

The formulae of Euler characteristics of Coxeter groups were simplified by I. M. Chiswell [7], which we now quote. Before doing this, we remark that, if a Coxeter group W is of finite order, then its Euler characteristic is given by $e(W) = 1/|W|$ (Example 4.1). Hence we may assume a Coxeter group W to be of infinite order.

Theorem 3 (Chiswell [7])**.** *The Euler characteristic $e(W)$ of a Coxeter group W of infinite order is given by*

$$e(W) = \sum_{\substack{T\subset S\\ |W_T|<\infty}} (-1)^{|T|} e(W_T) = \sum_{\substack{T\subset S\\ |W_T|<\infty}} (-1)^{|T|}\frac{1}{|W_T|}. \tag{1}$$

Thus the Euler characteristics of Coxeter groups are completely determined their parabolic subgroups of finite order. Since the order of a finite reflection group is easy to compute, so is the Euler characteristic of a Coxeter group.

Serre also obtained in [22] the relation between the Euler characteristic of a Coxeter group and the Poincaré series. Namely, for a Coxeter group (W,S), define

$$g(t) = \sum_{w\in W} t^{l(w)},$$

where $l(w)$ is the minimum of the length of reduced words in S representing w. The function $g(t)$ is known to be a rational function and is called *Poincaré series* of (W,S). Serre proved

$$e(W) = \frac{1}{g(1)}.$$

In general, Poincaré series of arbitrary finitely presented groups may not satisfy this property. See [11].

6. Poset of Parabolic Subgroups of Finite Order

Before continuing the discussion of the Euler characteristics of Coxeter groups, we introduce the simplicial complexes associated with Coxeter groups. Given a Coxeter group (W, S), define $\mathcal{F}(W)$ to be the poset of nontrivial subsets $F \subseteq S$ such that the order of the corresponding parabolic subgroup W_F is of finite order. If there is no ambiguity we write $\mathcal{F}$ instead of $\mathcal{F}(W)$. The poset $\mathcal{F}(W)$ can be regarded as an (abstract) simplicial complex with the set of vertices S.

Example 6.1. If (W, S) is a finite reflection group with $|S| = n$, then any nontrivial subset $F \subset S$ belongs to $\mathcal{F}(W)$ and hence

$$\mathcal{F} = \Delta^{n-1},$$

the standard $(n-1)$-simplex.

Example 6.2. If (W, S) is a full triangular group of infinite order, then

$$\mathcal{F} = \partial\Delta^2,$$

the boundary of the standard 2-simplex (i.e. a triangle).

Example 6.3. The list of Coxeter groups with $\mathcal{F}(W) = \partial\Delta^3$ can be found in [24].

Example 6.4. Let K be a finite simplicial complex. A finite simplicial complex K is called a *flag complex* if K satisfies the following condition: For any subset $V = \{v_0, \cdots, v_n\}$ of vertices of K, if any two element subset $\{v_i, v_j\}$ of V form an edge of K, then $V = \{v_0, \cdots, v_n\}$ spans an n-simplex. A barycentric subdivision $\operatorname{Sd} K$ of a finite simplicial complex K is an example of a flag complex.

If K is a flag complex, then there is a Coxeter group W for which $\mathcal{F}(W) = K$. Namely, let S be the set of vertices of K. Define $m : S \times S \to \mathbb{N} \cup \{\infty\}$ by

$$m(s_1, s_2) = \begin{cases} 1 & s_1 = s_2 \\ 2 & \{s_1, s_2\} \text{ forms a 1-simplex} \\ \infty & \text{otherwise.} \end{cases}$$

The resulting Coxeter group (W, S) satisfies $\mathcal{F}(W) = K$. In particular, given a finite simplicial complex K, there is a Coxeter group W with $\mathcal{F}(W) = \operatorname{Sd} K$.

Definition 6.1. A Coxeter group (W, S) with all $m(s, t) = 2$ or ∞ for distinct $s, t \in S$ is called *right-angled* Coxeter group.

Coxeter groups constructed in Example 6.4 are examples of right-angled Coxeter groups. Conversely, if W is a right-angled Coxeter group, then $\mathcal{F}(W)$ is a flag complex.

Remark 4. It is not known if there is a Coxeter group W for which $\mathcal{F}(W) = K$ for a given finite simplicial complex K.

7. Euler characteristics of Coxeter groups (II)

Now let us consider the Euler characteristic of W in terms of the structure of $\mathcal{F}(W)$. Proofs of statements of the following three sections will appear in [3]. If (W, S) is a finite reflection group, then

$$|W| \geq 2^{|S|}. \tag{2}$$

The equality holds if and only if W is isomorphic to the elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^{|S|}$ of rank $|S|$. Now let K be a finite simplicial complex. Then the Euler characteristic of any Coxeter group W with $K = \mathcal{F}(W)$ must satisfy

$$1 - \sum_{i:even} \frac{f_i(K)}{2^{i+1}} \le e(W) \le 1 + \sum_{i:odd} \frac{f_i(K)}{2^{i+1}}, \tag{3}$$

where $f_i(K)$ is the number of i-simplices of K. This follows from Theorem 3 and the equation (2). As in the following example, the inequality (3) is not best possible.

Example 7.1. Suppose $K = \partial\Delta^3$. Then Coxeter groups W with $\mathcal{F}(W) = K$ are precisely full triangular groups of infinite order. The inequality (3) implies

$$-\frac{1}{2} \le e(W) \le \frac{7}{4}.$$

On the other hand, from the formula in Example 4.6 one has

$$-\frac{1}{2} < e(W) \le 0,$$

which is best possible.

Example 7.1 shows that, for a fixed finite simplicial complex K, Euler characteristics of Coxeter groups with $\mathcal{F}(W) = K$ can vary. However, from the result of M. W. Davis [8], one has:

Theorem 4. *Let W be a Coxeter group such that $\mathcal{F}(W)$ is a generalized homology $2n$-sphere, then*

$$e(W) = 0.$$

Here a generalized homology $2n$-sphere is a simplicial complex K satisfying

1. K has the homology of a $2n$-sphere.
2. The link of an i-simplex of K has the homology of a $(2n - i - 1)$-sphere.

A simplicial complex satisfying the condition 1 and 2 is also called a *Cohen-Macaulay complex.* A triangulation of a homology sphere is an example of a generalized homology sphere.

Note that Davis actually proved that, if W is a Coxeter group such that $\mathcal{F}(W)$ is a generalized homology $2n$-sphere, then, for each torsion free subgroup Γ of finite index in W, there is a closed aspherical $(2n+1)$-manifold M with $\pi_1(M) \cong \Gamma$ [8, Theorem 10.1]. It follows that

$$e(W) = \frac{e(\Gamma)}{(W : \Gamma)} = \frac{\chi(M)}{(W : \Gamma)} = 0,$$

since M is odd dimensional and has homotopy type of a finite simplicial complex.

We (partially) generalize Theorem 4. A finite simplicial complex K is a PL-triangulation of a closed M if, for each simplex T of K, the link of T in K is a triangulation of $(\dim M - \dim T - 1)$-sphere. If K is a PL-triangulation of a homology sphere, then K is a generalized homology sphere.

Theorem 5 (T. Akita)**.** *Let W be a Coxeter group such that $\mathcal{F}(W)$ is a PL-triangulation of a closed $2n$-manifold, then*

$$e(W) = 1 - \frac{\chi(\mathcal{F}(W))}{2},$$

where $\chi(\mathcal{F}(W))$ is the Euler characteristic of the simplicial complex $\mathcal{F}(W)$.

Remark 5. Given a simplicial complex K, there is a Coxeter group W such that $\mathcal{F}(W)$ agrees with the barycentric subdivision of K (Example 6.4). Hence there are Coxeter groups for which Theorem 5 and Theorem 5 can be applied.

Remark 6. We should point out that the assumptions of Theorem 4 and 5 permit, for instance, K to be an *arbitrary* triangulation of $2n$-sphere. The significance becomes clear if we compare with the case that K is a triangulation of a circle S^1. Indeed, the Euler characteristics of Coxeter groups with $\mathcal{F}(W)$ a triangulation of a circle S^1 can be arbitrary small.

Remark 7. Under the assumption of Theorem 5, $2 \cdot e(W)$ is an integer. On the other hand, given a rational number q, there is a Coxeter group W with $e(W) = q$.

8. Application of Theorem 5

Let K be a flag complex. Let (W, S) be a Coxeter group with $\mathcal{F}(W) = K$ as in Example 6.4. Any parabolic subgroup W_F of finite order is isomorphic to the elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^{|F|}$ of rank $|F|$. Hence the Euler characteristic of W is determined by the number of simplices of K. Explicitly, let $f_i(K)$ be the number of i-simplices of K. Then

$$e(W) = 1 + \sum_i \left(-\frac{1}{2}\right)^{i+1} f_i(K), \tag{4}$$

Using this together with Theorem 5, one obtains

Theorem 6 (T. Akita). *Let K be a PL-triangulation of a closed $2n$-manifold. Assume K is a flag complex. Then*

$$\chi(K) = \sum_i \left(-\frac{1}{2}\right)^{i} f_i(K).$$

In particular, the barycentric subdivision of any finite simplicial complex is a flag complex. Thus

Corollary . *Let K be a PL-triangulation of a closed $2n$-manifold. Let $f_i(\mathrm{Sd}\,K)$ is the number of i-simplices of the barycentric subdivision $\mathrm{Sd}\,K$ of K. Then*

$$\chi(\mathrm{Sd}\,K) = \sum_i \left(-\frac{1}{2}\right)^{i} f_i(\mathrm{Sd}\,K).$$

In general, if K is a triangulation of a closed n-manifold, then

$$\begin{aligned} \chi(K) &= \sum_i (-1)^i f_i(K) \\ f_{n-1}(K) &= \frac{n+1}{2} f_n(K) \end{aligned} \tag{5}$$

hold. The equality in Theorem 6 is not the consequence of the equalities (5).

For a triangulation K of a sphere S^n (for arbitrary n), the Dehn-Sommerville equations give a set of equations for the $f_i(K)$'s. It would be interesting to investigate the relation between the equation in Theorem 6 and Dehn-Somerville equations.

9. Euler Characteristics of Aspherical Coxeter Groups and the Genus of a Graph

In this section, we consider the Euler characteristics of Coxeter groups W such that $\mathcal{F}(W)$ is a graph (1-dimensional simplicial complex).

Definition 9.1. A Coxeter group (W, S) is called *aspherical* (in [18]) if every three distinct elements of S generate a parabolic subgroup of infinite order.

In view of Example 2.4, a Coxeter group (W, S) is aspherical if and only if for every three distinct elements $s, t, u \in S$,

$$\frac{1}{m_{st}} + \frac{1}{m_{tu}} + \frac{1}{m_{us}} \leq 1$$

holds, where $1/\infty = 0$ by the convention. It is easy to see that a Coxeter group W is aspherical if and only if $\mathcal{F}(W)$ is a graph.

For a graph Γ, let $E(\Gamma)$ be the set of edges of Γ. When (W, S) is an aspherical Coxeter group with $\Gamma = \mathcal{F}(W)$, it follows from Chiswell's formula (1) that

$$1 - \frac{|S|}{2} < e(W) \leq 1 - \frac{|S|}{2} + \frac{|E(\Gamma)|}{4}. \tag{6}$$

One has another inequality for $e(W)$ using the genus of a graph. The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest number g such that the graph Γ imbeds in the closed orientable surface of genus g. For instance, a graph Γ is a planar graph if and only if $\gamma(\Gamma) = 0$.

Theorem 7 (T. Akita)**.** *Let (W, S) be a Coxeter group for which $\mathcal{F}(W)$ is a connected finite graph. Then*

$$e(W) \geq \gamma(\mathcal{F}).$$

Example 9.1. For any non-negative integer n, there is a Coxeter group W satisfying

1. $\mathcal{F}(W)$ is a graph of genus n.
2. $e(W) = n$.

The construction uses the complete bipartite graphs $K_{m,n}$.

Recall that a graph Γ is a *bipartite graph* if its vertex set can be partitioned into two subsets U and V such that the vertices in U are mutually nonadjacent and the vertices in V are mutually nonadjacent. If every vertex of U is adjacent to every vertex of V, then the graph is called *completely bipartite* on the sets U and V. A complete bipartite graph on sets of m vertices and n vertices is denoted by $K_{m,n}$.

Now the genus of the completely bipartite graph $K_{m,n}$ is given by

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

See [13, Theorem 4.5.3]. Now let $S = S_1 \sqcup S_2$ with $|S_1| = m$, $|S_2| = n$. Define $m : S \times S \to \mathbb{N} \cup \{\infty\}$ by

$$m(s,t) = \begin{cases} 1 & s = t \\ 2 & s \in S_i, t \in S_j, i \neq j \\ \infty & \text{otherwise.} \end{cases}$$

Then the resulting Coxeter group (W, S) is right-angled and satisfies $\mathcal{F}(W) = K_{m,n}$. Its Euler characteristic is given by

$$e(W) = \frac{(m-2)(n-2)}{4}.$$

Alternatively, one can construct similar examples by using complete graphs.

10. Cohomology of Coxeter Groups

In this section, we are concerned with the cohomology of Coxeter groups. The content of this section extends the earlier papers [1] and [2]. We restrict our attention to the relation between the cohomology of Coxeter groups and the cohomology of parabolic subgroups of finite order. Let (W, S) be a Coxeter group. Let k be a commutative ring with identity, regarded as a W-module with trivial W-action. Set

$$\mathcal{H}^*(W, k) = \varprojlim{}_{W_F} H^*(W_F, k),$$

where W_F runs all (possibly trivial) parabolic subgroups of finite order. The inverse limit is taken with respect to restriction maps $H^*(W_F) \to H^*(W'_F)$ for $F' \subset F$. Let

$$\rho : H^*(W, k) \to \mathcal{H}^*(W, k)$$

be the ring homomorphism induced by the restriction maps $H^*(W, k) \to H^*(W_F, k)$. The properties of ρ are the main topic of this section.

D. J. Rusin [20], M. W. Davis and T. Januszkiewicz [10] computed the mod 2 cohomology ring of certain Coxeter groups.

Theorem 8 ([20, Corollary 30])**.** *Let W be a Coxeter group with hyperbolic signature, with all rank-3 parabolic subgroups hyperbolic, and with even exponents $m(s, t)$. Then*

$$H^*(W, \mathbb{F}_2) \cong \mathbb{F}_2[u_r, w_{s,t}] \ (r, s, t \in S)$$

with relations $u_r w_{s,t} = 0$ if $r \neq s$ and $r \neq t$, $w_{r,s} w_{t,u} = 0$ unless $\{r, s\} = \{t, u\}$, and $u_s u_t = 0$ if 4 divides $m(s, t)$ but $u_s u_t = w_{s,t}$ otherwise.

Here we shall not explain the assumptions in Theorem 8. Instead we point out that if all $m(s, t)$ (with s, t distinct) are large enough (compared with the cardinality S), then the resulting Coxeter group has hyperbolic signature and its rank 3 parabolic subgroups are hyperbolic. Such a Coxeter group must be aspherical.

Theorem 9 ([10, Theorem 4.11])**.** *Let W be a right-angled Coxeter group. Then*

$$H^*(W, \mathbb{F}_2) \cong \mathbb{F}_2[v_1, \cdots, v_m]/I,$$

where I is the ideal generated by all square free monomials of the form $v_{i_1} \cdots v_{i_n}$, where at least two of the v_{i_j} do not commute when regarded as elements of W.

See Definition 6.1 for the definition of right-angled Coxeter group. From their results, one can show that ρ induces an isomorphism

$$H^*(W, \mathbb{F}_2) \cong \mathcal{H}^*(W, \mathbb{F}_2).$$

for a Coxeter group W which satisfies the assumptions in Theorem 8 or 9. Inspired by this observation, we proved

Theorem 10. *Let W be a Coxeter group and k a commutative ring with identity. Let $\rho : H^*(W,k) \to \mathcal{H}^*(W,k)$ be as above. Then the kernel and the cokernel of ρ consist of nilpotent elements.*

A homomorphism satisfying these properties is called an *F-isomorphism* in [19]. Notice that, unlike the famous result of Quillen [19] concerning of the mod p cohomology of groups of finite virtual cohomological dimension, the coefficient ring k can be the ring $\mathbb{Z}$ of rational integers.

Example 10.1. Let W be the full triangular group $T^*(3,3,3)$. Its mod 2 cohomology ring is given by $H^*(W,\mathbb{F}_2) \cong \mathbb{F}_2[u,v]/(u^2)$, where $\deg u = 2$ and $\deg v = 1$ [20, p.52], while $\mathcal{H}(W,\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[w]$ with $\deg w = 1$. Then $\rho(u) = 0$ and hence ρ has nontrivial kernel for $k = \mathbb{F}_2$. This shows the homomorphism ρ may not be an isomorphism in general.

Unfortunately, we do not know whether ρ may have a non-trivial cokernel. We give a sufficient condition for ρ to be surjective.

Theorem 11. *Suppose that W is an aspherical Coxeter group (see Definition 9.1). Then ρ is surjective for any abelian group A of coefficients (with trivial W-action).*

For example, Coxeter groups satisfying the assumptions in Theorem 8 must be aspherical.

In the case $k = \mathbb{F}_2$, there is more to say. By Theorem 10, the homomorphism ρ induces the homomorphism $H^*(W,k)/\sqrt{0} \to \mathcal{H}^*(W,k)/\sqrt{0}$, where $\sqrt{0}$ denotes nilradical. Rusin proved that the mod 2 cohomology ring of any finite Coxeter group (finite reflection group) has no nilpotent elements [20, Theorem 9]. Hence the nilradical of $\mathcal{H}^*(W,\mathbb{F}_2)$ is trivial. From this, together with Theorem 10 and 11 we obtain

Corollary . *For any Coxeter group W, ρ induces a monomorphism*

$$H^*(W,\mathbb{F}_2)/\sqrt{0} \to \mathcal{H}^*(W,\mathbb{F}_2),$$

which is an isomorphism if W is aspherical.

Remark 8. Another study of the relation between the cohomology of aspherical Coxeter groups and their parabolic subgroups of finite order can be found in [18].

Now we turn to our attention to detection by finite subgroups. An element $u \in H^*(W,k)$ is said to be *detected by finite subgroups* if the image of u by the map

$$\prod_H \mathrm{res}^W_H : H^*(W,k) \to \prod_H H^*(H,k)$$

is nontrivial, where H runs all the finite subgroups of W. It would be of interest to know which elements of $H^*(W,k)$ $\hat{H}^*(W,k)$ are detected by finite subgroups. One can reduce this question to the following proposition, which follows from Theorem 10 and Proposition 1.

Proposition 3. *An element $u \in H^*(W,k)$ is detected by finite subgroups if and only if $u \notin \ker \rho$.*

Finally, we give a example of elements of $H^*(W,k)$ which cannot be detected by finite subgroups.

Example 10.2. Let W be the full triangular group $T^*(3,3,3)$. Its mod 2 cohomology ring is given by in Example 10.1. One can check easily that uv^n $(n \geq 1)$ is contained in $\ker \hat{\rho}$. Thus uv^n $(n \geq 1)$ cannot be detected by finite subgroups as elements of $H^{2+n}(W, \mathbb{F}_2)$.

Remark 9. The virtual cohomological dimension of any Coxeter group W is known to be finite [22, p. 107], and its Farrell-Tate cohomology, written $\hat{H}^*(W,k)$, is defined. For the Farrell-Tate cohomology, the analogues of Theorem 10 and 11 and Proposition 3 are valid. See [1] and [2] for detail.

11. Outline of Proof

11.1. **Actions of Coxeter groups.** A suitable complex on which a Coxeter group acts is used in the proof of Theorem 10. We recall how this goes. Let (W,S) be a Coxeter group. Let X be a topological space, $(X_s)_{s\in S}$ be a family of closed subsets of X indexed by S. From these data, one can construct a space on which W acts as follows. Set

$$S(x) = \{s \in S : x \in X_s\},$$

and let $\mathcal{U} = \mathcal{U}(X) = W \times X/\sim$, W being discrete, where the equivalence relation $\sim$ is defined by

$$(w,x) \sim (w',x') \iff x = x' \& w^{-1}w' \in W_{S(x)}.$$

Then W acts on $\mathcal{U}(X)$ by $w' \cdot [w,x] = [w'w,x]$. The isotropy subgroup of $[w,x]$ is $wW_{S(x)}w^{-1}$.

11.2. **Proof of Theorem 10 (Outline).** Given a Coxeter group (W,S), let X be the barycentric subdivision of $c * \mathcal{F}(W)$, the cone of $\mathcal{F}(W)$ with the cone point c. Define X_s to be the closed star of $s \in S$ (here $s \in S$ is regarded as a vertex of $\mathcal{F}(W)$ and hence a vertex of X). Then one of the main results of M. W. Davis [8, §13.5] asserts that $\mathcal{U}(X)$ is contractible.

Consider the spectral sequence of the form

$$E_1^{pq} = \prod_{\sigma\in\Sigma_p} H^q(W_\sigma, k) \Longrightarrow H^{p+q}(W,k).$$

In the spectral sequence, one can prove that $E_2^{0,*}$ is isomorphic to $\mathcal{H}^*(W,k)$ and the homomorphism ρ is identified with the edge homomorphism $H^*(W,k) \to E_2^{0,*}$. Observe that

1. $E_2^{p,0} = 0$ if $p \neq 0$.
2. There is a natural number $n > 0$ such that $n \cdot E_r^{p,q} = 0$ for all p and $q > 0$.

Together with these observations, Theorem 10 follows from the formal properties of the differentials of the spectral sequence.

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COMPLETIONS OF STRATIFIED ENDS

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1. Introduction

1.1. A famous result of L. Siebenmann characterizes those topological manifolds which are the interiors of compact manifolds with boundary. Elsewhere we have recently shown that his theorem generalizes to the context of *stratified spaces*. Our purpose here is to explain the main results of our work briefly. See [1] for the full account.

2. Definitions

Let X^n, $n \geq 6$ be a tame ended topological n-manifold. Siebenmann proves that there is a single obstruction $\sigma(X)$, in $\tilde{K}_0(\mathbb{Z}\pi)$, with the property that $\sigma(X) = 0$ if and only if X^n is the interior of a compact manifold with boundary. By the work of Freedman and Quinn [3] one can also allow $n \geq 5$, if π is not too complicated. The group π denotes the fundamental group of the end of X, which can be described as the $Holink(\widehat{X}, \infty)$; here $\widehat{X}$ denotes the one-point compactification of X. The space, $Holink(X, A)$, (the "homotopy link" of A in X), is defined for any subspace A of a topological space X as:

$$Holink(X, A) = \{\sigma \in \mathrm{Map}([0,1], X) \mid \sigma^{-1}(A) = 0\}.$$

It is given the compact-open topology. It comes with two maps,

$$A \xleftarrow{p_X} Holink(X, A) \xrightarrow{j_X} X - A: \quad j_X(\sigma) = \sigma(1), p_X(\sigma) = \sigma(0).$$

It is used by Quinn [6] as a homotopical analogue for the normal sphere bundle of A in X.

F. Quinn generalizes Siebenmann's result greatly. For any locally compact pair (X, A), where A is closed and *tame* in X, $X - A$ is an n-manifold ($n \geq 6$) and A is an ANR, Quinn [4],[5],[6] defines an obstruction, $q_0(X, A) \in \tilde{K}_0^{lf}(A, p_X)$, which vanishes if and only if A has a mapping cylinder neighborhood in X. Here the map $p_X : Holink(X, A) \to A$ is the projection. This concept of tameness is discussed by many others at this conference. The foundational concepts surrounding controlled K-theory have recently been greatly clarified by the eminently readable paper of Ranicki and Yamasaki [7].

2.1. Quinn's obstruction, $q_0(X, A)$ can be localized in the following way: let A be a closed and tame subset of X, and X' an open subset of X. Then $A' := X' \cap A$ is tame in X' and $i^* q_0(X, A) = q_0(X', A')$ where $i^* : \tilde{K}_0^{lf}(A, p_X) \to \tilde{K}_0^{lf}(A', p_{X'})$ is the restriction map. Using these maps one can define, for every subset $B \subset A$,

$$K_0^{lf}((A, p_X)_{(B)}) = \varinjlim K_0^{lf}(A', p_{X'}|A')$$

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(the direct limit is over the $X-$ neighborhoods, X', of B). Then the image of any one of the obstructions, $q_0(X',A')$ in $K_0^{lf}((A,p_X)_{(B)})$ is independent of the X-neighborhood, X', chosen. We will write this image $q_0((X,A)_{(B)})$.

2.2. In geometric topology, the stratified analogue of a topological manifold is a *stratified space.* This concept was introduced by Quinn under the name "manifold homotopy stratified set"; our terminology is due to Hughes and Weinberger.

A *stratified space* is a locally compact space, finitely filtered by closed subsets $X = X^n \supset X^{n-1} \supset \cdots \supset X^{-1} = \emptyset$. Each *stratum* $X_i := X^i - X^{i-1}$ must be a manifold, and the *boundary of* X, defined by the rule, $\partial X := \cup_i(\partial X_i)$, must be closed in X. (It is customary, to arrange the indexing so that $dim(X_i) = i$). It is also required that X_i must be tame in $X_i \cup X_j$ for each $j > i$. The projection, $Holink(X_i \cup X_j, X_i) \xrightarrow{p} X_i$ must be a fibration, and the inclusion $Holink(\partial X_i \cup \partial X_j, \partial X_i) \to Holink(X_i \cup X_j, X_i)|_{\partial X_i}$ must be a fiber homotopy equivalence over ∂X_i.

3. Main results

Let X be a stratified space with empty boundary. We seek a *completion* of X, i.e. a compact stratified space $\bar{X}$ such that $X = \bar{X} - \partial\bar{X}$, and $\partial\bar{X}$ has a collar neighborhood in $\bar{X}$. It is easy to see that a necessary condition for X having a completion is to be *tame ended.* This means that the one point compactification of X, $\widehat{X} := X \cup \infty$, is again a stratified space. The stratification of the one-point compactification is the following;

$$\widehat{X}_0 = X_0 \cup \infty; \quad \widehat{X}_j = X_j,\, \forall j > 0.$$

An equivalent formulation is that $\{\infty\}$ is tame in $X_j \cup \{\infty\}$ for each j. Notice that, by reverse tameness each X^j can have only finitely many ends.

A completion may not always exist; a weaker requirement would be an *exhaustion* of X. This is defined to be an increasing sequence of compact stratified subspaces of X, with bicollared boundaries in X, whose union is X. An exhaustion is also obstructed in the category of stratified spaces.

Our main results, 3.2, 3.3, and 3.7, say that a completion (or an exhaustion) exists if a single obstruction vanishes.

3.1. *The End Obstruction.* Let X be a tame ended stratified space. For each integer $m > 0$ we define, (as in 2.1, above),

$$\gamma_m(X) = q_0((\widehat{X}^m, \widehat{X}^{m-1})_{(\infty)}) \;\in\; K_0^{lf}((\widehat{X}^{m-1}, p_{\widehat{X}^m})_{(\infty)})$$

As before, the map $p_{\widehat{X}^m} : Holink(\widehat{X}^m, \widehat{X}^{m-1}) \to \widehat{X}^{m-1}$ denotes the Holink projection. Set also:

$$\gamma_*(X) = \bigoplus_m \gamma_m(X) \;\in\; \bigoplus_m K_0^{lf}((\widehat{X}^{m-1}, p_{\widehat{X}^m})_{(\infty)}).$$

3.2. Theorem. *Suppose X is a stratified space, with empty boundary, which admits a completion. Then $\gamma_*(X) = 0$.*

3.3. Theorem. *Let X be a tame ended stratified space with empty boundary. Let A be any closed pure subset of X, containing X^5, such that A admits a completion, $\bar{A}$. Suppose $\gamma_*(X) = 0$. Then X admits a completion $\bar{X}$ such that $Cl_{\bar{X}}(A) = \bar{A}$.*

(A *pure* subset is one which is the union of components of strata.)

3.4. Note *This result reduces to Siebenmann's theorem when X has only one stratum.*

3.5. Note Following Weinberger, we say that a finite group action on a manifold, (M, G) is a *stratified G manifold* if the fixed set of each subgroup, M^H, is a manifold, and M^H is locally flat in M^K for each $K \subset H$. By (1.4, 1.5 and 1.6 of [6]), this is equivalent to saying that $X = M/G$ is a stratified space when it is stratified by its orbit type components. A corollary of our main theorem is an end-completion result for G-manifolds:

Corollary 3.6. *Let (M, G) be a stratified G-manifold with $\partial M = \emptyset$. Then (M, G) is the interior of a compact stratified G-manifold with collared boundary iff $X = M/G$ is tame ended, $\gamma_*(X) = 0$, and X^5 has a completion.*

The obstruction to finding an exhaustion for the stratified space X turns out to have the form $\partial\gamma_*(X)$, where ∂ is a map we will not define here in complete generality. Instead we will give the definition of $\partial\gamma_n(X)$ in the special case when X^{n-1} admits a completion. In this case an ∞- neighborhood in X^{n-1} has the form $B \times [0, \infty)$, for some stratified space B. Then the *open cone* of B, OB which can be thought of as $B \times (0, \infty]/B \times \infty$ is a neighborhood of ∞ in $\widehat{X}^{n-1}$; moreover ∞ has a cofinal sequence of such neighborhoods, $B \times (k, \infty]/B \times \infty$, $k = 0, 1, 2, \ldots$. The restriction maps connecting the K-theory of these are isomorphisms. This implies that the obstruction $\gamma_*(X)$ reduces to $\gamma_n(X)$, and moreover, that $\tilde{K}_0^{lf}((\widehat{X}^{n-1}, p_{\widehat{X}})_{(\infty)})$ can be identified to $\tilde{K}_0^{lf}(OB, p_{\widehat{X}}|_{OB})$, where $Holink(\widehat{X}, \widehat{X}^{n-1})|_{OB} \xrightarrow{p_{\widehat{X}}|_{OB}} OB$ is the projection map. The inclusion map induces a restriction map:

$$\tilde{K}_0^{lf}(\, OB, p) \longrightarrow \tilde{K}_0^{lf}(B \times (0, \infty), p|_{B \times (0, \infty)})$$

which amounts then to a map:

$$\partial_n : K_0^{lf}((\widehat{X}^{n-1}, p_{\widehat{X}})_{(\infty)}) \to K_{-1}(B, p_B)$$

where p_B denotes the restriction of the holink projection over B.

This is the map we seek. It turns out that $\partial_n\gamma_n(X) \in K_{-1}(B, p_B)$ is the obstruction to finding an exhaustion of X:

Theorem 3.7 (Exhaustibility Theorem)**.** *Let X be a tame ended n-dimensional stratified space with empty boundary for which X^{n-1} admits a completion. Assume that $\partial\gamma_n(X) = 0$. Then X admits an exhaustion.*

Conversely, if X admits an exhaustion, and all the fundamental groups of the fibers of the map, $Holink(X, X^{n-1}) \xrightarrow{p_X} X^{n-1}$ are good*, then $\partial\gamma_n(X) = 0$.*

3.8. We say a group G is *good* if $K_i(\mathbb{Z}[G]) = 0$ for $i \leq -2$.
No example of a group which is not good is known. Moreover, a recent theorem of Farrell and Jones [2] shows that any subgroup of a uniform discrete subgroup of a virtually connected Lie group must be good.

3.9. There are stratified G-manifolds which are not exhaustable, but are tame ended. In fact, there is a semifree action of $G = \mathbb{Z}/6\mathbb{Z}$ on $M_1' = \mathbb{R}^{2n+1} - \{0\}$, $n \geq 2$ with fixed set $\mathbb{R}^1 - \{0\}$, for which $\partial\gamma_{2n+1}(M/G) \neq 0$ in $K_{-1}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G)$. Furthermore if $M_1 = S^{2n} \times S^1$ and $M_1' \xrightarrow{\pi} M_1$ is the usual covering map, then π is equivariant with respect to a stratified G-action on M_1. This G-manifold, (M_1, G) is h-cobordant (stratified and equivariant) to some (M_0, G), whose infinite cyclic cover (M_0', G) has the form $(V - \{0\}, G)$, where (V, G) is a linear representation of G. This example and the more general question of realizability of the obstruction $\gamma_*(X)$ are thoroughly analyzed in the 1996 Ph.D. thesis of B. Vajiac.

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THE BRAID STRUCTURE OF MAPPING CLASS GROUPS

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0. Introduction

It was shown by Stasheff([13]) and MacLane([7]) that monoidal categories give rise to loop spaces. A recognition principle specifies an internal structure such that a space X has such a structure if and only if X is of the weak homotopy type of n-fold loop space. It has been known for years that there is a relation between coherence problems in homotopy theory and in categories. May's recognition theorem([9]) states that for little n-cube operad $\mathcal{C}_n$, $n \geq 2$, every n-fold loop space is a $\mathcal{C}_n$-space and every connected $\mathcal{C}_n$-space has the weak homotopy type of an n-fold loop space.

E. Miller([9]) observed that there is an action of the little square operad on the disjoint union of $BDiff^+(S_{g,1})$'s extending the F-product which is induced by a kind of connected sum of surfaces. We hence have that the group completion of $\amalg_{g\geq 0} BDiff^+(S_{g,1})$ is a double loop space up to homotopy. Miller applied this result to the calculation of the homology groups of mapping class groups. However his description of the action of the little square operad is somewhat obscure. On the other hand the first author proved([4]) that the group completion of the nerve of a braided monoidal category is the homotopy type of a double loop space. This result implies that there exists a strong connection between braided monoidal category and the mapping class groups $\Gamma_{g,1}$ in view of Miller's result.

We, in this paper, show that the disjoint union of $\Gamma_{g,1}$'s is a braided monoidal category with the product induced by the connected sum. Hence the group completion of $\amalg_{g\geq 0} B\Gamma_{g,1}$ is the homotopy type of a double loop space. We explicitly describe the braid structure of $\amalg_{g\geq 0}\Gamma_{g,1}$, regarding $\Gamma_{g,1}$ as the subgroup of the automorphism group of $\pi_1 S_{g,1}$ that consists of the automorphisms fixing the fundamental relator. We provide the formula for the braiding (Lemma 2.1) which is useful in dealing with the related problems. Using this braiding formula (2.2), we can make a correction on Cohen's diagram. We also show that the double loop space structure of the disjoint union of classifying spaces of mapping class groups cannot be extended to the triple loop space structure (Theorem 2.5). It seems important to note the relation between the braid structure and the double loop space structure in an explicit way.

Turaev and Reshetikhin introduced an invariant of ribbon graphs which is derived from the theory of quantum groups and is a generalization of Jones polynomial. This invariant was extended to those of 3-manifolds and of mapping class groups(cf.[11],[12],[6]). The definitions are abstract and a little complicated since they are defined through quantum groups. G.

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Wright([16]) computed the Reshetikhin-Turaev invariant of mapping class group explicitly in the case $r = 4$, that is, at the sixteenth root of unity. For each $h \in \Gamma_{g,0}$ we can find the corresponding (colored) ribbon graph, whose Reshetikhin-Turaev invariant turns out to be an automorphism of the 1-dimensional summand of $V^{k_1} \otimes V^{k_1}{}^* \otimes \cdots \otimes V^{k_g} \otimes V^{k_g}{}^*$ which we denote by $V_{r,g}$. We get this ribbon graph using the Heegaard splitting and the surgery theory of 3-manifolds. Wright showed as a result of her calculation that the restriction of this invariant to the Torelli subgroup of $\Gamma_{g,0}$ is equal to the sum of the Birman-Craggs homomorphisms. $\dim(V_{4,g}) = 2^{g-1}(2^g + 1)$, so the Reshetikhin-Turaev invariant of $h \in \Gamma_{g,0}$, when $r = 4$, is a $2^{g-1}(2^g + 1) \times 2^{g-1}(2^g + 1)$ matrix with entries of complex numbers. Wright proved a very interesting lemma that there is a natural one-to-one correspondence between the basis vectors of $V_{4,g}$ and the $\mathbb{Z}/2$-quadratic forms of Arf invariant zero. It would be interesting to check if the Reshetikhin-Turaev representation preserves the braid structure.

1. Mapping class groups and monoidal structure

Let $S_{g,k}$ be an orientable surface of genus g obtained from a closed surface by removing k open disks. The mapping class group $\Gamma_{g,k}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms of $S_{g,k}$ fixing the boundary of $S_{g,k}$ that consists of k disjoint circles. Let $Diff^+(S_{g,k})$ be the group of orientation preserving self-diffeomorphisms of $S_{g,k}$. We also have the following alternative definition :

$$\Gamma_{g,k} = \pi_o Diff^+(S_{g,k})$$

We will mainly deal with the case $k = 1$ and $k = 0$. $\Gamma_{g,1}$ and $\Gamma_{g,0}$ are generated by $2g+1$ Dehn twists(cf.[14]). There is a surjective map $\Gamma_{g,1} \to \Gamma_{g,0}$.

Figure 1. Dehn twists

Many topologists are interested in the homology of mapping class groups. An interesting observation is that there is a product on the disjoint union of $Diff^+(S_{g,1})$'s. It is known by Stasheff([13]) and MacLane([7]) that if a category $\mathcal{C}$ has a monoidal structure then its classifying space gives rise to a space which has the homotopy type of a loop space. Fiedorowicz showed([4]) that a braid structure gives rise to a double loop space structure. We now recall the definition of (strict) braided monoidal category.

Definition 1.1. A *(strict) monoidal (or tensor) category* $(\mathcal{C}, \otimes, E)$ is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the *product*) and an object E (called the *unit object*) satisfying

(a) $\otimes$ is strictly associative
(b) E is a strict 2-sided unit for $\otimes$

Definition 1.2. A monoidal category $(\mathcal{C}, \otimes, E)$ is called a *(strict) braided monoidal category* if there exists a natural commutativity isomorphism $C_{A,B}$: $A\otimes B \to B\otimes A$ satisfying

(c) $C_{A,E} = C_{E,A} = 1_A$
(d) The following diagrams commute:

$$\begin{tikzcd} A\otimes B\otimes C \arrow[rr, "C_{A\otimes B,C}"] \arrow[dr, "1_A\otimes C_{B,C}"'] & & C\otimes A\otimes B \\ & A\otimes C\otimes B \arrow[ur, "C_{A,C}\otimes 1_B"'] & \end{tikzcd}$$

$$\begin{tikzcd} A\otimes B\otimes C \arrow[rr, "C_{A,B\otimes C}"] \arrow[dr, "C_{A,B}\otimes 1_C"'] & & B\otimes C\otimes A \\ & B\otimes A\otimes C \arrow[ur, "1_B\otimes C_{A,C}"'] & \end{tikzcd}$$

The first author recently gave a proof of the following lemma([4]).

Lemma 1.3. *The group completion of the nerve of a braided monoidal category is the homotopy type of a double loop space. The converse is true.*

Miller claimed in [10] that there is an action of the little square operad of disjoint squares in D^2 on the disjoint union of the $B\Gamma_{g,1}$'s extending the F-product that is induced by the connected sum. Here the F-product $\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1}$ is obtained by attaching a pair of *pants* (a surfaces obtained from a sphere by removing three open disks) to the surfaces $S_{g,1}$ and $S_{h,1}$ along the fixed boundary circles and extending the identity map on the boundary to the whole pants. Hence, according to May's recognition theorem on the loop spaces([9]), the group completion of $\amalg_{g\geq 0} B\Gamma_{g,1}$ is homotopy equivalent to a double loop space. Miller's proposition seems correct, although the details are not so transparent. In view of lemma 1.3, the disjoint union of $\Gamma_{g,1}$'s should be related to a braided monoidal category. Here we regard $\amalg_{g\geq 0}\Gamma_{g,1}$ as a category whose objects are $[g]$, $g \in \mathbb{Z}_+$ and morphisms satisfy

$$Hom([g],[h]) = \begin{cases} \Gamma_{g,1} & \text{if } g = h \\ \emptyset & \text{if } g \neq h \end{cases}$$

Without speaking of the action of little square operad, we are going to show that the group completion of $\amalg_{g\geq 0} B\Gamma_{g,1}$ is homotopy equivalent to a double loop space by showing that the disjoint union of $\Gamma_{g,1}$'s is a braiding monoidal category.

Lemma 1.4. *The disjoint union of $\Gamma_{g,1}$'s is a braided monoidal category with the product induced by the F-product.*

Proof. Let $x_1, y_1, \ldots, x_g, y_g$ be generators of the fundamental group of $S_{g,1}$ which are induced by the Dehn twists $a_1, b_1, \cdots, a_g, b_g$, respectively. The mapping class group $\Gamma_{g,1}$ can be identified with the subgroup of the automorphism group of the free group on $x_1, y_1, \cdots, x_g, y_g$ that consists of the automorphisms fixing the fundamental relator $R = [x_1, y_1][x_2, y_2]\cdots[x_g, y_g]$. The binary operation on $\amalg_{g\geq 1}\Gamma_{g,1}$ induced by the F-product can be identified with the operation taking the free product of the automorphisms. The (r, s)-braiding on the free group on $x_1, y_1, \cdots, x_g, y_g$ can be expressed by:

$$\begin{aligned}
x_1 &\longmapsto x_{s+1}\\
y_1 &\longmapsto y_{s+1}\\
&\vdots\\
x_r &\longmapsto x_{s+r}\\
y_r &\longmapsto y_{s+r}\\
x_{r+1} &\longmapsto S^{-1}x_1 S\\
y_{r+1} &\longmapsto S^{-1}y_1 S\\
&\vdots\\
x_{r+s} &\longmapsto S^{-1}x_s S\\
y_{r+s} &\longmapsto S^{-1}y_s S
\end{aligned}$$

where $S = [x_{s+1}, y_{s+1}][x_{s+2}, y_{s+2}]\cdots[x_{s+r}, y_{s+r}]$.
It is easy to see that the (r, s)-braiding fixes the fundamental relator R
Moreover, the (r, s)-braiding makes the diagrams in (d) of Definition 1.2 commute. □

Lemma 1.4 explains the pseudo double loop space structure on the union of the classifying spaces of the mapping class groups observed by E. Miller. Lemma 1.3 and Lemma 1.4 imply the following:

Theorem 1.5. *The group completion of $\amalg_{g\geq 0} B\Gamma_{g,1}$ is the homotopy type of a double loop space.*

2. Braid structure

Let B_n denote Artin's braid group. B_n has $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ and is specified by the following presentation:

$$\begin{aligned}
\sigma_i\sigma_j &= \sigma_j\sigma_j && \text{if } |i-j|\geq 2\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i = 1, \cdots, n-2
\end{aligned}$$

It has been observed for many years that there are certain connections between the braid groups and the mapping class groups. In this section we introduce a new kind of braid structure in the mapping class groups $\Gamma_{g,1}$'s in an explicit form. This explicit expression enables us to deal with a kind of Dyer-Lashof operation (or Browder operation) in an explicit form. It seems possible for us to get further applications of the formula of the braid structure in the future. First let us express explicitly the (1,1)-braiding on genus 2 surface. $\Gamma_{2,1}$ is generated by the Dehn twists $a_1, b_1, a_2, b_2, \omega_1$. Let x_1, y_1, x_2, y_2 be generators of $\pi_1 S_{g,1}$ which are induced by a_1, b_1, a_2, b_2, respectively. Regard $a_1, b_1, a_2, b_2, \omega_1$ as automorphisms on $F_{\{x_1,y_1,x_2,y_2\}}$. Then we have

$$\begin{aligned}
a_1 &: y_1 \longmapsto y_1 x_1^{-1} \\
b_1 &: x_1 \longmapsto x_1 y_1 \\
a_2 &: y_2 \longmapsto y_2 x_2^{-1} \\
b_2 &: x_2 \longmapsto x_2 y_2 \\
\omega_1 &: x_1 \longmapsto x_1 [x_2, y_2] x_2^{-1} x_1 x_2 [y_2, x_2] x_1^{-1} \\
&\quad\; y_1 \longmapsto x_1 [x_2, y_2] x_2^{-1} x_1^{-1} x_2 [y_2, x_2] y_1 x_2 [y_2, x_2] x_1^{-1} \\
&\quad\; y_2 \longmapsto x_2^{-1} x_1 x_2 y_2 x_2^{-1}
\end{aligned}$$

These automorphisms fix the generators that do not appear in the above list.

The (1,1)-braiding in genus 2 should be expressed in terms of the elements $a_1, b_1, a_2, b_2, \omega_1$ and should be specified on the generators of $\pi_1 S_{g,1}$ by the formulas:

$$\begin{aligned}
x_1 &\longmapsto x_2 \\
y_1 &\longmapsto y_2 \\
x_2 &\longmapsto [y_2, x_2] x_1 [x_2, y_2] \\
y_2 &\longmapsto [y_2, x_2] y_1 [x_2, y_2]
\end{aligned}$$

We need a hard calculation to get such a braiding. By using a computer program, we could get the following explicit formula for the braid structure.

Lemma 2.1. *The (1,1)-braiding for the monoidal structure in genus 2 is given by*

$$\beta_1 = (b_1 a_1 a_1 b_1 a_1 \omega_1 (a_1 b_1 a_1)^{-1} b_2 a_2)^{-3} (a_1 b_1 a_1)^4 \tag{2.2}$$

The braid group of all braidings in the mapping class group of genus g is generated by

$$\beta_i = (b_i a_i a_i b_i a_i \omega_i (a_i b_i a_i)^{-1} b_{i+1} a_{i+1})^{-3} (a_i b_i a_i)^4 \tag{2.3}$$

for $i = 1, 2, \cdots, g-1$. We can obtain the following formula for the (r,s)-braiding in terms of the braiding generators:

$$(\beta_r \beta_{r+1} \cdots \beta_{r+s-1})(\beta_{r-1} \beta_r \cdots \beta_{r+s-2}) \cdots (\beta_1 \beta_2 \cdots \beta_s)$$

or alternatively as

$$(\beta_r\beta_{r-1}\cdots\beta_1)(\beta_{r+1}\beta_r\cdots\beta_2)\cdots(\beta_{r+s-1}\beta_{r+s-2}\cdots\beta_s)$$

Remark 2.4 The braid structure gives rise to the double loop space structure, so it is supposed to be related to the Dyer-Lashof operation. Let $D : B_{2g} \longrightarrow \Gamma_{g,1}$ be the obvious map given by

$$D(\sigma_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ \omega_{\frac{i}{2}} & \text{if } i \text{ is even} \end{cases}$$

F. Cohen in [3] dealt with this map D. He said that the homology homomorphism D_* induced by D is trivial, because D preserves the Dyer-Lashof operation. Precisely speaking, he made a commutative diagram

$$\begin{array}{ccc} B_p \int B_{2g} & \xrightarrow{\theta'} & B_{2pg} \\ {\scriptstyle B_p \int D}\Big\downarrow & & \Big\downarrow{\scriptstyle D} \\ B_p \int \Gamma_{g,1} & \xrightarrow{\theta} & \Gamma_{pg,1} \end{array}$$

where θ is the analogue of the Dyer-Lashof operation (it should be rather Browder operation). According to his definition, $(\sigma;1,1) \in B_2 \int \Gamma_{1,1}$ is mapped by θ to $\omega_1 b_2 b_1 \omega_1$. His definition of θ, however, is not well-defined. This can be detected by mapping $\Gamma_{2,1}$ to $Sp(4;\mathbb{Z})$. Here $Sp(4;\mathbb{Z})$ is the automorphism group of $H_1(S_{g,1};\mathbb{Z})$. The map $\phi : \Gamma_{2,1} \to Sp(4;\mathbb{Z})$ is described as follows:

$$a_1 \to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b_1 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_2 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b_2 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\omega_1 \to \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The map ϕ sends $\omega_1 b_2 b_1 \omega_1$ to $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

We have

$$(\sigma_1;1,1)^{-1}(1;a_1,1)(\sigma_1;1,1) = (1;1,a_1) = (\sigma_1;1,1)(1;a_1,1)(\sigma_1;1,1)^{-1}$$

This element must commute with $(1; a_1, 1)$. $(\sigma_1; 1, 1)^{-1}(1; a_1, 1)(\sigma_1; 1, 1)$ is mapped to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $(\sigma_1; 1, 1)(1; a_1, 1)(\sigma_1; 1, 1)^{-1}$ is mapped to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. However neither of these two matrices commutes with $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ which corresponds to a_1.

The braiding structure (2.2) plays a key role in the correct formula for θ which should be the following:

$$(\sigma_1; 1, 1) \stackrel{\theta}{\longmapsto} (b_1 a_1 a_1 b_1 a_1 \omega_1 (a_1 b_1 a_1)^{-1} b_2 a_2)^{-3} (a_1 b_1 a_1)^4$$

Let $\mathcal{C}_n$ be the little n-cube operad. Let Y be an n-fold loop space. Then Y is a $\mathcal{C}_n$-space, so there is a map

$$\mathcal{C}_n(2) \times Y^2 \longrightarrow Y$$

It is known that $\mathcal{C}_n(2)$ has the same homotopy type as S^{n-1}. Hence the above map induces a homology operation

$$H_i(Y) \otimes H_j(Y) \longrightarrow H_{i+j+n-1}(Y)$$

which is called the Browder operation. It is easy to see that if Y is a $\mathcal{C}_{n+1}$-space, then the Browder operation equals zero.

Let X be the group completion of $\amalg_{g \geq 0} B\Gamma_{g,1}$. Since X is homotopy equivalent to a Ω^2-space, it is, up to homotopy, a $\mathcal{C}_2$-space. It is natural to raise the question whether X is a $\mathcal{C}_3$-space, or not. The answer is negative. In the proof of the following theorem the braid formula (2.2) again plays a key role.

Theorem 2.5. *Let X be the group completion of $\amalg_{g \geq 0} B\Gamma_{g,1}$. The double loop space structure cannot be extended to the triple loop space structure.*

Proof. We show that the Browder operation

$$\theta_* : H_i(X) \otimes H_j(X) \longrightarrow H_{i+j+1}(X)$$

is nonzero for X. We have the map

$$\phi : \mathcal{C}_2(2) \times X^2 \longrightarrow X$$

Note that $\mathcal{C}_2(2)$ has the same homotopy type as S^1. By restricting the map ϕ to each connected component we get

$$S^1 \times B\Gamma_{g,1} \times B\Gamma_{g,1} \longrightarrow B\Gamma_{2g,1}$$

This map is, in the group level, denoted by the map

$$\theta : B_2 \int \Gamma_{g,1} \longrightarrow \Gamma_{2g,1}$$

which is same as described in Remark 2.4. In order to show that θ_* is nonzero it suffices to show that

$$\tilde{\theta_*} : H_0(B\Gamma_{1,1}) \otimes H_0(B\Gamma_{1,1}) \longrightarrow H_1(B\Gamma_{2,1})$$

is nonzero. The image of the map $\tilde{\theta_*}$ equals the image of the homology homomorphism $\alpha : H_1(S^1) \to H_1(B\Gamma_{2,1})$ induced by the map $S^1 \to B\Gamma_{2,1}$ which is the restriction of the map $S^1 \times B\Gamma_{1,1} \times B\Gamma_{1,1} \to B\Gamma_{2,1}$. The map α sends the generator of $H_1(S^1)$ to the abelianization class of

$$(b_1a_1a_1b_1a_1\omega_1(a_1b_1a_1)^{-1}b_2a_2)^{-3}(a_1b_1a_1)^4$$

which is nonzero, since the isomorphism $H_1(\;) \cong (\;)_{ab}$ is natural. $\square$

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CONTROLLED TOPOLOGICAL EQUIVALENCE OF MAPS IN THE THEORY OF STRATIFIED SPACES AND APPROXIMATE FIBRATIONS

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Abstract. Ideas from the theory of topological stability of smooth maps are transported into the controlled topological category. For example, the controlled topological equivalence of maps is discussed. These notions are related to the classification of manifold approximate fibrations and manifold stratified approximate fibrations. In turn, these maps form a bundle theory which can be used to describe neighborhoods of strata in topologically stratified spaces.

1. Introduction

We explore some connections among the theories of topological stability of maps, controlled topology, and stratified spaces. The notions of topological equivalence of maps and locally trivial families of maps play an important role in the theory of topological stability of smooth maps. We formulate the analogues of these notions in the controlled topological category for two reasons. First, the notion of controlled topological equivalence of maps is a starting point for formulating a topological version of Mather's theory of the topological stability of smooth maps. Recall that Mather proved that the topologically stable maps are generic for the space of all smooth maps (with the C^∞ topology) between closed smooth manifolds (see Mather [22], Gibson, Wirthmüller, du Plessis, and Looijenga [9]). The hope is to identify an analogous generic class for the space of all maps (with the compact-open topology) between closed topological manifolds. Controlled topology at least gives a place to begin speculations. Second, the controlled analogue of local triviality for families of maps is directly related to the classification of approximate fibrations between manifolds due to Hughes, Taylor and Williams [17], [18]. We elucidate that relation in §8.

Another important topic in the theory of topological stability of smooth maps is that of smoothly stratified spaces (cf. Mather [21]). Quinn [26] initiated the study of topologically stratified spaces and Hughes [12], [13] has shown that 'manifold

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stratified approximate fibrations' form the correct bundle theory for those spaces. The classification of manifold approximate fibrations via controlled topology mentioned above extends to manifold stratified approximate fibrations; hence, we have another connection between controlled topology and stratified spaces. This classification of manifold stratified approximate fibrations is the main new result of this paper.

Two essential tools in stability theory are Thom's two isotopy lemmas [21]. In §9 we formulate an analogue of the first of these lemmas for topologically stratified spaces. A non-proper version is also stated.

It should be noted that in his address to the International Congress in 1986, Quinn predicted that controlled topology would have applications to the topological stability of smooth maps [25]. In particular, controlled topology should be applicable to the problem of characterizing the topologically stable maps among all smooth maps. More recently, Cappell and Shaneson [1] suggested that topologically stratified spaces should play a role in the study of the local and global topological type of topologically smooth maps (the connection is via the mapping cylinder of the smooth map). While the speculations in this paper are related to these suggestions, they differ in that it is suggested here that controlled topology might be used to study a generic class of topological, rather than smooth, maps.

2. Topological equivalence and locally trivial families of maps

We recall some definitions from the theory of topological stability of smooth maps (see Damon [3], du Plessis and Wall [5], Gibson, Wirthmüller, du Plessis, and Looijenga [9], Mather [21], [22]).

Definition 2.1. Two maps $p_0 : X_0 \to Y_0$, $p_1 : X_1 \to Y_1$ are *topologically equivalent* if there exist homeomorphisms $h : X_0 \to X_1$ and $g : Y_0 \to Y_1$ such that $p_1 h = g p_0$, so that there is a commuting diagram:

$$\begin{array}{ccc} X_0 & \xrightarrow{h} & X_1 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_1} \\ Y_0 & \xrightarrow{g} & Y_1. \end{array}$$

Definition 2.2. A smooth map $p_0 : M \to N$ between smooth manifolds is *topologically stable* if there exists a neighborhood V of p_0 in the space of all smooth maps $C^\infty(M, N)$ such that for all $p_1 \in V$, p_0 is topologically equivalent to p_1.

The space $C^\infty(M, N)$ is given the Whitney C^∞ topology. Thom conjectured and Mather proved that the topologically stable maps are generic in $C^\infty(M, N)$; in fact, they form an open dense subset (see [9], [21], [22]). The proof yields a stronger result, namely that the *strongly* topologically stable maps are dense (see [9]).

Definition 2.3. A smooth map $p_0 : M \to N$ between smooth manifolds is *strongly topologically stable* if there exists a neighborhood V of p_0 in $C^\infty(M, N)$ such that for all $p_1 \in V$, there exists a (*topologically*) *trivial* smooth one-parameter family $p : M \times I \to N$ joining p_0 to p_1. This means there exist continuous families

$\{h_t : M \to M \mid 0 \le t \le 1\}$ and $\{g_t : N \to N \mid 0 \le t \le 1\}$ of homeomorphisms such that $p_0 = g_t^{-1} \circ p_t \circ h_t$ for all $t \in I$, so that there is a commuting diagram:

$$\begin{tikzcd} M \arrow[r, "h_t"] \arrow[d, "p_0"'] & M \arrow[d, "p_t"'] \\ N \arrow[r, "g_t"] & N. \end{tikzcd}$$

The notion of triviality for the one-parameter family of maps in the definition above can be generalized to arbitrary families of maps. We now recall that definition and the related notion of local triviality (cf. [21]).

Definition 2.4. Consider a commuting diagram of spaces and maps:

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B \arrow[r, "\mathrm{id}_B"] & B \end{tikzcd}$$

(1) f is *trivial* over B if there exist spaces F_1 and F_2, a map $q : F_1 \to F_2$ and homeomorphisms $h : E_1 \to F_1 \times B$, $g : E_2 \to F_2 \times B$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[d, "\mathrm{id}_B"'] & E_1 \arrow[l, "p_1"'] \arrow[r, "f"] \arrow[d, "h"'] & E_2 \arrow[r, "p_2"] \arrow[d, "g"'] & B \arrow[d, "\mathrm{id}_B"] \\ B & F_1 \times B \arrow[l, "\mathrm{proj}"'] \arrow[r, "q \times \mathrm{id}_B"] & F_2 \times B \arrow[r, "\mathrm{proj}"] & B \end{tikzcd}$$

(2) f is *locally trivial* over B if for every $x \in B$ there exists an open neighborhood U of x in B such that $f| : p_1^{-1}(U) \to p_2^{-1}(U)$ is trivial over U.
(3) In either case, $q : F_1 \to F_2$ is the *model* of the family f.

Remarks 2.5.

(1) The model $q : F_1 \to F_2$ is well-defined up to topological equivalence.
(2) Both $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are fibre bundle projections with fibre F_1 and F_2, respectively.
(3) For every $x \in B$, $f_x = f| : p_1^{-1}(x) \to p_2^{-1}(x)$ is topologically equivalent to $q : F_1 \to F_2$.
(4) One step in Mather's proof that the topologically stable smooth maps form an open dense subset is to show that certain families of maps are locally trivial. Thom's second isotopy lemma is used for this.

A *fibre preserving map* is a map which preserves the fibres of maps to a given space, usually a k-simplex or an arbitrary space B. Specifically, if $\rho : X \to B$ and $\sigma : Y \to B$ are maps, then a map $f : X \to Y$ is fibre preserving over B if $\sigma f = \rho$.

There is a notion of equivalence for families of maps over B.

Definition 2.6.

(1) Two locally trivial families of maps over B

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_2'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

are *topologically equivalent* provided there exist homeomorphisms

$$h_1 : E_1 \to E_1' \quad \text{and} \quad h_2 : E_2 \to E_2'$$

which are fibre preserving over B and $f'h_1 = h_2 f$; that is, the following diagram commutes:

$$\begin{array}{ccccccc} B & \xleftarrow{p_1} & E_1 & \xrightarrow{f} & E_2 & \xrightarrow{p_2} & B \\ {\scriptstyle \mathrm{id}_B}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\ B & \xleftarrow{p_1'} & E_1' & \xrightarrow{f'} & E_2' & \xrightarrow{p_2'} & B \end{array}$$

(2) Let $\mathcal{A}_1(q, B)$ denote the set of topological equivalence classes of locally trivial families of maps over B with model $q : F_1 \to F_2$.

The set $\mathcal{A}_1(q, B)$ can be interpreted as a set of equivalence classes of certain fibre bundles over B as follows. Let $\mathrm{TOP}(q)$ be the topological group given by the pull-back diagram

$$\begin{array}{ccc} \mathrm{TOP}(q) & \longrightarrow & \mathrm{TOP}(F_2) \\ \downarrow & & \downarrow{\scriptstyle q^\sharp} \\ \mathrm{TOP}(F_1) & \xrightarrow{q_\sharp} & \mathrm{Map}(F_1, F_2) \end{array}$$

where $q_\sharp(h) = q \circ h$ and $q^\sharp(g) = g \circ q$. That is,

$$\mathrm{TOP}(q) = \{(h, g) \in \mathrm{TOP}(F_1) \times \mathrm{TOP}(F_2) \mid qh = gq\}.$$

Note that $\mathrm{TOP}(q)$ is naturally a subgroup of $\mathrm{TOP}(F_1 \amalg F_2)$ via $(h, g) \mapsto h \amalg g$. Let $\mathcal{A}_2(q, B)$ denote the set of bundle equivalence classes of fibre bundles over B with fibre $F_1 \amalg F_2$ and structure group $\mathrm{TOP}(q)$.

Proposition 2.7. *There is a bijection $\alpha : \mathcal{A}_1(q, B) \to \mathcal{A}_2(q, B)$. In particular, if B is a separable metric space, then there is a bijection $\mathcal{A}_1(q, B) \to [B, \mathrm{BTOP}(q)]$.*

The function α is defined by sending a locally trivial family

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

to the fibre bundle $p_1 \amalg p_2 : E_1 \amalg E_2 \to B$ whose total space is the disjoint union of E_1 and E_2. The fact that α is a bijection is fairly straightforward to prove. At any rate, it follows from a more general result in §5 (see Theorem 5.5 and the comments following it).

3. Controlled topological equivalence

We propose a definition of topological equivalence in the setting of controlled topology and use it to make some speculations about generic maps between topological manifolds.

The mapping cylinder of a map $p : X \to Y$ is the space

$$\mathrm{cyl}(p) = (X \times I \amalg Y)/\{(x,1) \sim p(x) \mid x \in X\}.$$

There is a natural map $\pi : \mathrm{cyl}(p) \to I$ defined by

$$\begin{cases} \pi([x,t]) = t, & \text{if } (x,t) \in X \times I \\ \pi([y]) = 1, & \text{if } y \in Y. \end{cases}$$

For clarification the map π will sometimes be denoted $\pi_p : \mathrm{cyl}(p) \to I$. If $p : X \to Y$ and $p' : X' \to Y'$ are maps and $\pi_p : \mathrm{cyl}(p) \to I$ and $\pi_{p'} : \mathrm{cyl}(p') \to I$ are the natural maps, then a homeomorphism $h : \mathrm{cyl}(p) \to \mathrm{cyl}(p')$ is *level* if $\pi_p = \pi_{p'} h$. Let $\mathrm{TOP}^{\mathrm{level}}(p)$ denote the simplicial group of level homeomorphisms from $\mathrm{cyl}(p)$ onto itself. That is, a k–simplex of $\mathrm{TOP}^{\mathrm{level}}(p)$ consists of a Δ^k–parameter family of level homeomorphisms $h : \mathrm{cyl}(p) \times \Delta^k \to \mathrm{cyl}(p) \times \Delta^k$. The group $\mathrm{TOP}(p)$ as defined in the previous section has a simplicial version (the singular complex of the topological group) and, as such, is a simplicial subgroup of $\mathrm{TOP}^{\mathrm{level}}(p)$. For example, a pair of homeomorphisms $(h : X \to X, g : Y \to Y)$ such that $ph = gp$ induces a level homeomorphism

$$\mathrm{cyl}(p) \to \mathrm{cyl}(p), \quad \begin{cases} [x,t] \mapsto [h(x),t], & \text{if } x \in X \\ [y] \mapsto [g(y)], & \text{if } y \in Y \end{cases}$$

Definition 3.1. Two maps $p_0 : X_0 \to Y_0$, $p_1 : X_1 \to Y_1$ are *controlled topologically equivalent* if there exists a level homeomorphism $h : cyl(p_0) \to \mathrm{cyl}(p_1)$.

Note that a level homeomorphism $h : \mathrm{cyl}(p_0) \to \mathrm{cyl}(p_1)$ induces (by restriction) a one-parameter family $h_t : X_0 \to X_1, 0 \le t < 1$, of homeomorphisms and a homeomorphism $g : Y_0 \to Y_1$. If all the spaces involved are compact metric, then

$$gp_0 = \lim_{t \to 1} p_1 h_t$$

and such data is equivalent to having a level homeomorphism (cf. [16], [17], [19], [20]). This formulation should be compared with the formulation of topological equivalence in Definition 2.1.

Definition 3.2. Two maps $p_0 : X_0 \to Y_0$, $p_1 : X_1 \to Y_1$ between compact metric spaces are *weakly controlled topologically equivalent* if there exist continuous families $\{h_t : X_0 \to X_1 \mid 0 \le t < 1\}$ and $\{g_t : Y_0 \to Y_1 \mid 0 \le t < 1\}$ of homeomorphisms such that $p_0 = \lim_{t \to 1} g_t^{-1} \circ p_1 \circ h_t$.

The limit above is taken in the sup metric which is the metric for the compact-open topology. The space $C(X,Y)$ of maps from X to Y is given the compact-open topology.

Definition 3.3. A map $p_0 : X \to Y$ between compact metric spaces is *weakly controlled topologically stable* if there exists a neighborhood V of p_0 in $C(X,Y)$ such that for all $p_1 \in V$ and $\epsilon > 0$, there exists a map $p_1' : X \to Y$ such that p_0 is weakly controlled topologically equivalent to p_1' and p_1' is ϵ–close to p_1.

Many of the results in the theory of singularities have a mixture of smooth and topological hypotheses and conclusions. This is the case in Mather's theorem on the genericness of topologically stable maps among smooth maps. One direction that controlled topology is likely to take is in finding the topological underpinnings in this area. The following speculation is meant to be a step towards formulating what might be true.

Speculation 3.4. *If M and N are closed topological manifolds, then the weakly controlled topologically stable maps from M to N are generic in $C(M,N)$.*

This might be established by showing that the stratified systems of approximate fibrations are dense and also weakly controlled topologically stable (see Hughes [14] and Quinn [27] for stratified systems of approximate fibrations). As evidence for this line of reasoning, note that Chapman's work [2] shows that manifold approximate fibrations are weakly controlled topologically stable.

Another line of speculation concerns polynomial maps between euclidean spaces. It is known that the classification of polynomial maps up to smooth equivalence differs from their classification up to topological equivalence (cf. Thom [32], Fakuda [8], Nakai [23]). What can be said about the classification of polynomial maps up to controlled topological equivalence?

4. Controlled locally trivial families of maps

Analogues in controlled topology of locally trivial families of maps are defined. In fact, we define a moduli space of all such families.

Definition 4.1. Consider a commuting diagram of spaces and maps:

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B \arrow[r, "\mathrm{id}_B"] & B \end{tikzcd}$$

(1) f is *controlled trivial* over B if there exist spaces F_1 and F_2, a map $q : F_1 \to F_2$ and a homeomorphism $H : \mathrm{cyl}(f) \to \mathrm{cyl}(q) \times B$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[d, "\mathrm{id}_B"'] & \mathrm{cyl}(f) \arrow[l, "c"'] \arrow[r, "\pi_f"] \arrow[d, "H"] & I \arrow[d, "\mathrm{id}_I"] \\ B & \mathrm{cyl}(q) \times B \arrow[l, "\mathrm{proj}"'] \arrow[r, "\pi_q'"] & I \end{tikzcd}$$

where $c : \mathrm{cyl}(f) \to B$ is given by

$$\begin{cases} c([x,t]) = p_1(x) = p_2 f(x), & \text{if } (x,t) \in E_1 \times I \\ c([y]) = p_2(y), & \text{if } y \in E_2 \end{cases}$$

and π'_q is the composition $\mathrm{cyl}(q) \times B \xrightarrow{\mathrm{proj}} \mathrm{cyl}(q) \xrightarrow{\pi_q} I$.

(2) f is *controlled locally trivial* over B if for every $x \in B$ there exists an open neighborhood U of x in B such that $f| : p_1^{-1}(U) \to p_2^{-1}(U)$ is controlled trivial over U.

(3) In either case, $q : F_1 \to F_2$ is the *model* of the family f.

Remarks 4.2.

(1) The model $q : F_1 \to F_2$ is well-defined up to controlled topological equivalence.

(2) Both $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are fibre bundle projections with fibre F_1 and F_2, respectively.

(3) For every $x \in B$, $f_x = f| : p_1^{-1}(x) \to p_2^{-1}(x)$ is controlled topologically equivalent to $q : F_1 \to F_2$.

There is a notion of controlled equivalence for families of maps over B.

Definition 4.3.

(1) Two controlled locally trivial families of maps over B

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} E'_1 & \xrightarrow{f'} & E'_2 \\ {\scriptstyle p'_1}\downarrow & & \downarrow{\scriptstyle p'_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

are *controlled topologically equivalent* provided there exists a level homeomorphism

$$H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$$

which is fibre preserving over B in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{cyl}(f) & \xrightarrow{H} & \mathrm{cyl}(f') \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

where c is given by

$$\begin{cases} c([x,t]) = p_2 f(x) = p_1(x), & \text{if } (x,t) \in E_1 \times I \\ c([y]) = p_2(y), & \text{if } y \in E_2 \end{cases}$$

and c' is given by

$$\begin{cases} c'([x,t]) = p'_2 f'(x) = p'_1(x), & \text{if } (x,t) \in E'_1 \times I \\ c'([y]) = p'_2(y), & \text{if } y \in E'_2. \end{cases}$$

(2) Let $\mathcal{B}_1(q, B)$ denote the set of controlled topological equivalence classes of locally trivial families of maps over B with model $q : F_1 \to F_2$.

In the next section we will show that the set $\mathcal{B}_1(q,B)$ can be interpreted as a set of equivalence classes of certain fibre bundles over B in analogy with Proposition 2.7 (see Theorem 5.5). But first we will define the moduli space of all controlled locally trivial families of maps over B with model $q : F_1 \to F_2$. This is done in the setting of simplicial sets as follows.

Define a simplicial set $\mathbf{B}_1(q,B)$ so that a typical k–simplex of $\mathbf{B}_1(q,B)$ consists of a commuting diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B\times\Delta^k & \xrightarrow{\mathrm{id}_{B\times\Delta^k}} & B\times\Delta^k \end{array}$$

which is a controlled locally trivial family of maps over $B\times\Delta^k$ with model $q : F_1 \to F_2$. Thus, a vertex of $\mathbf{B}_1(q,B)$ is a controlled locally trivial family of maps over B with model $q : F_1 \to F_2$. (As in [17], [18] we also need to require that these spaces are reasonably embedded in an ambient universe, but we will ignore that technicality in this paper.) Face and degeneracy operations are induced from those on the standard simplexes. As in [18], this simplicial set satisfies the Kan condition.

Definition 4.4. The *mapping cylinder construction* μ takes a controlled locally trivial family of maps

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

to the mapping cylinder $\mathrm{cyl}(f)$ together with the natural map $\mu(f) : \mathrm{cyl}(f) \to B$.

Note that the controlled locally trivial condition on f means that $\mu(f) : \mathrm{cyl}(f) \to B$ is a fibre bundle with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$ where q is the model of f. If

$$\begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_2'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

is another controlled locally trivial family of maps over B with model q, then to have a controlled topological equivalence $H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$ as in Definition 4.3 means precisely to have a bundle isomorphism from $\mu(f)$ to $\mu(f')$.

Proposition 4.5. *There is a bijection $\pi_0\mathbf{B}_1(q,B) \approx \mathcal{B}_1(q,B)$.*

Proof. In order to see that the natural function $\pi_0\mathbf{B}_1(q,B) \to \mathcal{B}_1(q,B)$ is well-defined, suppose

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B\times\Delta^1 & \xrightarrow{\mathrm{id}_{B\times\Delta^1}} & B\times\Delta^1 \end{array}$$

is a locally trivial family of maps with model $q : F_1 \to F_2$. Then by the remarks above $\mu(f) : \mathrm{cyl}(f) \to B \times \Delta^1$ is a fibre bundle with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$. Thus, there is a bundle isomorphism from the restriction of $\mu(f)$ over $B \times \{0\}$ to the restriction of $\mu(f)$ over $B \times \{1\}$, and the remarks above further show that this isomorphism gives a controlled topological equivalence from

$$\begin{array}{ccc} p_1^{-1}(B\times\{0\}) & \xrightarrow{f|} & p_2^{-1}(B\times\{0\}) \\ {\scriptstyle p_1|}\downarrow & & \downarrow{\scriptstyle p_2|} \\ B\times\{0\} & \xrightarrow{\mathrm{id}_B} & B\times\{0\} \end{array} \quad \text{to} \quad \begin{array}{ccc} p_1^{-1}(B\times\{1\}) & \xrightarrow{f|} & p_2^{-1}(B\times\{1\}) \\ {\scriptstyle p_1|}\downarrow & & \downarrow{\scriptstyle p_2|} \\ B\times\{1\} & \xrightarrow{\mathrm{id}_B} & B\times\{1\} \end{array}$$

showing that the function is well-defined. The function is obviously surjective, so it remains to see that it is injective. To this end suppose that

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_2'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

are controlled topologically equivalent with a level homeomorphism $H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$ as in Definition 4.3. Let $h_0 : E_1 \to E_1'$ and $h_1 : E_2 \to E_2'$ be the restrictions of H to the top and bottom of the mapping cylinders, respectively. Then there is an induced commutative diagram

$$\begin{array}{ccc} \mathrm{cyl}(h_0) & \longrightarrow & \mathrm{cyl}(h_1) \\ \downarrow & & \downarrow \\ B\times\Delta^1 & \longrightarrow & B\times\Delta^1 \end{array}$$

which is a 1–simplex in $\mathbf{B}_1(q,B)$ from f to f'. $\square$

5. Bundles with mapping cylinder fibres

In this section we show that controlled locally trivial families of maps over B can be interpreted as fibre bundles over B with fibre the mapping cylinder of the model. Reduced structure groups are discussed as well as a relative situation in which the target bundle over B is fixed.

Let B be a fixed separable metric space. Let $\mathcal{B}_2(q,B)$ denote the set of bundle equivalence classes of fibre bundles over B with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$. Define $\mathbf{B}_2(q,B)$ to be the simplicial set whose k–simplices are fibre bundles over $B\times\Delta^k$ with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$. The following result is well-known (cf. [17]).

Proposition 5.1. *There are bijections*

$$\pi_0\mathbf{B}_2(q,B) \approx \mathcal{B}_2(q,B) \approx [B, \mathrm{BTOP}^{\mathrm{level}}(q)].$$

The mapping cylinder construction of Definition 4.4 has the following simplicial version.

Definition 5.2. *The mapping cylinder construction* is the simplicial map

$$\mu : \mathbf{B}_1(q, B) \to \mathbf{B}_2(q, B)$$

defined by sending a diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ B \times \Delta^k & \xrightarrow{\mathrm{id}_{B\times\Delta^k}} & B \times \Delta^k \end{array}$$

to $\mathrm{cyl}(f) \to B \times \Delta^k$. Note that the local triviality condition on f implies that $\mathrm{cyl}(f) \to B \times \Delta^k$ is a fibre bundle projection with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$.

The first part of the following result is proved in [18]. The second part follows from the first part together with Propositions 4.5 and 5.1.

Theorem 5.3. *The mapping cylinder construction defines a homotopy equivalence* $\mu : \mathbf{B}_1(q, B) \to \mathbf{B}_2(q, B)$. *In particular,* $\mathcal{B}_1(q, B) \approx \mathcal{B}_2(q, B) \approx [B, \mathrm{BTOP}^{\mathrm{level}}(q)]$.

Reduced structure groups. Let G be a simplicial subgroup of $\mathrm{TOP}^{\mathrm{level}}(q)$. We will now generalize the discussion above to the situation where the structure group is reduced to G.

Definition 5.4. Consider a controlled locally trivial family

$$\begin{array}{ccc} E_1 & \xrightarrow{f_1} & E_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

with model $q : F_1 \to F_2$. Then f is *G–locally trivial* over B provided there exists an open cover $\mathcal{U}$ of B such that f is controlled trivial over U for each $U \in \mathcal{U}$ via a trivializing homeomorphism

$$H_U : \mathrm{cyl}(f| : p_1^{-1}(U) \to p_2^{-1}(U)) \to \mathrm{cyl}(q) \times B.$$

These trivializing homeomorphisms are required to have the property that if $U, V \in \mathcal{U}$ and $x \in U \cap V$, then

$$H_V \circ H_U^{-1}| : \mathrm{cyl}(q) \times \{x\} \to \mathrm{cyl}(q) \times \{x\}$$

is an element of G.

Let $\mathbf{B}_1(q, B, G)$ be the simplicial set whose k–simplices are the G–locally trivial families of maps over $B \times \Delta^k$ with model $q : F_1 \to F_2$. For example,

$$\mathbf{B}_1(q, B, \mathrm{TOP}^{\mathrm{level}}(q)) = \mathbf{B}_1(q, B).$$

Definition 4.3 can be extended in the obvious way to define what it means for two G–locally trivial families to be *G–controlled topologically equivalent* (the homeomorphism H is required to be a family of homeomorphisms in the group G) and $\mathcal{B}_1(q, B, G)$ denotes the set of equivalence classes. In analogy with Proposition 4.5 there is a bijection

$$\pi_0\mathbf{B}_1(q, B, G) \approx \mathcal{B}_1(q, B, G).$$

Likewise $\mathcal{B}_2(q, B, G)$ denotes the set of bundle equivalence classes of fibre bundles over B with fibre $\mathrm{cyl}(q)$ and structure group G, and $\mathbf{B}_2(q, B, G)$ is the simplicial set whose k–simplices are fibre bundles over $B \times \Delta^k$ with fibre $\mathrm{cyl}(q)$ and structure group G. In analogy with Proposition 5.1 there are bijections

$$\pi_0\mathbf{B}_2(q, B, G) \approx \mathcal{B}_2(q, B, G) \approx [B, \mathrm{B}G].$$

Moreover, the proof of Theorem 5.3 can be seen to give a proof of the following result (cf. [18, §2]).

Theorem 5.5. *The mapping cylinder construction defines a homotopy equivalence $\mu : \mathbf{B}_1(q, B, G) \to \mathbf{B}_2(q, B, G)$. In particular, $\mathcal{B}_1(q, B, G) \approx \mathcal{B}_2(q, B, G) \approx [B, \mathrm{B}G]$.*

As an example, consider the group $\mathrm{TOP}(q)$ of §2. It was pointed out at the beginning of §3 that $\mathrm{TOP}(q)$ is naturally a subgroup of $\mathrm{TOP}^{\mathrm{level}}(q)$. Note that $\mathcal{B}_1(q, B, \mathrm{TOP}(q)) = \mathcal{A}_1(q, B)$ and $\mathcal{B}_2(q, B, \mathrm{TOP}(q)) = \mathcal{A}_2(q, B)$, so that Proposition 2.7 follows directly from Theorem 5.5.

Fixed target bundle. There are also relative versions of the preceding results in which the bundle $p_2 : E_2 \to B$ is fixed. For example, $\mathcal{B}_1(q \text{ rel } p_2 : E_2 \to B)$ is the set of controlled locally trivial families of maps of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B. \end{array}$$

Two such families $f : E_1 \to E_2$ and $f' : E_1' \to E_2$ are *controlled topologically equivalent* rel p_2 if the homeomorphism $H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$ of Definition 4.3 is required to be the identity on E_2. There are analogous definitions of the following:

(1) $\mathbf{B}_1(q \text{ rel } p_2 : E_2 \to B)$,
(2) $\mathcal{B}_2(q \text{ rel } p_2 : E_2 \to B)$,
(3) $\mathbf{B}_2(q \text{ rel } p_2 : E_2 \to B)$.

Definition 5.6. The group of *controlled homeomorphisms of* q is the subgroup $\mathrm{TOP}^c(q)$ of $\mathrm{TOP}^{\mathrm{level}}(q)$ consisting of all level homeomorphisms $h : \mathrm{cyl}(q) \times \Delta^k \to \mathrm{cyl}(q) \times \Delta^k$ such that $h|F_2 \times \Delta^k = \mathrm{id}_{F_2 \times \Delta^k}$.

Note that $\mathrm{TOP}^c(q)$ is the kernel of the restriction homomorphism

$$\mathrm{TOP}^{\mathrm{level}}(q) \to \mathrm{TOP}(F_2).$$

Let $\widehat{p_2} : B \to \mathrm{BTOP}(F_2)$ be the classifying map for the bundle p_2. Thus, $\mathcal{B}_2(q \text{ rel } p_2 : E_2 \to B)$ is in one-to-one correspondence with the set of vertical homotopy classes of lifts of $\widehat{p_2} : B \to \mathrm{BTOP}(F_2)$ to $\mathrm{BTOP}^{\mathrm{level}}(q) \to \mathrm{BTOP}(F_2)$:

$$\begin{array}{ccc} & & \mathrm{BTOP}^{\mathrm{level}}(q) \\ & & \downarrow \\ B & \xrightarrow{\widehat{p_2}} & \mathrm{BTOP}(F_2). \end{array}$$

The following result follows from the proofs of the preceding results.

Proposition 5.7.

(1) $\pi_0 \mathbf{B}_1(q \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_1(q \text{ rel } p_2 : E_2 \to B)$,
(2) $\pi_0 \mathbf{B}_2(q \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_2(q \text{ rel } p_2 : E_2 \to B)$,
(3) *the mapping cylinder construction defines a homotopy equivalence*

$$\mu : \mathbf{B}_1(q \text{ rel } p_2 : E_2 \to B) \to \mathbf{B}_2(q \text{ rel } p_2 : E_2 \to B).$$

Reduced structure group and fixed target bundle. There are versions of these relative results when the structure groups are reduced to G as before. The sets and simplicial sets involved are denoted as follows:

(1) $\mathcal{B}_1(q, G \text{ rel } p_2 : E_2 \to B)$,
(2) $\mathbf{B}_1(q, G \text{ rel } p_2 : E_2 \to B)$,
(3) $\mathcal{B}_2(q, G \text{ rel } p_2 : E_2 \to B)$,
(4) $\mathbf{B}_2(q, G \text{ rel } p_2 : E_2 \to B)$.

The following result records the analogous bijections and homotopy equivalences.

Proposition 5.8.

(1) $\pi_0 \mathbf{B}_1(q, G \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_1(q, G \text{ rel } p_2 : E_2 \to B)$,
(2) $\pi_0 \mathbf{B}_2(q, G \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_2(q, G \text{ rel } p_2 : E_2 \to B)$,
(3) *the mapping cylinder construction defines a homotopy equivalence*

$$\mu : \mathbf{B}_1(q, G \text{ rel } p_2 : E_2 \to B) \to \mathbf{B}_2(q, G \text{ rel } p_2 : E_2 \to B).$$

6. Manifold stratified spaces

There are many naturally occurring spaces which are not manifolds but which are composed of manifold pieces, those pieces being called the *strata* of the space. Examples include polyhedra, algebraic varieties, orbit spaces of many group actions on manifolds, and mapping cylinders of maps between manifolds. Quinn [26] has introduced a class of stratified spaces called by him 'manifold homotopically stratified sets' with the objective 'to give a setting for the study of purely topological stratified phenomena' as opposed to the smooth and piecewise linear phenomena previously studied.

Roughly, the stratified spaces of Quinn are spaces X together with a finite filtration by closed subsets

$$X = X^m \supseteq X^{m-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset,$$

such that the strata $X_i = X^i \setminus X^{i-1}$ are manifolds with neighborhoods in $X_i \cup X_k$ (for $k > i$) which have the local homotopy properties of mapping cylinders of fibrations. These spaces include the smoothly stratified spaces of Whitney [35], Thom [31] and Mather [21] (for historical remarks on smoothly stratified spaces see Goresky and MacPherson [10]) as well as the locally conelike stratified spaces of Siebenmann [29] and, hence, orbit spaces of finite groups acting locally linearly on manifolds.

Cappell and Shaneson [1] have shown that mapping cylinders of 'smoothly stratified maps' between smoothly stratified spaces are in this class of topologically stratified spaces even though it is known that such mapping cylinders need not be smoothly stratified (see [1] and [32]). Hence, the stratified spaces of Quinn arise naturally in the category of smoothly stratified spaces. For a comprehensive survey of the classification and applications of stratified spaces, see Weinberger [34].

Smoothly stratified spaces have the property that strata have neighborhoods which are mapping cylinders of fibre bundles, a fact which is often used in arguments involving induction on the number of strata. Such neighborhoods fail to exist in general for Siebenmann's locally conelike stratified spaces. For example, it is known that a (topologically) locally flat submanifold of a topological manifold (which is an example of a locally conelike stratified space with two strata) may fail to have a tubular neighborhood (see Rourke and Sanderson [28]). However, Edwards [6] proved that such submanifolds do have neighborhoods which are mapping cylinders of manifold approximate fibrations (see also [18]). On the other hand, examples of Quinn [24] and Steinberger and West [30] show that strata in orbit spaces of finite groups acting locally linearly on manifolds may fail to have mapping cylinder neighborhoods. In Quinn's general setting, mapping cylinder neighborhoods may fail to exist even locally.

The main result announced in [12] (and restated here in §8) gives an effective substitute for neighborhoods which are mapping cylinders of bundles. Instead of fibre bundles, we use 'manifold stratified approximate fibrations,' and instead of mapping cylinders, we use 'teardrops'. This result should be thought of as a tubular neighborhood theorem for strata in manifold stratified spaces.

We now recall the concepts needed to precisely define the manifold stratified spaces of interest (see [26], [12], [15], [16]). A subset $Y \subseteq X$ is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I, h_1(U) = Y$, and $h((U \setminus Y) \times [0,1)) \subseteq X \setminus Y$.

Define the *homotopy link* of Y in X by

$$\text{holink}(X,Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{iff } t = 0\}.$$

Evaluation at 0 defines a map $q : \text{holink}(X,Y) \to Y$ called *holink evaluation*.

Let $X = X^m \supseteq X^{m-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset$ be a space with a finite filtration by closed subsets. Then X^i is the *i-skeleton* and the difference $X_i = X^i \setminus X^{i-1}$ is called the *i-stratum*.

A subset A of a filtered space X is called a *pure* subset if A is closed and a union of components of strata of X. For example, the skeleta are pure subsets.

The *stratified homotopy link* of Y in X, denoted $\mathrm{holink}_{\mathrm{s}}(X,Y)$ consists of all ω in $\mathrm{holink}(X,Y)$ such that $\omega((0,1])$ lies in a single stratum of X. The stratified homotopy link has a natural filtration with i–skeleton

$$\mathrm{holink}_{\mathrm{s}}(X,Y)^i = \{\omega \in \mathrm{holink}_{\mathrm{s}}(X,Y) \mid \omega(1) \in X^i\}.$$

The holink evaluation (at 0) restricts to a map $q : \mathrm{holink}_{\mathrm{s}}(X,Y) \to Y$.

If X is a filtered space, then a map $f : Z \times A \to X$ is *stratum preserving along* A if for each $z \in Z$, $f(\{z\} \times A)$ lies in a single stratum of X. In particular, a map $f : Z \times I \to X$ is a *stratum preserving homotopy* if f is stratum preserving along I.

Definition 6.1. A filtered space X is a *manifold stratified space* if the following four conditions are satisfied:

(1) **Manifold strata.** X is a locally compact, separable metric space and each stratum X_i is a topological manifold (without boundary).
(2) **Forward tameness.** For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.
(3) **Normal fibrations.** For each $k > i$, the holink evaluation $q : \mathrm{holink}(X_i \cup X_k, X_i) \to X_i$ is a fibration.
(4) **Finite domination.** For each i there exists a closed subset K of the stratified homotopy link $\mathrm{holink}_{\mathrm{s}}(X, X^i)$ such that the holink evaluation map $K \to X^i$ is proper, together with a stratum preserving homotopy

$$h : \mathrm{holink}_{\mathrm{s}}(X, X^i) \times I \to \mathrm{holink}_{\mathrm{s}}(X, X^i)$$

which is also fibre preserving over X^i (i.e., $qh_t = q$ for each $t \in I$) such that $h_0 = \mathrm{id}$ and $h_1(\mathrm{holink}_{\mathrm{s}}(X, X^i)) \subseteq K$.

7. Manifold stratified approximate fibrations

The definition of an approximate fibration (as given in [17]) was generalized in [12] to the stratified setting. Let $X = X^m \supseteq \cdots \supseteq X^0$ and $Y = Y^n \supseteq \cdots \supseteq Y^0$ be filtered spaces and let $p : X \to Y$ be a map (p is not assumed to be stratum preserving). Then p is said to be a *stratified approximate fibration* provided given any space Z and any commuting diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & Y \end{array}$$

where F is a stratum preserving homotopy, there exists a *stratified controlled solution*; i.e., a map $\tilde{F} : Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\tilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\bar{F} : Z \times I \times [0,1] \to Y$ defined by $\bar{F}|Z \times I \times [0,1) = p\tilde{F}$ and $\bar{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous.

A stratified approximate fibration between manifold stratified spaces is a *manifold stratified approximate fibration* if, in addition, it is a proper map (i.e., inverse images of compact sets are compact).

8. Teardrop neighborhoods

Given spaces X, Y and a map $p : X \to Y \times \mathbb{R}$, the *teardrop* of p (see [16]) is the space denoted by $X \cup_p Y$ whose underlying set is the disjoint union $X \amalg Y$ with the minimal topology such that

(1) $X \subseteq X \cup_p Y$ is an open embedding, and
(2) the function $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ defined by

$$c(x) = \begin{cases} p(x), & \text{if } x \in X \\ (x, +\infty), & \text{if } x \in Y. \end{cases}$$

is continuous.

The map c is called *the tubular map of the teardrop* or *the teardrop collapse*. The tubular map terminology comes from the smoothly stratified case (see [4], [21], [33]). This is a generalization of the construction of the open mapping cylinder of a map $g : X \to Y$. Namely, $\overset{\circ}{\mathrm{cyl}}(g)$ is the teardrop $(X \times \mathbb{R}) \cup_{g \times \mathrm{id}} Y$.

Theorem 8.1. *If X and Y are manifold stratified spaces and $p : X \to Y \times \mathbb{R}$ is a manifold stratified approximate fibration, then $X \cup_p Y$ is a manifold stratified space with Y a pure subset.*

In this statement, $Y \times \mathbb{R}$ and $X \cup_p Y$ are given the natural stratifications.

The next result from [12] is a kind of converse to this proposition. First, some more definitions. A subset Y of a space X has a *teardrop neighborhood* if there exist a neighborhood U of Y in X and a map $p : U \setminus Y \to Y \times \mathbb{R}$ such that the natural function $(U \setminus Y) \cup_p Y \to U$ is a homeomorphism. In this case, U is the *teardrop neighborhood* and p is the restriction of the tubular map.

Theorem 8.2 (Teardrop Neighborhood Existence). *Let X be a manifold stratified space such that all components of strata have dimension greater than 4, and let Y be a pure subset. Then Y has a teardrop neighborhood whose tubular map*

$$c : U \to Y \times (-\infty, +\infty]$$

is a manifold stratified approximate fibration.

A complete proof of this result will be given in [13], but special cases are in [15] and [16].

The next result from [12] concerns the classification of neighborhoods of pure subsets of a manifold stratified space. Given a manifold stratified space Y, a *stratified neighborhood* of Y consists of a manifold stratified space containing Y as a pure subset. Two stratified neighborhoods X, X' of Y are *equivalent* if there exist neighborhoods U, U' of Y in X, X', respectively, and a stratum preserving homeomorphism $h : U \to U'$ such that $h|Y = \mathrm{id}$. A *neighborhood germ* of Y is an equivalence class of stratified neighborhoods of Y.

Theorem 8.3 (Neighborhood Germ Classification). *Let Y be a manifold stratified space such that all components of strata have dimension greater than 4. Then the teardrop construction induces a one-to-one correspondence from controlled, stratum preserving homeomorphism classes of manifold stratified approximate fibrations over $Y \times \mathbb{R}$ to neighborhood germs of Y.*

9. Applications of Teardrop Neighborhoods

Teardrop neighborhoods can also be used in conjunction with the geometric theory of manifold approximate fibrations [11] to study the geometric topology of manifold stratified pairs. Examples of results proved using teardrop technology are stated in this section. Details will appear in [13].

Theorem 9.1 (Parametrized Isotopy Extension). *Let X be a manifold stratified space such that all components of strata have dimension greater than 4, let Y be a pure subset of X, let U be a neighborhood of Y in X, and let $h : Y \times \Delta^k \to Y \times \Delta^k$ be a k-parameter stratum preserving isotopy. Then there exists a k-parameter isotopy $\tilde{h} : X \times \Delta^k \to X \times \Delta^k$ extending h and supported on $U \times \Delta^k$.*

This generalizes results of Edwards and Kirby [7], Siebenmann [29] and Quinn [26].

The next result is a topological analogue of Thom's First Isotopy Theorem [31] and can be viewed as a first step towards a topological theory of topological stability.

Theorem 9.2 (First Topological Isotopy). *Let X be a manifold stratified space and let $p : X \to \mathbb{R}^n$ be a map such that*

(i) *p is proper,*
(ii) *for each stratum X_i of X, $p| : X_i \to \mathbb{R}^n$ is a topological submersion,*
(iii) *for each $t \in \mathbb{R}^n$, the filtration of X restricts to a filtration of $p^{-1}(t)$ giving $p^{-1}(t)$ the structure of a manifold stratified space such that all components of strata have dimension greater than 4.*

Then p is a bundle and can be trivialized by a stratum preserving homeomorphism; that is, there exists a stratum preserving homeomorphism $h : p^{-1}(0) \times \mathbb{R}^n \to X$ such that ph is projection.

Here is a non-proper version of the preceding result.

Theorem 9.3 (Non-proper First Topological Isotopy). *Let X be a manifold stratified space and let $p : X \to \mathbb{R}^n$ be a map such that*

(i) *if $\rho : X \to [0, \infty)$ is a proper map and $p' = \rho \times p : X \to \mathbb{R}^n \times [0, \infty)$, then the teardrop $X \cup_{p'} \mathbb{R}^n$ is a manifold stratified space,*
(ii) *for each stratum X_i of X, $p| : X_i \to \mathbb{R}^n$ is a topological submersion,*
(iii) *for each $t \in \mathbb{R}^n$, the filtration of X restricts to a filtration of $p^{-1}(t)$ giving $p^{-1}(t)$ the structure of a manifold stratified space such that all components of strata have dimension greater than 4.*

Then p is a bundle and can be trivialized by a stratum preserving homeomorphism; that is, there exists a stratum preserving homeomorphism $h : p^{-1}(0) \times \mathbb{R}^n \to X$ such that ph is projection.

10. Classifying manifold stratified approximate fibrations

Some applications of teardrop neighborhoods are combined with the material in §5 on bundles with mapping cylinder fibres in order to present a classification of manifold stratified approximate fibrations, at least when the range is a manifold, generalizing the classification of manifold approximate fibrations in [17] and [18].

For notation, let B be a connected i-manifold without boundary and let $q : V \to \mathbb{R}^i$ be a manifold stratified approximate fibration where all components of strata of V have dimension greater than 4. A stratified manifold approximate fibration $p : X \to B$ has *fibre germ* q if there exists an embedding $\mathbb{R}^i \subseteq B$ such that $p| : p^{-1}(\mathbb{R}^i) \to \mathbb{R}^i$ is controlled, stratum preserving homeomorphic to q; that is, there exists a stratum preserving, level homeomorphism $\mathrm{cyl}(q) \to \mathrm{cyl}(p| : p^{-1}(\mathbb{R}^i) \to \mathbb{R}^i)$ where the mapping cylinders have the natural stratifications.

The following result shows that fibre germs are essentially unique. For notation, let $r : \mathbb{R}^i \to \mathbb{R}^i$ be the orientation reversing homeomorphism defined by $r(x_1, x_2, \ldots, x_i) = (-x_1, x_2, \ldots, x_i)$.

Theorem 10.1. *Let $p : X \to B$ be a manifold stratified approximate fibration such that all components of strata have dimension greater than 4. Let $g_k : \mathbb{R}^i \to B$, $k = 1, 2$, be two open embeddings. Then $p| : p^{-1}(g_0(\mathbb{R}^i)) \to g_0(\mathbb{R}^i)$ is controlled, stratum preserving homeomorphic to either $p| : p^{-1}(g_1(\mathbb{R}^i)) \to g_1(\mathbb{R}^i)$ or $p| : p^{-1}(g_1(\mathbb{R}^i)) \to rg_1(\mathbb{R}^i)$.*

Proof. The proof follows that of the corresponding result for manifold approximate fibrations in [17, Cor. 14.6]. The stratified analogues of the straightening phenomena are consequences of the teardrop neighborhood results [12], [13]. The use of Siebenmann's Technical Bundle Theorem is replaced with the non-proper topological version of Thom's First Isotopy Lemma in §9. □

There is a moduli space $\mathrm{MSAF}(B)_q$ of all manifold stratified approximate fibrations over B with fibre germ q. It is defined as a simplicial set with a typical k–simplex given by a map $p : X \to B \times \Delta^k$ such that for each $t \in \Delta^k$, $p| : p^{-1}(t) \to B \times \{t\}$ is a manifold stratified approximate fibration with fibre germ q and there exists a stratum preserving homeomorphism $p^{-1}(0) \times \Delta^k \to X$ which is fibre preserving over Δ^k. (There is also a technical condition giving an embedding in an ambient universe; cf. [17]).

The proof of the next proposition follows that of the corresponding result for manifold approximate fibrations in [17]. The necessary stratified versions of the manifold approximate fibration tools are in [12] and [13] and follow from teardrop technology.

Proposition 10.2. *$\pi_0 \,\mathrm{MSAF}(B)_q$ is in one-to-one correspondence with the set of controlled, stratum preserving homeomorphism classes of stratified manifold approximate fibrations over B with fibre germ q.*

Let $\mathrm{TOP}_s^{\mathrm{level}}(q)$ denote the simplicial group of self homeomorphisms of the mapping cylinder $\mathrm{cyl}(p)$ which preserve the mapping cylinder levels and are stratum preserving with respect to the induced stratification of $\mathrm{cyl}(q)$. Note that there is a restriction homomorphism $\mathrm{TOP}_s^{\mathrm{level}}(q) \to \mathrm{TOP}_i$.

Let $\tau B \to B$ denote the topological tangent bundle of B. Consider τB as an open neighborhood of the diagonal in $B \times B$ so that $\tau B \to B$ is first coordinate projection. As in §5 we can form the simplicial set $\mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B \to B)$ which we denote simply by $\mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B)$.

The *differential*

$$d : \mathrm{MSAF}(B)_q \to \mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B)$$

is a simplicial map whose definition is illustrated on vertices as follows (for higher dimensional simplices, the construction is analogous; cf. [17]). If $p : X \to B$ is a vertex of $\mathrm{MSAF}(B)_q$, then form

$$\mathrm{id}_B \times p : B \times X \to B \times B$$

and let

$$\hat{p} = p| : E = p^{-1}(\tau B) \to \tau B.$$

Thus, there is a commuting diagram

$$\begin{array}{ccc} E & \xrightarrow{\hat{p}} & \tau B \\ \downarrow & & \downarrow \\ B & \xrightarrow{\mathrm{id}_B} & B. \end{array}$$

It follows from the stratified straightening phenomena [13] that the local triviality condition is satisfied, so that the diagram is a vertex of

$$\mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B).$$

Once again the proof of the following result follows that of the corresponding manifold approximate fibration result in [17] using the stratified results of [12] and [13].

Theorem 10.3 (MSAF Classification). *The differential*

$$d : \mathrm{MSAF}(B)_q \to \mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B)$$

is a homotopy equivalence.

Corollary 10.4. *Controlled, stratum preserving homeomorphism classes of stratified manifold approximate fibrations over B with fibre germ q are in one-to-one correspondence with homotopy classes of lifts of the map $\tau : B \to \mathrm{BTOP}_i$ which classifies the tangent bundle of B, to $\mathrm{BTOP}_s^{\mathrm{level}}(q)$:*

$$\begin{array}{ccc} & & \mathrm{BTOP}_s^{\mathrm{level}}(q) \\ & & \downarrow \\ B & \xrightarrow{\tau} & \mathrm{BTOP}_i\,. \end{array}$$

Proof. Combine Theorem 10.3, Proposition 10.2 and Proposition 5.8. $\square$

Finally, observe that Corollary 10.4 can be combined with Theorem 8.3 to give a classification of neighborhood germs of B with fixed local type.

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THE ASYMPTOTIC METHOD IN THE NOVIKOV CONJECTURE

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The famous Hirzebruch signature theorem asserts that the signature of a closed oriented manifold is equal to the integral of the so called L-genus. An immediate corollary of this is the homotopy invariance of $< L(M), [M] >$. The L-genus is a characteristic class of tangent bundles, so the above remark is a non-trivial fact. The problem of higher signatures is a generalization of the above consideration. Namely we investigate whether the higher signatures are homotopy invariants or not. The problem is called the Novikov conjecture. The characteristic numbers are closely related to the fundamental groups of manifolds.

There are at least two proofs of the signature theorem. One is to use the cobordism ring. The other is to use the Atiyah-Singer index theorem. Recall that the signature is equal to the index of the signature operators. The higher signatures are formulated as homotopy invariants of bordism groups of $B\Gamma$. The problem was solved using the Atiyah-Singer index theorem in many partial solutions. Here we have the index-theoretic approach in mind when considering the higher signatures. Roughly speaking, a higher signature is an index for a signature operator with some coefficients. To interpret the number as a generalized signature, one considers homology groups with rational group ring coefficients. By doing surgery on the homology groups, we obtain non degenerate symmetric form $\sigma \in L(\Gamma)$ over the group ring. It is called the Mishchenko-Ranicki symmetric signature. This element is a homotopy invariant of manifolds. Mishchenko introduced Fredholm representations, obtaining a number $\sigma(\mathbb{F})$ from a Fredholm representation $\mathbb{F}$ and σ. On the other hand, one can construct a virtual bundle over $K(\Gamma, 1)$ from a Fredholm representation. By pulling back the bundle through maps from the base manifolds to $K(\Gamma, 1)$, we can make a signature operator with coefficients. Mishchenko discovered the generalized signature theorem which asserts the coincidence of the index of the operators and $\sigma(\mathbb{F})$. Thus a higher signature coming from a Fredholm representation is an oriented homotopy invariant.

In [CGM] the authors showed that all higher signatures come from Fredholm representations for large class of discrete groups, including word hyperbolic groups. They formulated the notion of a proper Lipschitz cohomology class in group cohomology. It corresponds to a Fredholm representation in K-theory. In fact for many discrete groups, any class of group cohomology can be represented by a proper Lipschitz cohomology class. Their method depends on the existence of finite dimensional spaces of $\mathbb{Q}$ type $K(\Gamma, 1)$.

On the other hand for larger classes of discrete groups, we cannot expect existence of such good spaces. In [G], Gromov introduced a very large class of discrete groups, the quasi geodesic bicombing groups. This class is characterized by convexity of the Cayley graph. Hyperbolic groups are contained in the class. For the class we cannot ensure the existence of good spaces as above. Moreover it is unknown whether $H^n(\Gamma;\mathbb{Q})$ is zero for sufficiently large n. To overcome this difficulty, the following is shown in [K]. We realize $K(\Gamma,1)$ by infinite dimensional space and approximate it by a family of finite dimensional spaces. By applying the method of [CGM] for finite dimensional spaces iteratively, it turns out that any cohomology class comes from a Fredholm representation asymptotically. It suffices for the Novikov conjecture because of finite dimensionality of manifolds.

§1 Geometric interpretation

To indicate the geometric features of higher signatures, let us consider signatures of submanifolds(see[G2]). Let Γ be a discrete group, M be a closed manifold and $f : M \to K(\Gamma,1)$ be a smooth map. Let us assume that $K(\Gamma,1)$ is realized by a closed manifold V ($\dim M \geq \dim V$). Then for a regular value $m \in V$, the cobordism class of $W = f^{-1}(m)$ is defined uniquely up to homotopy class of f. Moreover the Poincaré dual class of $[W] \in H_*(M)$ is $f^*([V])$ where $[V] \in H^{\dim V}(V)$ is the fundamental cohomology class. Notice that the normal bundle of W is trivial. Thus

$$\sigma(W) \ = < L(W),[W] > = < L(M),[W] > = < L(M)f^*([V]),[M] > .$$

$\sigma(W)$ is a higher signature of M which we now define as follows.

Definition 1-1. Let M be a closed manifold and Γ be a discrete group. Then a higher signature of M is a characteristic number

$$< L(M)f^*(x),[M] >$$

where $f : M \to K(\Gamma,1)$ is a continuous map and $x \in H^*(\Gamma;\mathbb{Q})$.

It is conjectured that these characteristic numbers are all homotopy invariants.

Let us see another geometric interpretation. Let $F \to X \to M$ be a smooth fiber bundle over M and assume F is $4k$ dimensional. Then the flat bundle induced from the fibration $H \to M$ has a natural involution $*$. Thus H splits as $H = H_+ \oplus H_-$ and by the index theorem for families, it follows

$$\sigma(X) \ = < L(M)ch(H_+ - H_-),[M] > .$$

As a corollary, we see that the right hand side is a homotopy invariant of fiber bundles over M(see[At]).

It is not necessary to construct a fiber bundle corresponding to each higher signature. To induce the homotopy invariance, we only need a flat bundle and an involution over M. From the point of view, Lusztig succeeded in verifying Novikov conjecture for free abelian groups by the analytic method ([L]). Let Y be a compact topological space and X be $2k$ dimensional compact manifold. Let $\rho : Y \times \pi_1(X) \to U(p,q)$ be a family of $U(p,q)$ representations of the fundamental group of X. Then one can construct a vector bundle E over $Y \times X$ which is flat in the X direction. E is admitted a non degenerate hermitian form $<,>$ and E splits as $E = E_+ \oplus E_-$. Using the splitting, we obtain a family of quadratic forms

$$\sigma_y : H^k(X; E) \times H^k(X; E) \to \mathbb{C}.$$

Naturally there corresponds $\sigma(X, \rho) \in K(Y)$ which is homotopy invariant of X. Lusztig discovered the index theorem as follows. Let $\pi : Y \times X \to Y$ be the projection. Then

$$\pi_*(L(X)ch(E_+ - E_-)) \ = \ ch(\sigma(X, \rho)).$$

In particular we can take as Y the representation space of the fundamental group of X. In the special case of the free abelian group $\mathbb{Z}^n$, $U(1)$ the representation space is the dual torus which is topologically isomorphic to the torus T^n. In the case of a single $U(1)$ representation, one can only obtain the signature. However Lusztig found the following. There exist bases $\{a_i\}$ and $\{b_i\}$ of $H^{2*}(T^n; \mathbb{Z})$ and $H^{2n}(\hat{T}^n; \mathbb{Z})$ such that

$$ch(\sigma(X, L)) \ = \ \sum_i < L(X) f^*(a_i), [X] > b_i$$

where $f : X \to T^n$ induces an isomorphism of the fundamental groups. This is enough to verify the Novikov conjecture for free abelian groups. In the case of general noncommutative discrete groups, the representation space will be too complicated and it will be very difficult to apply this method to general noncommutative discrete groups.

§2 Fredholm representation

Mishchenko discovered the infinite dimensional version of the method of flat vector bundles.

Definition 2-1. Let Γ be a discrete group. Then a Fredholm representation of Γ is a set $(H_1, H_2, \rho_1, \rho_2, F)$ where

(1) H_1, H_2 are Hilbert spaces,

(2) $F : H_1 \to H_2$ is a Fredholm map,

(3) $\rho_i : \Gamma \to U(H_i, H_i)$ is a unitary representation such that $\rho_2(\gamma)F - F\rho_1(\gamma)$ is a compact operator for any $\gamma \in \Gamma$.

Using a Fredholm representation, we can construct a virtual bundle over $K(\Gamma, 1)$ as follows. From the condition (3), we can construct an equivariant continuous map $f : E\Gamma \to B(H_1, H_2)$ which satisfies

(1) for some point $x \in E\Gamma$, $f(x) = F$,

(2) for any points $x, y \in E\Gamma$, $f(x) - f(y)$ is a compact operator.

Notice that f is unique up to homotopy. Then the virtual bundle is $(f : E\Gamma \times_\Gamma H_1 \to E\Gamma \times_\Gamma H_2)$ and we write

$$\mu : \{ \text{ Fredholm representations } \} / \text{ homotopy} \to \text{ Virtual bundles over } B\Gamma.$$

Theorem 2-2 (Mishchenko). *Let $f : M \to B\Gamma$ be a continuous map. Then*

$$< L(M)f^*(ch(\mu(F))), [M] >$$

is an oriented homotopy invariant of M.

Let us interpret this theorem as an infinite version of the one of Lusztig. By doing surgery on the homology groups with local coefficient, we have the resulting homology only on the middle dimension. Poincaré duality on the homology gives a symmetric form σ. This is an element of the Wall L-group $L(\Gamma)$ of the fundamental group Γ, represented by a group ring valued nondegenerate symmetric matrix. If there is a unitary representation of Γ, then the matrix can be regarded as an invertible self adjoint operator on an infinite dimensional Hilbert space. The Fredholm operator F of a Fredholm representation decomposes into an operator valued 2 by 2 matrix $\{F_{i,j}\}_{i,j=1,2}$ corresponding to the decomposition of the Hilbert space into positive and negative parts of the self adjoint operators. It turns out that the diagonal parts F_{11} and F_{22} are also Fredholm operators and F_{12}, F_{21} are compact operators. This follows essentially from the almost commutativity of the unitary representations and the Fredholm operator in the definition of Fredholm representation. Thus we obtain a number $\text{index}F_{11} - \text{index}F_{22}$. Mishchenko discovered the generalized signature theorem which asserts the coincidence of this number and the characteristic number of the above theorem. The process is parallel to the signature theorem in the case of the simply connected spaces.

§3 Novikov conjecture for word hyperbolic groups

It is natural to ask how large $ch^*(\mu($ Fredholm representations $))$ is in $H^{2*}(\Gamma;\mathbb{Q})$. By a celebrated work by A.Connes, M.Gromov and H.Moscovici, it is shown that if Γ is hyperbolic, then they occupy in $H^{2*}(\Gamma;\mathbb{Q})$.

In some cases of discrete groups, Eilenberg-Maclane spaces are realized by (compact) smooth manifolds. In particular compact negatively curved manifolds themselves are Eilenberg-Maclane spaces. Hyperbolic groups are introduced by Gromov. The class is characterized by the essential properties which are possessed by the fundamental groups of compact negatively curved manifolds. Though the class is very large, they have reasonable classifying spaces which are enough to work instead of Eilenberg-Maclane spaces, at least for the Novikov conjecture. The spaces are called Rips complexes.

Fact 3-1. Let Γ be a discrete group. Then there exists a family of finite dimensional simplicial complexes $\{P_n(\Gamma)\}_{1\leq n}$. They satisfy the following:

(1) Γ acts on each $P_n(\Gamma)$ proper discontinuously with compact quotient,
(2) if Γ is torsion free, then the action is also free,
(3) $P_1(\Gamma) \subset \cdots \subset P_n(\Gamma) \subset P_{n+1}(\Gamma) \dots$,
(4) if Γ is hyperbolic, then $P_n(\Gamma)$ is contractible for sufficiently large n.

In particular, torsion free hyperbolic groups have $B\Gamma$ represented by finite dimensional simplicial complexes. In the following, we shall write $\tilde{P}_n/\Gamma$ as a tubular neighborhood in an embedding $P_n(\Gamma)/\Gamma \to \mathbb{R}^N$. $\tilde{P}_n/\Gamma$ is an open manifold with the induced metric from $\mathbb{R}^N$. In the following, Γ is a hyperbolic group.

Kasparov KK-groups.

Before explaining the method of [CGM], we shall quickly review Kasparov's KK-theory. The KK-groups are used effectively to prove Novikov conjecture. KK is a bifunctor from a pair of distinct spaces (X, Y) to abelian groups which is covariant on X and contravariant on Y. The KK-groups include both K-cohomology and K-homology.

Roughly speaking K-homology consists of the set of Dirac operators on spaces. Precisely an element of $K_0(X)$ is represented by (M, E, φ) where

(1) M is an even dimensional spin^c manifold which need not be compact or connected,

(2) E is a complex vector bundle over M,

(3) φ is a proper map from M to X.

$K_0(X)$ is the set of the above triples quotiented by a certain equivalence relation. It is dual to K-cohomology and the pairing is to take the index on twisted vector bundles. Let S be the spin^c vector bundle over M and $D_E : S_+ \otimes E \to S_- \otimes E$ be the Dirac operator on M. Then the pairing of K theory is $< F, (M, E, \varphi) >=$ index$D_{E \otimes F}$.

Fact 3-2. There exists a Chern character isomorphism,

$$ch_* : K_0(X) \otimes \mathbb{Q} = H_{2*}^{\inf}(X; \mathbb{Q})$$

by $\varphi_*(ch^*(E) \cup td(M) \cap [M])$ where $H_*^{\inf}$ is the homology with locally finite infinite support.

Roughly speaking $KK(X, Y)$ is the set of sections over a family of elements of $K_0(X)$ over Y. Thus if Y is a point,

$$KK_*(X, \mathrm{pt}) = K_*(X).$$

There is an analytical interpretation of topological K-homology. Let $C_0(X)$ be the set of the continuous functions on X vanishing at infinity. $C_0(X)$ is C^* algebra whose C^* norm is to take pointwise supremum. The analytical K-homology $\hat{K}(X)$ is the set $(H_0 \oplus H_1, \rho_0, \rho_1, T)$ quotiented by an equivalence relation, where

(1) H_i is a Hilbert space,

(2) $\rho_i : C_0(X) \to B(H_i)$ is a $*-$homomorphism,

(3) $T : H_0 \to H_1$ is a bounded operator such that $I - T^*T$, $I - TT^*$, $\rho_1(a)T - T\rho_0(a)$, are all compact operators.

The explicit map $K_0(X) \to \hat{K}_0(X)$ is to take L^2 sections of twisted spin^c vector bundles, $(L^2(M, S\hat{\otimes}E), D_E, \varphi)$. Though D_E is an unbounded operator, by making pseudo differential calculus, we can construct a bounded operator. If M is compact, then it is $D_E(I + D_E^* D_E)^{-\frac{1}{2}}$. As φ is a proper map, it pulls back $C_0(X)$ to $C_0(M)$ and the $*-$homomorphism is the multiplication by $\varphi^*(a)$, $a \in C_0(X)$. If X is a point, then an element of $\hat{K}_0(X)$ is represented by a Fredholm operator over Hilbert spaces. $K_0($ pt $)$ is naturally isomorphic to $\mathbb{Z}$ by taking Fredholm indices. $KK($ pt $, Y)$ is a family of Fredholm operators over Y. Thus

$$KK_*(\mathrm{pt}, Y) = K^*(Y).$$

Now let us define the KK-groups. First, let us recall the definition of the analytical K homology $(1),(2),(3)$ and consider the family version. (1) The set of sections over the family of Hilbert spaces over Y admits a natural $C_0(Y)$-module structure. (2) As the $*-$homomorphism ρ_i action is fiberwise, it commutes with that of $C_0(Y)$. (3) A family of compact operators will be formulated as an element of a norm closure of finite rank projections in the set of endomorphisms of the $C_0(Y)$-module. Soon we define this precisely.

Let us consider the triple (E, ϕ, F) where

(1) E is a $\mathbb{Z}_2$-graded right $C_0(Y)$-module with a $C_0(Y)$ valued inner product. It is complete with respect to C^* norm of $C_0(Y)$. E is called a Hilbert module over $C_0(Y)$.

(2) ϕ is a degree 0 $*-$homomorphism from $C_0(X)$ to $B(E)$ where $B(E)$ is the set of $C_0(Y)$-module endomorphisms. $C_0(X)$ acts on E from the left.

(3) $F \in B(E)$ is of degree 1 such that $[F-F^*]\phi(a)$, $[\phi(a), F]$ and $(F^2-1)\phi(a)$ are all compact endomorphisms. A compact endomorphism is an element of $B(E)$ which lies in the closure of linear span of the rank one projections $\theta_{x,y} \in B(E)$, $\theta_{x,y}(z) = x < y, z >$. We denote the set of compact endomorphisms by $K(E)$

If $C_0(Y)$ itself is considered as $C_0(Y)$-module, then $B(C_0(Y))$ is the set of bounded continuous functions on Y. $K(C_0(Y))$ is also $C_0(Y)$.

Let us denote the set of the above triples (E, ϕ, F) by $E(X, Y)$. Notice that we can replace $C_0(X)$ and $C_0(Y)$ by any C^*-algebras A, B and write $E(A, B)$ for the set of triples which satisfy the above (1), (2), (3) replacing $C_0(X)$ by A and $C_0(Y)$ by B.

Now let us introduce a homotopy equivalence relation as follows. (E_1, ϕ_1, F_1) is equivalent to (E_2, ϕ_2, F_2) if there exists $(E, \phi, F) \in E(A, C([0,1], B))$ such that $(E \hat{\otimes}_{f_i} B, f_i \circ \phi, (f_i)_* F)$ is isomorphic to (E_i, ϕ_i, F_i) where $f_i : C([0,1], B) \to B$ is the evaluation maps.

Definition 3-3. $KK(X, Y) = E(X, Y)/$ homotopy .

It turns out that $KK(X, Y)$ is a group. $KK(A, B)$ is defined similarly.

Notice that $KK(\text{ pt }, \mathbb{R}^n)$ is isomorphic to the K-homology of $\mathbb{R}^n$ with compact support which is isomorphic to $\mathbb{Z}$. The generator of $KK(\text{ pt }, \mathbb{R}^n)$ is expressed using Clifford algebra. Let $n = 2k$ be even. Then by identifying $\mathbb{R}^n$ with $\mathbb{C}^k$, any vector in $\mathbb{R}^n$ acts on $\wedge \mathbb{C}^k$ by Clifford multiplication. Then the generator is

$$\{C_0(\mathbb{R}^n, \wedge \mathbb{C}^k), F(x) = \frac{x}{1+|x|}\}$$

in $KK(\text{ pt }, \mathbb{R}^n)$.

There is also equivariant KK-theory. Let A and B admit automorphisms of Γ. If X and Y are Γ spaces, then $C_0(X)$ and $C_0(Y)$ have natural Γ actions. Let $E_\Gamma(A, B)$ be the set of triples $(E, \phi, F) \in E(A, B)$ such that there exists an action of Γ on E which satisfy

(1) $g(a\zeta b) = (ga)(g\zeta)(gb)$, $< g\zeta, g\zeta' >= g < \zeta, \zeta' >$

(2) $\phi(a)(gFg^{-1} - F)$ is a compact endomorphism of E.

Homotopy equivalence is defined similarly.

Definition 3-4. $KK_\Gamma(X,Y) = E_\Gamma(X,Y)/$ homotopy .

Notice that $KK_\Gamma(\mathrm{pt},\mathrm{pt})$ is the set of Fredholm representations quotiented by homotopy equivalence.

There is a very important operation in KK-theory, called the intersection product pairing (see [Bl])

$$KK_\Gamma(X,Y) \times KK_\Gamma(Y,Z) \to KK_\Gamma(X,Z).$$

Lipschitz geometry.

In the essential step, [CGM] constructs an element $\varphi \in KK_\Gamma(\mathrm{pt},\tilde{P}_n)$. Roughly speaking, the construction is as follows.

First of all, using the contractibility of $\tilde{P}_n$, one constructs a map which induces Poincaré duality,

$$\alpha : \tilde{P}_n \times_\Gamma \tilde{P}_n \to T\tilde{P}_n/\Gamma.$$

Namely for $\beta \in H^*(\tilde{P}_n/\Gamma;\mathbb{Q})$, $z \in H_*^{\mathrm{inf}}(\tilde{P}_n/\Gamma;\mathbb{Q})$,

$$\alpha\cap : H^{\mathrm{inf}}(\tilde{P}_n/\Gamma;\mathbb{Q}) \to H^*(P_n/\Gamma;\mathbb{Q})$$
$$\alpha\cap(z) = \int_{\tilde{z}\times_\Gamma\tilde{\beta}} \alpha.$$

Proposition 3-5[CGM]. *If α is fiberwise proper Lipschitz, then one can construct φ.*

Let $\alpha : \tilde{P}_n \times_\Gamma \tilde{P}_n \to T\tilde{P}_n/\Gamma = \tilde{P}_n/\Gamma \times \mathbb{R}^N$ be the fiberwise proper Lipschitz map which induces Poincaré duality. Let us take $e \in \tilde{P}_n$ and restrict α on $\tilde{P}_n \times e$. Then

$$\varphi = \{C_0(\tilde{P}_n, \wedge\mathbb{C}^K), \frac{\alpha(x)}{1+|\alpha(x)|}\}$$

in $KK_\Gamma($ pt $,\tilde{P}_n)$ is the desired one. If α is not fiberwise Lipschitz, then the above φ does not define an element of the equivariant KK-group. To ensure $\gamma F\gamma^{-1} - F$ is a compact endomorphism, it is enough to see that $|\gamma F(x)\gamma^{-1} - F(x)|$ goes to zero when x goes to infinity. This follows, by simple calculation, from the Lipschitzness of α.

A priori, we only have a fiberwise proper map which induces Poincaré duality. It is natural to try to deform the map so that it becomes fiberwise proper Lipschitz by a proper homotopy. To do so, first using the hyperbolicity, we have the following map.

Proposition 3-6. *Let us take a sufficiently large $n > 0$ and a sufficiently small constant $0 < \mu < 1$. The there exists a map*

$$F : \tilde{P}_n \times_\Gamma \tilde{P}_n \to \tilde{P}_n \times_\Gamma \tilde{P}_n$$

such that

(1) F is fiberwise μ Lipschitz,

(2) there exists a fiberwise proper homotopy F_t which connects F to the identity.

Remark 3-7. The existence of such F implies that $\tilde{P}_n$ must be contractible.

Let us take sufficiently large r and put $D = \{(x,y) \in \tilde{P}_n \times_\Gamma \tilde{P}_n; d(x,y) \leq r\}$. By modifying α slightly, we may assume that $\mu^{-1}\alpha \circ F|_{\partial F^{-1}(D)} = \alpha|_{\partial F^{-1}(D)}$.

Let us put

$$B_i = \{(x,y) \in \tilde{P}_n \times_\Gamma \tilde{P}_n; F^i(x,y) = F \circ F \dots F(x,y) \in D\}$$

$$D_i = B_i - B_{i-1}.$$

Let us define

$$\alpha_\infty : \tilde{P}_n \times_\Gamma \tilde{P}_n \to T\tilde{P}_n$$

$$\alpha_\infty|_{D_i} = \mu^{-i+1}\alpha \circ F^{i-1}.$$

It is not difficult to see that α_∞ is fiberwise proper Lipschitz and it is fiberwise proper homotopic to α.

Using the Kasparov intersection product, we have a map

$$\varphi : KK_\Gamma(\tilde{P}_n, \text{ pt }) \to KK_\Gamma(\text{ pt }, \text{ pt })$$

$$\varphi(x) = \varphi \times x.$$

Theorem 3-8[CGM]. *There exists the following commutative diagram.*

$$\begin{array}{ccc} KK_\Gamma(\tilde{P}_n(\pi), \mathbb{Q}) & \xrightarrow{\phi \circ \mu} & K^{2*}(B\Gamma) \\ \Big\downarrow{ch_*} & & \Big\downarrow{ch^*} \\ H^{inf}_{2*}(\tilde{P}_n(\Gamma)/\Gamma) & \xrightarrow{PD} & H^{2*}(B\Gamma) \end{array}$$

where PD is the Poincare duality.

In the case of cohomology groups of odd degrees, we can reduce to the case of even ones by considering $\mathbb{Z} \times \Gamma$. Thus

Corollary 3-9. *Let Γ be a hyperbolic group and $f : M \to K(\Gamma, 1)$ be a continuous map. Then $< L(M)f^*(x), [M] >$ is an oriented homotopy invariant for any $x \in H^*(\Gamma; \mathbb{Q})$. Namely let $p : M_1 \to M_2$ be an oriented homotopy equivalence. Then*

$$< L(M_1)(p \circ f)^*(x), [M_1] > = < L(M_2)f^*(x), [M_2] > .$$

Notice that in the case of hyperbolic groups, we have used the fact that $K(\Gamma, 1)$ was realized by a finite dimensional simplicial complex over $\mathbb{Q}$. On the other hand, we cannot expect it on more large classes of discrete groups, in particular quasi geodesic bicombing groups which we shall treat in the next section. For the class, we cannot expect even that the ranks of cohomology over $\mathbb{Q}$ are finite.

§4 Quasi geodesic bicombing groups

In [ECHLPT], a very large class of discrete groups is defined. The elements of the class are called combing groups. It contains hyperbolic groups and quasi geodesic bicombing groups defined later.

Theorem 4-1[ECHLPT]. *If Γ is a combable group, then $K(\Gamma, 1)$ space can be realized by a CW complex such that the number of cells in each dimension is finite.*

As an immediate corollary of this, we can see that $\dim H^n(\Gamma; \mathbb{Q}) < \infty$ for each n. Using this fact, in the following construction, we shall make an analogy of the case of hyperbolic groups on spaces which approximates $K(\Gamma, 1)$.

A set of generators of a discrete group determines a 1 dimensional simplicial complex called Cayley graph $G(\Gamma)$. $G(\Gamma)$ has a natural metric. Notice that the universal covering spaces of non positively curved manifolds have the convex property. With this in mind, we shall define the following.

Definition 4-2[G1]. If Γ has the following properties, we call it a bicombing group. Let us fix a generating set of Γ. Then there exists a continuous and Γ equivariant map
$$S : \Gamma \times \Gamma \times [0, 1] \to G(\Gamma)$$
such that for some $k \geq 1, C \geq 0$, it satisfies
$$S(\gamma_1, \gamma_2, 0) = \gamma_1, \quad S(\gamma_1, \gamma_2, 1) = \gamma_2,$$
$$d(S_t(\gamma_1, \gamma_2), S_t(\gamma_1', \gamma_2')) \leq k(td(\gamma_2, \gamma_2') + (1-t)d(\gamma_1, \gamma_1')) + C.$$

Though $S(\gamma_1, \gamma_2, \quad) : [0, 1] \to G(\pi)$ connects γ_1 and γ_2, we shall require balanced curves.

Definition 4-3[G1]. Let Γ be bicombing. We say that Γ is bounded if for some $k \geq 1, C \geq 0$, it satisfies
$$d(S_t(\gamma_1, \gamma_2), S_{t'}(\gamma_1, \gamma_2)) \leq k|t - t'|d(\gamma_1, \gamma_2) + C$$

Definition 4-4. Γ : bounded bicombing is quasi geodesic if for every γ, a sufficiently small ϵ and $0 \leq t < s < t + \epsilon \leq 1$, $S(e, \gamma, t) \neq S(e, \gamma, s)$. Moreover let us denote a unit speed path of $S(e, \gamma, \quad)$ by ω_γ.

$\omega_\gamma : [0, |S(e, \gamma, \quad)|] \to G(\Gamma)$. Then for $d(\gamma_1, \gamma_2) \leq 1$,
$$Ud(\omega_{\gamma_1}, \omega_{\gamma_2}) = sup_t d(\omega_{\gamma_1}(t), \omega_{\gamma_2}(t)) \leq C,$$
$$|S(e, \gamma, \quad)| \leq k|\gamma| + C$$
Furthermore, for some $A > 0, B \geq 0$, S_t satisfies
$$d(\gamma, S_t(\gamma, \gamma')) \geq Atd(\gamma, \gamma') - B.$$

Using S, it is easy to prove the following lemma.

Lemma 4-5[Al]. *If Γ is a quasi geodesic bicombing group, then each Rips complex $P_i(\Gamma)$ is contractible in $P_n(\Gamma)$ for large $n = n(i)$.*

From this, we see that for a torsion free quasi geodesic bicombing group Γ, $P_\infty(\Gamma) = lim_i P_i(\Gamma)$ is a realization of $E\Gamma$. Unlike to the case of hyperbolic groups, we cannot make $F : \tilde{P}_n \times_\Gamma \tilde{P}_n(\Gamma) \to \tilde{P}_n \times_\Gamma \tilde{P}_n(\Gamma)$ as before, because P_n is not contractible in itself. However we have the following family of maps.

Proposition 4-6. *Let us take an arbitrary family of small constants $\{\mu_i\}_{0\leq i}$, $1 >> \dots \mu_i >> \mu_{i-1} \cdots >> \mu_0 > 0$. Then for some family of Rips complexes $\{P_{n(i)}\}_{0\leq i}$, there exists a family of maps*

$$F_i : \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(i)} \to \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(i+1)}$$

such that
(1) F_i is fiberwise μ_i Lipschitz,
(2) F_i is fiberwise proper homotopic to the inclusion.

Let $\alpha_0 : \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(0)} \to \mathbb{R}^N$ be a fiberwise proper map which induces Poincaré duality. Using the above family of maps, we can construct the following commutative diagram of maps.

$$\begin{array}{ccc}
\tilde{P}_0 \times_\Gamma \tilde{P}_0 & \xrightarrow{\alpha_0} & \mathbb{R}^{N_0} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\tilde{P}_0 \times_\Gamma \tilde{P}_1 & \xrightarrow{\alpha_1} & \mathbb{R}^{N_1} \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle incl} \\
\dots & \longrightarrow & \dots
\end{array}$$

To produce a proper Lipschitz map, we need to control growth of these maps at infinity. In this case, we can construct α_i which satisfy the following. There exist families of constants $\{C_i\}$, $\{a_i\}$ such that the Lipschitz constant of α_i on $N_i(r) = \{(x,y)|x \in \tilde{P}_0, y \in \tilde{P}_i, d_i(x,y) \leq r\}$, for sufficiently large r, is bounded by $C_i H(a_i r)$ where H is a Lipschitz function on $[0,\infty)$.

Proposition 4-7. *Using these maps, we have*

$$\alpha_\infty : \tilde{P}_{n(0)} \times_\pi \tilde{P}_{n(0)} \to \mathbb{R}^\infty$$

which is fiberwise proper homotopic to α. Moreover let

$$pr : \mathbb{R}^\infty \to \mathbb{R}^N, \qquad N = dim\tilde{P}_{n(0)}.$$

Then $pr \circ \alpha_\infty$ is fiberwise proper Lipschitz.

Let us recall that homology commutes with spaces under the direct limit operation. Thus $H_*(P_\infty/\Gamma) = lim_n H_*(P_n/\Gamma)$. With the fact that the rank of $H_N(P_n/\Gamma)$ is finite for every N, we have

Lemma 4-8. *Let us take any large N. Then for $* < N$, there exists n such that*

$$i_* : H_*(P_n/\Gamma;\mathbb{Q}) \to H_*(P_\infty/\Gamma;\mathbb{Q})$$

is surjective where $i : P_n(\pi)/\pi \to P_\infty(\pi)/\pi$ is the inclusion.

As before, we can construct $\varphi \in KK_\Gamma(\text{ pt }, \tilde{P}_n)$.

Theorem 4-9. *The following diagram commutes.*

$$\begin{array}{ccc} KK_\Gamma(\tilde{P}_n, \ pt\ ) & \xrightarrow{\mu\circ\psi} & K^{2*}(B\Gamma) \\ \Big\downarrow{\scriptstyle ch_*} & & \Big\downarrow{\scriptstyle i^*\circ ch^*} \\ H_{2*}^{inf}(\tilde{P}_n/\Gamma) & \xrightarrow{PD} & H^{2*}(P_n/\Gamma). \end{array}$$

We cannot construct the homotopy between pr$\circ\alpha_\infty$ and α_0 through the map to $\mathbb{R}^N$. Let $2K = N$ and $Z = \tilde{P}_{n(0)} \times P_{n(0)}$. To show the commutativity of the diagram, we construct a homotopy between the following two elements in $KK(\text{ pt }, Z)$.

$$\alpha_0^*(\beta) = \{C_0(Z, \wedge\mathbb{C}^K), F_0(x) = \frac{\alpha_0(x)}{1+|\alpha_0(x)|}\}$$
$$(\text{ pr } \circ\alpha_\infty)^*(\beta) = \{C_0(Z, \wedge\mathbb{C}^K), F_\infty(x) = \frac{\text{pr}\circ\alpha_\infty(x)}{1+|\text{pr}\circ\alpha_\infty(x)|}\}.$$

Let $\Delta_N = \wedge\mathbb{C}^N$. Then there are natural inclusions $\Delta_N \subset \Delta_{N+1} \subset \dots$ which preserves the metrics. Let $[\wedge C^\infty]$ be the infinite dimensional Hilbert space which is the completion of the union. By adding degenerate elements, we can express

$$\alpha_0^*(\beta) = \{C_0(Z, [\wedge\mathbb{C}^\infty]), F_0 \oplus G_0\}$$
$$(\text{ pr } \alpha_\infty)^*(\beta) = \{C_0(Z, [\wedge\mathbb{C}^\infty]), F_\infty \oplus G_\infty\}$$

Using the proper homotopy between α_0 and pr$\circ\alpha_\infty$ through maps to $\mathbb{R}^\infty$, we can construct the homotopy between the elements in $KK(\text{ pt }, Z)$.

Corollary 4-10. *Let Γ be a torsion free quasi geodesic bicombing group. Then for arbitrary large N and $x \in H^{2*}(B\Gamma;\mathbb{Q})$, there exists a Fredholm representation F such that $x - ch(\mu(F)) \in H^*(B\Gamma;\mathbb{Q})$, $* \geq N$.*

Corollary 4-11. *For torsion free quasi geodesic bicombing groups, the higher signatures are oriented homotopy invariants.*

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A NOTE ON EXPONENTIALLY NASH G MANIFOLDS AND VECTOR BUNDLES

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1. Introduction.

Nash manifolds have been studied for a long time and there are many brilliant works (e.g. [2], [3], [10], [19], [20], [21], [22], [23]).

The semialgebraic subsets of $\mathbb{R}^n$ are just the subsets of $\mathbb{R}^n$ definable in the standard structure $\mathbf{R}_{stan} := (\mathbb{R}, <, +, \cdot, 0, 1)$ of the field $\mathbb{R}$ of real numbers [24]. However any non-polynomially bounded function is not definable in $\mathbf{R}_{stan}$, where a polynomially bounded function means a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ admitting an integer $N \in \mathbb{N}$ and a real number $x_0 \in \mathbb{R}$ with $|f(x)| \leq x^N$, $x > x_0$. C. Miller [17] proved that if there exists a non-polynomially bounded function definable in an o-minimal expansion $(\mathbb{R}, <, +, \cdot, 0, 1, ...)$ of $\mathbf{R}_{stan}$, then the exponential function $exp : \mathbb{R} \longrightarrow \mathbb{R}$ is definable in this structure. Hence $\mathbf{R}_{exp} := (\mathbb{R}, <, +, \cdot, exp, 0, 1)$ is a natural expansion of $\mathbf{R}_{stan}$. There are a number of results on $\mathbf{R}_{exp}$ (e.g. [11], [12], [13], [14], [26]). Note that there are other structures with properties similar to those of $\mathbf{R}_{exp}$ ([5], [6], [25]).

We say that a C^r manifold $(0 \leq r \leq \omega)$ is *an exponentially C^r Nash manifold* if it is definable in $\mathbf{R}_{exp}$ (See Definition 2.5). Equivariant such manifolds are defined in a similar way (See Definition 2.6).

In this note we are concerned with exponentially C^r Nash manifolds and equivariant exponentially C^r Nash manifolds.

Theorem 1.1. *Any compact exponentially C^r Nash manifold* $(0 \leq r < \infty)$ *admits an exponentially C^r Nash imbedding into some Euclidean space.*

Note that there exists an exponentially C^ω Nash manifold which does not admit any exponentially C^ω imbedding into any Euclidean space [8]. Hence an exponentially C^ω Nash manifold is called *affine* if it admits an exponentially C^ω Nash imbedding into some Euclidean space (See Definition 2.5). In the usual Nash category, Theorem 1.1 is a fundamental theorem and it holds true without assuming compactness of the Nash manifold [19].

Equivariant exponentially Nash vector bundles are defined as well as Nash ones (See Definition 2.8).

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Theorem 1.2. *Let G be a compact affine Nash group and let X be a compact affine exponentially C^ω Nash G manifold with $\dim X^G \geq 2$. Then for any $C^\infty G$ vector bundle η of positive rank over X, there exist two exponentially C^ω Nash G vector bundle structures of η such that they are exponentially C^∞ Nash G vector bundle isomorphic but not exponentially C^ω Nash G vector bundle isomorphic.*

Theorem 1.3. *Let G be a compact affine exponentially Nash group and let X be a compact $C^\infty G$ manifold. If $\dim X \geq 3$ and $\dim X^G \geq 2$, then X admits two exponentially C^ω Nash G manifold structures which are exponentially C^∞ Nash G diffeomorphic but not exponentially C^ω Nash G diffeomorphic.*

In the usual equivariant Nash category, any C^∞ Nash G vector bundle isomorphism is a C^ω Nash G one, and moreover every C^∞ Nash G diffeomorphism is a C^ω Nash G one. Note that Nash structures of $C^\infty G$ manifolds and $C^\infty G$ vector bundles are studied in [9] and [7], respectively.

In this note, all exponentially Nash G manifolds and exponentially Nash G vector bundles are of class C^ω and manifolds are closed unless otherwise stated.

2. Exponentially Nash G manifolds and exponentially Nash G vector bundles.

Recall the definition of exponentially Nash G manifolds and exponentially Nash G vector bundles [8] and basic properties of exponentially definable sets and exponentially Nash manifolds [8].

Definition 2.1. (1) An $\mathbf{R}_{exp}$*-term* is a finite string of symbols obtained by repeated applications of the following two rules:
[1] Constants and variables are $\mathbf{R}_{exp}$-terms.
[2] If f is an m-place function symbol of $\mathbf{R}_{exp}$ and $t_1, \dots, t_m$ are $\mathbf{R}_{exp}$-terms, then the concatenated string $f(t_1, \dots, t_m)$ is an $\mathbf{R}_{exp}$-term.
(2) An $\mathbf{R}_{exp}$-*formula* is a finite string of $\mathbf{R}_{exp}$-terms satisfying the following three rules:
[1] For any two $\mathbf{R}_{exp}$-terms t_1 and t_2, $t_1 = t_2$ and $t_1 > t_2$ are $\mathbf{R}_{exp}$-formulas.
[2] If ϕ and ψ are $\mathbf{R}_{exp}$-formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are $\mathbf{R}_{exp}$-formulas.
[3] If ϕ is an $\mathbf{R}_{exp}$-formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are $\mathbf{R}_{exp}$-formulas.
(3) *An exponentially definable set* $X \subset \mathbb{R}^n$ is the set defined by an $\mathbf{R}_{exp}$-formula (with parameters).
(4) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets. A map $f : X \longrightarrow Y$ is called *exponentially definable* if the graph of $f \subset \mathbb{R}^n \times \mathbb{R}^m$ is exponentially definable.

On the other hand, using [12] any exponentially definable subset of $\mathbb{R}^n$ is the image of an $\mathfrak{R}_{n+m}$-semianalytic set by the natural projection $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ for some m. Here a subset X of $\mathbb{R}^n$ is called $\mathfrak{R}_n$*-semianalytic* if X is a finite union of sets of the following form:

$$\{x \in \mathbb{R}^n \,|\, f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq k, 1 \leq j \leq l\},$$

where $f_i, g_j \in \mathbb{R}[x_1, \dots, x_n, exp(x_1), \dots, exp(x_n)]$.

The following is a collections of properties of exponentially definable sets (cf. [8]).

Proposition 2.2 (cf. [8]). *(1) Any exponentially definable set consists of only finitely many connected components.*

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets.
(2)The closure $Cl(X)$ and the interior $Int(X)$ of X are exponentially definable.
(3) The distance function $d(x, X)$ from x to X defined by $d(x, X) = \inf\{||x-y|| | y \in X\}$ is a continuous exponentially definable function, where $||\cdot||$ denotes the standard norm of $\mathbb{R}^n$.
(4) Let $f : X \longrightarrow Y$ be an exponentially definable map. If a subset A of X is exponentially definable then so is $f(A)$, and if $B \subset Y$ is exponentially definable then so is $f^{-1}(B)$.
(5) Let $Z \subset \mathbb{R}^l$ be an exponentially definable set and let $f : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ be exponentially definable maps. Then the composition $h \circ f : X \longrightarrow Z$ is also exponentially definable. In particular for any two polynomial functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, the function $h : \mathbb{R} - \{f = 0\} \longrightarrow \mathbb{R}$ defined by $h(x) = e^{g(x)/f(x)}$ is exponentially definable.
(6) The set of exponentially definable functions on X forms a ring.
(7) Any two disjoint closed exponentially definable sets X and $Y \subset \mathbb{R}^n$ can be separated by a continuous exponentially definable function. $\square$

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open exponentially definable sets. A C^r $(0 \le r \le \omega)$ map $f : U \longrightarrow V$ is called *an exponentially C^r Nash map* if it is exponentially definable. An exponentially C^r Nash map $g : U \longrightarrow V$ is called *an exponentially C^r Nash diffeomorphism* if there exists an exponentially C^r Nash map $h : V \longrightarrow U$ such that $g \circ h = id$ and $h \circ g = id$. Note that the graph of an exponentially C^r Nash map may be defined by an $\mathbf{R}_{exp}$-formula with quantifiers.

Theorem 2.3 [14]. *Let $S_1, \dots, S_k \subset \mathbb{R}^n$ be exponentially definable sets. Then there exists a finite family $\mathfrak{W} = \{\Gamma^d_\alpha\}$ of subsets of $\mathbb{R}^n$ satisfying the following four conditions:*
(1) Γ^d_α are disjoint, $\mathbb{R}^n = \cup_{\alpha,d}\Gamma^d_\alpha$ and $S_i = \cup\{\Gamma^d_\alpha | \Gamma^d_\alpha \cap S_i \neq \emptyset\}$ for $1 \le i \le k$.
(2) Each Γ^d_α is an analytic cell of dimension d.
(3) $\overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ is a union of some cells Γ^e_β with $e < d$.
(4) If $\Gamma^d_\alpha, \Gamma^e_\beta \in \mathfrak{W}, \Gamma^e_\beta \subset \overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ then $(\Gamma^d_\alpha, \Gamma^e_\beta)$ satisfies Whitney's conditions (a) and (b) at all points of Γ^e_β. $\square$

Theorem 2.3 allows us to define the *dimension* of an exponentially definable set E by

$$\dim E = \max\{\dim \Gamma | \Gamma \text{ is an analytic submanifold contained in } E\}.$$

Example 2.4. (1) The C^∞ function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\lambda(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ e^{(-1/x)} & \text{if} \quad x > 0 \end{cases}$$

is exponentially definable but not exponentially Nash. This example shows that an exponentially definable C^∞ map is not always analytic. This phenomenon does not occur in the usual Nash category. We will use this function in section 3.

(2) The Zariski closure of the graph of the exponential function $exp : \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathbb{R}^2$ is the whole space $\mathbb{R}^2$. Hence the dimension of the graph of exp is smaller than that of its Zariski closure.
(3) The continuous function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h(x) = \begin{cases} e^{x-n} & \text{if} \quad n \leq x \leq n+1 \\ e^{n+2-x} & \text{if} \quad n+1 \leq x \leq n+2 \end{cases}, \text{ for } n \in 2\mathbb{Z},$$

is not exponentially definable, but the restriction of h on any bounded exponentially definable set is exponentially definable. $\square$

Definition 2.5. Let r be a non-negative integer, ∞ or ω.
(1) *An exponentially C^r Nash manifold X of dimension d* is a C^r manifold admitting a finite system of charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^d\}$ such that for each i and j $\phi_i(U_i \cap U_j)$ is an open exponentially definable subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is an exponentially C^r Nash diffeomorphism. We call these charts *exponentially C^r Nash.* A subset M of X is called *exponentially definable* if every $\phi_i(U_i \cap M)$ is exponentially definable.
(2) An exponentially definable subset of $\mathbb{R}^n$ is called *an exponentially C^r Nash submanifold of dimension d* if it is a C^r submanifold of dimension d of $\mathbb{R}^n$. An exponentially C^r $(r > 0)$ Nash submanifold is of course an exponentially C^r Nash manifold [8].
(3) Let X (resp. Y) be an exponentially C^r Nash manifold with exponentially C^r Nash charts $\{\phi_i : U_i \longrightarrow \mathbb{R}^n\}_i$ (resp. $\{\psi_j : V_j \longrightarrow \mathbb{R}^m\}_j$). A C^r map $f : X \longrightarrow Y$ is said to be *an exponentially C^r Nash map* if for any i and j $\phi_i(f^{-1}(V_j) \cap U_i)$ is open and exponentially definable in $\mathbb{R}^n$, and that the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \longrightarrow \mathbb{R}^m$ is an exponentially C^r Nash map.
(4) Let X and Y be exponentially C^r Nash manifolds. We say that X is *exponentially C^r Nash diffeomorphic to Y* if one can find exponentially C^r Nash maps $f : X \longrightarrow Y$ and $h : Y \longrightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.
(5) An exponentially C^r Nash manifold is said to be *affine* if it is exponentially C^r Nash diffeomorphic to some exponentially C^r Nash submanifold of $\mathbb{R}^l$.
(6) A group G is called *an exponentially Nash group* (resp. *an affine exponentially Nash group*) if G is an exponentially Nash manifold (resp. an affine exponentially Nash manifold) and that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ are exponentially Nash maps.

Definition 2.6. Let G be an exponentially Nash group and let $0 \leq r \leq \omega$.
(1) An exponentially C^r Nash submanifold in a representation of G is called *an exponentially C^r Nash G submanifold* if it is G invariant.
(2) An exponentially C^r Nash manifold X is said to be *an exponentially C^r Nash G manifold* if X admits a G action whose action map $G \times X \longrightarrow X$ is exponentially C^r Nash.
(3) Let X and Y be exponentially C^r Nash G manifolds. An exponentially C^r Nash map $f : X \longrightarrow Y$ is called *an exponentially C^r Nash G map* if it is a G map. An exponentially C^r Nash G map $g : X \longrightarrow Y$ is said to be *an exponentially C^r Nash G diffeomorphism* if there exists an exponentially C^r Nash G map $h : Y \longrightarrow X$ such that $g \circ h = id$ and $h \circ g = id$.

(4) We say that an exponentially C^r Nash G manifold is *affine* if it is exponentially C^r Nash G diffeomorphic to an exponentially C^r Nash G submanifold of some representation of G.

We have the following implications on groups:

an algebraic group$\Longrightarrow$an affine Nash group$\Longrightarrow$an affine exponentially Nash group $\Longrightarrow$ an exponentially Nash group $\Longrightarrow$ a Lie group .

Let G be an algebraic group. Then we obtain the following implications on G manifolds:

a nonsingular algebraic G set $\Longrightarrow$ an affine Nash G manifold $\Longrightarrow$ an affine exponentially Nash G manifold $\Longrightarrow$ an exponentially Nash G manifold $\Longrightarrow$ a $C^\infty G$ manifold .

Moreover, notice that a Nash G manifold is not always an affine exponentially Nash G manifold.

In the equivariant exponentially Nash category, the equivariant tubular neighborhood result holds true [8].

Proposition 2.7 [8]. *Let G be a compact affine exponentially Nash group and let X be an affine exponentially Nash G submanifold possibly with boundary in a representation Ω of G. Then there exists an exponentially Nash G tubular neighborhood (U,p) of X in Ω, namely U is an affine exponentially Nash G submanifold in Ω and the orthogonal projection $p : U \longrightarrow X$ is an exponentially Nash G map.* $\square$

Definition 2.8. Let G be an exponentially Nash group and let $0 \le r \le \omega$.
(1) A C^rG vector bundle (E,p,X) of rank k is said to be *an exponentially C^r Nash G vector bundle* if the following three conditions are satisfied:

(a) The total space E and the base space X are exponentially C^r Nash G manifolds.

(b) The projection p is an exponentially C^r Nash G map.

(c) There exists a family of finitely many local trivializations $\{U_i, \phi_i : U_i \times \mathbb{R}^k \longrightarrow p^{-1}(U_i)\}_i$ such that $\{U_i\}_i$ is an open exponentially definable covering of X and that for any i and j the map $\phi_i^{-1} \circ \phi_j|(U_i \cap U_j) \times \mathbb{R}^k : (U_i \cap U_j) \times \mathbb{R}^k \longrightarrow (U_i \cap U_j) \times \mathbb{R}^k$ is an exponentially C^r Nash map.

We call these local trivializations *exponentially C^r Nash*.
(2) Let $\eta = (E,p,X)$ (resp. $\zeta = (F,q,X)$) be an exponentially C^r Nash G vector bundle of rank n (resp. m). Let $\{U_i, \phi_i : U_i \times \mathbb{R}^n \longrightarrow p^{-1}(U_i)\}_i$ (resp. $\{V_j, \psi_j : V_j \times \mathbb{R}^m \longrightarrow q^{-1}(V_j)\}_j$) be exponentially C^r Nash local trivializations of η (resp. ζ). A C^rG vector bundle map $f : \eta \longrightarrow \zeta$ is said to be *an exponentially C^r Nash G vector bundle map* if for any i and j the map $(\psi_j)^{-1} \circ f \circ \phi_i|(U_i \cap V_j) \times \mathbb{R}^n : (U_i \cap V_j) \times \mathbb{R}^n \longrightarrow (U_i \cap V_j) \times \mathbb{R}^m$ is an exponentially C^r Nash map. A C^rG section s of η is called *exponentially C^r Nash* if each $\phi_i^{-1} \circ s|U_i : U_i \longrightarrow U_i \times \mathbb{R}^n$ is exponentially C^r Nash.
(3) Two exponentially C^r Nash G vector bundles η and ζ are said to be *exponen-*

tially C^r Nash G vector bundle isomorphic if there exist exponentially C^r Nash G vector bundle maps $f : \eta \longrightarrow \zeta$ and $h : \zeta \longrightarrow \eta$ such that $f \circ h = id$ and $h \circ f = id$.

Recall universal G vector bundles (cf. [7]).

Definition 2.9. Let Ω be an n-dimensional representation of G and B the representation map $G \longrightarrow GL_n(\mathbb{R})$ of Ω. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \longrightarrow B(g)^{-1}AB(g) \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\},$$

$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\},$$

$$u : E(\Omega, k) \longrightarrow G(\Omega, k) : u((A, v)) = A,$$

where A' denotes the transposed matrix of A and TrA stands for the trace of A. Then $\gamma(\Omega, k)$ is an algebraic set. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it *the universal G vector bundle associated with Ω and k.* Since $G(\Omega, k)$ and $E(\Omega, k)$ are nonsingular, $\gamma(\Omega, k)$ is a Nash G vector bundle, hence it is an exponentially Nash one.

Definition 2.10. An exponentially C^r Nash G vector bundle $\eta = (E, p, X)$ of rank k is said to be *strongly exponentially C^r Nash* if the base space X is affine and that there exist some representation Ω of G and an exponentially C^r Nash G map $f : X \longrightarrow G(\Omega, k)$ such that η is exponentially C^r Nash G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$.

Let G be a Nash group. Then we have the following implications on G vector bundles over an affine Nash G manifold:

a Nash G vector bundle $\Longrightarrow$ an exponentially Nash G vector bundle $\Longrightarrow$ a $C^\omega G$ vector bundle, and
a strongly Nash G vector bundle $\Longrightarrow$ a strongly exponentially Nash G vector bundle $\Longrightarrow$ an exponentially Nash G vector bundle.

3. Proof of results.

A subset of $\mathbb{R}^n$ is called *locally closed* if it is the intersection of an open set $\subset \mathbb{R}^n$ and a closed set $\subset \mathbb{R}^n$.

To prove Theorem 1.1, we recall the following.

Proposition 3.1 [8]. *Let $X \subset \mathbb{R}^n$ be a locally closed exponentially definable set and let f and g be continuous exponentially definable functions on X with $f^{-1}(0) \subset g^{-1}(0)$. Then there exist an integer N and a continuous exponentially definable function $h : X \longrightarrow \mathbb{R}$ such that $g^N = hf$ on X. In particular, for any compact subset K of X, there exists a positive constant c such that $|g^N| \leq c|f|$ on K* $\square$

Proof of Theorem 1.1. Let X be an exponentially C^r Nash manifold. If dim $X = 0$ then X consists of finitely many points. Thus the result holds true.

Assume that dim $X \geq 1$. Let $\{\phi_i : U_i \longrightarrow \mathbb{R}^m\}_{i=1}^l$ be exponentially C^r Nash charts of X. Since X is compact, shrinking U_i, if necessarily, we may assume that every $\phi_i(U_i)$ is the open unit ball of $\mathbb{R}^m$ whose center is the origin. Let f be the function on $\mathbb{R}^m$ defined by $f(x) = ||x|| - 1$. Then $f^{-1}(0) = \overline{\phi_i(U_i)} - \phi_i(U_i)$. Hence replacing the graph of $1/f$ on $\phi_i(U_i)$ by $\phi_i(U_i)$, each $\phi_i(U_i)$ is closed in $\mathbb{R}^m$. Consider the stereographic projection $s : \mathbb{R}^m \longrightarrow S^m \subset \mathbb{R}^m \times \mathbb{R}$. Composing ϕ_i and s, we have an exponentially C^ω Nash imbedding $\phi'_i : \phi_i(U_i) \longrightarrow \mathbb{R}^{m'}$ such that the image is bounded in $\mathbb{R}^{m'}$ and

$$\overline{\phi'_i \circ \phi_i(U_i)} - \phi'_i \circ \phi_i(U_i)$$

consists of one point, say 0. Set

$$\eta : \mathbb{R}^{m'} \longrightarrow \mathbb{R}^{m'}, \eta(x_1, \dots, x_{m'}) = (\sum_{j=1}^{m'} x_j^{2k} x_1, \dots, \sum_{j=1}^{m'} x_j^{2k} x_{m'}),$$

$$g_i : U_i \longrightarrow \mathbb{R}^{m'}, \eta \circ \phi'_i \circ \phi_i,$$

for a sufficiently large integer k. Then g_i is an exponentially C^r Nash imbedding of U_i into $\mathbb{R}^{m'}$. Moreover the extension $\tilde{g_i} : X \longrightarrow \mathbb{R}$ of g_i defined by $\tilde{g_i} = 0$ on $X - U_i$. We now prove that $\tilde{g_i}$ is of class exponentially C^r Nash. It is sufficient to see this on each exponentially C^r Nash coordinate neighborhood of X. Hence we may assume that X is open in $\mathbb{R}^m$. We only have to prove that for any sequence $\{a_j\}_{j=1}^\infty$ in U_i convergent to a point of $X - U_i$ and for any $\alpha \in \mathbb{N}^m$ with $|\alpha| < r$, $\{D^\alpha g_i(a_j)\}_{j=1}^\infty$ converges to 0. On the other hand $g_i = (\sum_{j=1}^{m'} \phi_{ij}^{2k} \phi_{i1}, \cdots \sum_{j=1}^{m'} \phi_{ij}^{2k} \phi_{im'})$, where $\phi'_i \circ \phi_i = (\phi_{i1}, \dots, \phi_{im'})$. Each ϕ_{ij} is bounded, and every $\{\phi_{ij}(a_i)\}_{i=1}^\infty$ converges to zero, and

$$|D^\alpha(\phi_{ij}^{2k} \phi_{is})| = |\sum_{\beta+\gamma=\alpha} (\alpha!/(\beta!\gamma!)) D^\beta \phi_{ij}^{2k} D^\gamma \phi_{is}| \leq$$

$$C \sum_{\beta_1+\cdots+\beta_{l'}+\gamma=\alpha, \beta_i \neq 0} |\phi_{ij}^{2k-l'} D^{\beta_1} \phi_{ij} \cdots D^{\beta_{l'}} \phi_{ij} D^\gamma \phi_{is}| \leq C' |\phi_{ij}^{2k-\gamma}| \psi,$$

where C, C' are constants, and ψ is the positive continuous exponentially definable function defined by

$$\psi(x) = \max\{1, \sum_{\beta_1+\cdots+\beta_{l'}+\gamma=\alpha} |D^{\beta_1} \phi_{ij}(x) \cdots D^{\beta_{l'}} \phi_{ij}(x) D^\gamma \phi_{is}(x)|\}.$$

Define

$$\theta_{ij}(x) = \begin{cases} \min\{|\phi_{ij}(x)|, 1/\psi(x)\} & \text{on } U_i \\ 0 & \text{on } X - U_i, \end{cases} \quad \tilde{\phi_{ij}} = \begin{cases} \phi_{ij} & \text{on } U_i \\ 0 & \text{on } X - U_i. \end{cases}$$

Then θ_{ij} and $\tilde{\phi_{ij}}$ are continuous exponentially definable functions on X such that

$$X - U_i \subset \theta_{ij}^{-1}(0) = \tilde{\phi_{ij}}^{-1}(0).$$

Hence by Proposition 3.1 we have $|\tilde{\phi_{ij}}^{l''}| \leq d\theta_{ij}$ on some open exponentially definable neighborhood V of $X - U_i$ in X for some integer l'', where d is a constant.

On the other hand, by the definition of θ_{ij} $|\psi\theta_{ij}| \leq 1$. Hence the above argument proves that

$$|D^\alpha(\phi_{ij}^{2k}\phi_{is})| \leq c'|\phi_{ij}^{2k-r-l''}|$$

on $U_i \cap V$, where c' is a constant and we take k such that $2k \geq r + l'' + 1$. Hence each $\tilde{g}_i$ is of class exponentially C^r Nash. It is easy to see that

$$\prod_{i=1}^{l} \tilde{g}_i : X \longrightarrow \mathbb{R}^{lm'}$$

is an exponentially C^r Nash imbedding. □

By the similar method of [7], we have the following.

Theorem 3.2 [8]**.** *Let G be a compact affine exponentially Nash group and let X be a compact affine exponentially Nash G manifold.*
(1) For every $C^\infty G$ vector bundle η over X, there exists a strongly exponentially Nash G vector bundle ζ which is $C^\infty G$ vector bundle isomorphic to η.
(2) For any two strongly exponentially Nash G vector bundles over X, they are exponentially Nash G vector bundle isomorphic if and only if they are C^0G vector bundle isomorphic. □

We prepare the following results to prove Theorem 1.2.

Proposition 3.3 [8]**.** *Let M be an affine exponentially Nash G manifold in a representation Ω of G.*
(1) The normal bundle (L, q, M) in Ω realized by

$$L = \{(x, y) \in M \times \Omega | y \text{ is orthogonal to } T_xM\}, q : L \longrightarrow M, q(x, y) = x$$

is an exponentially Nash G vector bundle.
(2) If M is compact, then some exponentially Nash G tubular neighborhood U of M in Ω obtained by Proposition 2.7 is exponentially Nash G diffeomorphic to L. □

Proposition 3.4 [8]**.** *Let G be a compact affine exponentially Nash group and let $\eta = (E, p, Y)$ be an exponentially Nash G vector bundle of rank k over an affine exponentially Nash G manifold Y. Then η is strongly exponentially Nash if and only if E is affine.* □

Lemma 3.5. *Let D_1 and D_2 be open balls of $\mathbb{R}^n$ which have the same center x_0, and let a (resp. b) be the radius of D_1 (resp. D_2) with $a < b$. Suppose that A and B are two real numbers. Then there exists a C^∞ exponentially definable function f on $\mathbb{R}^n$ such that $f = A$ on D_1 and $f = B$ on $\mathbb{R}^n - \overline{D_2}$.*

Proof. We can assume that $A = 1$, $B = 0$ and $x_0 = 0$.

At first we construct such a function when $n = 1$. Then we may assume that $D_1 = (-a, a)$ and $D_2 = (-b, b)$ be open intervals. Recall the exponentially definable C^∞ function λ defined in Example 2.4. The function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\phi(x) = \lambda(b - x)\lambda(b + x)/(\lambda(b - x)\lambda(b + x) + \lambda(x^2 - a^2))$$

is the desired function. Therefore $f : \mathbb{R}^n \longrightarrow \mathbb{R}, f(x) = \phi(|x|)$ is the required function, where $|x|$ denotes the standard norm of $\mathbb{R}^n$. $\square$

proof of Theorem 1.2. By Theorem 3.2 we may assume that η is a strongly exponentially C^ω Nash G vector bundle. We only have to find an exponentially C^ω Nash G vector bundle ζ which is exponentially C^∞ Nash G vector bundle isomorphic to η but not exponentially C^ω Nash G vector bundle isomorphic to η.

As well as the usual equivariant Nash category, X^G is an exponentially Nash G submanifold of X. Take an open exponentially definable subset U of X such that $\eta|U$ is exponentially C^ω Nash vector bundle isomorphic to the trivial bundle and that $X^G \cap U \neq \emptyset$. Since $\dim X^G \geq 2$, there exists a one-dimensional exponentially Nash G submanifold S of U which is exponentially Nash diffeomorphic to the unit circle S^1 in $\mathbb{R}^2$. Moreover there exist two open G invariant exponentially definable subsets V_1 and V_2 of U such that $V_1 \cup V_2 \supset S$ and $V_1 \cap V_2$ consists of two open balls Z_1 and Z_2. We define the exponentially C^ω Nash G vector bundle $\zeta' := (E, r, V_1 \cup V_2)$ over $V_1 \cup V_2$ to be the bundle obtained by the coordinate transformation

$$g_{12} : V_1 \cap V_2 \longrightarrow GL(\Xi), g_{12} = \begin{cases} I & \text{on } Z_1 \\ (1+\epsilon)I & \text{on } Z_2, \end{cases}$$

where I denotes the unit matrix, $\epsilon > 0$ is sufficiently small and Ξ stands for the fiber of $\eta|U$. This construction is inspired by the proof of 4.2.8 [23].

Let $\phi_i : V_i \times \Xi \longrightarrow p^{-1}(V_i), i = 1, 2$ be exponentially Nash G coordinate functions of ζ'. Consider an extension of the exponentially C^ω Nash section f on $S \cap V_1$ defined by $\phi_1^{-1} \circ f(x) = (x, I)$. If we extend f through Z_1, then the analytic extension $\tilde{f}$ to $S \cap V_2$ satisfies $\phi_2^{-1} \circ \tilde{f} = (x, I), x \in S \cap V_2$. However the analytic extension $\tilde{f}$ to $S \cap V_2$ through Z_2 satisfies $\phi_2^{-1} \circ \tilde{f} = (x, 1/(1+\epsilon)I)$. Thus the smallest analytic set containing the graph of f spins infinitely over S. Hence $\zeta'|S$ is not exponentially C^ω Nash G vector bundle isomorphic to $\eta|S$. By Theorem 3.2 $\zeta'|S$ is not strongly exponentially C^ω Nash. Thus the exponentially C^ω Nash G vector bundle ζ over X obtained by replacing $\eta|V_1 \cup V_2$ by ζ' is not exponentially C^ω Nash G vector bundle isomorphic to η.

On the other hand, by Lemma 3.5 we can construct an exponentially C^∞ Nash G map H from a G invariant exponentially definable neighborhood of $U \cap X^G$ in U to $GL(\Xi)$ such that $H|Z_2 = (1+\epsilon)I$ and $H = I$ outside of some G invariant exponentially definable neighborhood of Z_2. Since ϵ is sufficiently small, using this map, we get an exponentially C^∞ Nash G vector bundle isomorphism $\eta \longrightarrow \zeta$. $\square$

Proof of Theorem 1.3. By the proof of Theorem 1 (1) [9], X is $C^\infty G$ diffeomorphic to some affine exponentially Nash G manifold. Hence we may assume that X is an affine exponentially C^ω Nash G manifold.

Since X^G is an exponentially C^ω Nash G submanifold of X, there exists an exponentially Nash G tubular neighborhood (T, q) of X^G in X by Proposition 2.7. Moreover we may assume that T is exponentially C^ω Nash G diffeomorphic to the total space of the normal bundle η of X^G in X because of Proposition 3.3. Note that η is a strongly exponentially C^ω Nash G vector bundle over X^G and that each fiber is a representation of G. Take an open G invariant exponentially definable subset U of X^G such that $\eta|U$ is exponentially C^ω Nash G vector bundle isomorphic to the trivial bundle $U \times \Xi$, where Ξ denotes the fiber of $\eta|U$.

By the proof of Theorem 1.2, there exists an exponentially C^ω Nash G vector bundle η' over U such that η' is not exponentially C^ω Nash G vector bundle isomorphic to $\eta|U$ and that there exists an exponentially C^∞ Nash G vector bundle isomorphism $H : \eta|U \longrightarrow \eta'$ such that H is the identity outside of some open G invariant exponentially definable set.

Replacing the total space of $\eta|U$ by that of η', we have an exponentially C^ω Nash G manifold Y which is not exponentially C^ω Nash G diffeomorphic to X. Moreover using H, one can find an exponentially C^∞ Nash G diffeomorphism from X to Y. □

Note that Y is not exponentially C^ω Nash G affine but exponentially C^∞ Nash G affine by Proposition 3.4.

4. Remarks.

It is known in [1] that every compact Lie group admits one and exactly one algebraic group structure up to algebraic group isomorphism. Hence it admits an affine Nash group structure. Notice that all connected one-dimensional Nash groups and locally Nash groups are classified by [16] and [22], respectively. In particular, the unit circle S^1 in $\mathbb{R}^2$ admits a nonaffine Nash group structure.

But the analogous result concerning nonaffine exponentially Nash group structures of centerless Lie groups does not hold.

Remark 4.1. *Let G be a compact centerless Lie group. Then G does not admit any nonaffine exponentially Nash group structure.*

Proof. Let G' be an exponentially Nash group which is isomorphic to G as a Lie group. Then the adjoint representation $Ad : G' \longrightarrow Gl_n(\mathbb{R})$ is exponentially definable by the similar method of Lemma 2.2 [15] and it is C^ω, where n denotes the dimension of G. Hence Ad is an exponentially Nash one and its kernel is the center of G'. Therefore the image G" of Ad is an affine exponentially Nash group and Ad is an exponentially Nash group isomorphism from G' to G". □

It is known that any two disjoint closed semialgebraic sets X and Y in $\mathbb{R}^n$ can be separated by a C^ω Nash function on $\mathbb{R}^n$ [18], namely there exists a C^ω Nash function f on $\mathbb{R}^n$ such that

$$f > 0 \text{ on } X \text{ and } f < 0 \text{ on } Y.$$

The following is a weak equivariant version of Nash category and exponentially Nash category.

Remark 4.2. *Let G be a compact affine Nash (resp. a compact affine exponentially Nash) group. Then any two disjoint closed G invariant semialgebraic (resp. disjoint closed G invariant exponentially definable) sets in a representation Ω of G can be separated by a G invariant continuous semialgebraic (resp. a G invariant continuous exponentially definable) function on Ω.*

Proof. By the distance $d(x, X)$ of x between X is semialgebraic (resp. exponentially definable). Since G is compact, $d(x, X)$ is equivariant. Hence $F : \Omega \longrightarrow \mathbb{R}, F(x) = d(x, Y) - d(x, X)$ is the desired one. □

Remark 4.3. *Under the assumption of 4.2, if one of the above two sets is compact, then they are separated by a G invariant entire rational function on Ω, where an entire rational function means a fraction of polynomial functions with nowhere vanishing denominator.*

Proof. Assume that X is compact and Y is noncompact. Let $s : \Omega \longrightarrow S \subset \Omega \times \mathbb{R}$ be the stereographic projection and let $S = \Omega \cup \{\infty\}$. Since X is compact, $s(X)$ and $s(Y) \cup \{\infty\}$ are compact and disjoint. Applying Remark 4.2, we have a G invariant continuous semialgebraic (resp. a G invariant continuous exponentially definable) function f on $\Omega \times \mathbb{R}$. By the classical polynomial approximation theorem and Lemma 4.1 [4], we get a G invariant polynomial F on $\Omega \times \mathbb{R}$ such that $F|S$ is an approximation of f. Since $s(X)$ and $s(Y) \cup \{\infty\}$ are compact, $F \circ s$ is the required one. $\square$

Remark 4.4. *Let $X \subset \mathbb{R}^n$ be an open (resp. a closed) exponentially definable set. Suppose that X is a finite union of sets of the following form:*

$$\{x \in \mathbb{R}^n | f_1(x) = \cdots = f_i(x) = 0, g_1(x) > 0, \dots, g_j(x) > 0\},$$

$$(\textit{resp.}\ \{x \in \mathbb{R}^n | f_1(x) = \cdots = f_i(x) = 0, g_1(x) \geq 0, \dots, g_j(x) \geq 0\},)$$

where $f_1, \dots, f_i$ and $g_1, \dots, g_j$ are exponentially Nash functions on $\mathbb{R}^n$. Then X is a finite union of sets of the following form:

$$\{x \in \mathbb{R}^n | h_1(x) > 0, \dots, h_k(x) > 0\},$$

$$(\textit{resp.}\ \{x \in \mathbb{R}^n | h_1(x) \geq 0, \dots, h_k(x) \geq 0\},)$$

where $h_1, \dots, h_i$ are exponentially Nash functions on $\mathbb{R}^n$.

Note that any exponentially definable set in $\mathbb{R}^n$ can be described as a finite union of sets of the following form [8]:

$$\{x \in \mathbb{R}^n | F_1(x) = \cdots = F_s(x) = 0, G_1(x) > 0, \dots, G_t(x) > 0\}.$$

Here each of $F_1, \dots, F_s$ and $G_1, \dots, G_t$ is an exponentially Nash function defined on some open exponentially definable subset of $\mathbb{R}^n$, however its domain is not always the whole space $\mathbb{R}^n$.

We define $exp_n(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by $exp_0(x) = x$ and $exp_{n+1}(x) = exp(exp_n(x))$. The following is a bound of the growth of continuous exponentially definable functions

Proposition 4.5 [8]. *Let F be a closed exponentially definable set in $\mathbb{R}^k$ and let $f : F \longrightarrow \mathbb{R}$ be a continuous exponentially definable function. Then there exist $c > 0, n, m \in \mathbb{N}$ such that*

$$|f(x)| \leq c(1 + exp_n(||x||^m)) \textit{ for any } x \in F,$$

where $|| \cdot ||$ denotes the standard norm of $\mathbb{R}^k$. $\square$

Proof of Remark 4.4. It suffices to prove the result when X is open because the other case follows by taking complements.

Let

$$B = \{x \in \mathbb{R}^n | f_1(x) = \cdots = f_u(x) = 0, g_1(x) > 0, \ldots, g_v(x) > 0\},$$

where all f_i and all g_j are exponentially Nash functions on $\mathbb{R}^n$. Set $f := f_1^2 + \cdots + f_u^2$ and $g(x) := \prod_{i=1}^v (|g_i(x)| + g_i(x))$. On $\mathbb{R}^n - X$, $g(x) = 0$ if $f(x) = 0$. By Proposition 3.1 there exists an integer N and a continuous exponentially definable function h on $\mathbb{R}^n - X$ such that $g^N = hf$ on $\mathbb{R}^n - X$. By Proposition 4.5 we have some $c \in \mathbb{R}$ and some $m, n \in \mathbb{N}$ such that $|h(x)| \leq c(1 + exp_n(||x||^m)$ on $\mathbb{R}^n - X$. Define $B_1 = \{x \in \mathbb{R}^n | cf(x)(1 + exp_n(||x||^m)) < (2^m \prod_{i=1}^m g_i(x))^N, g_1(x) > 0, \ldots, g_m(x) > 0\}$. Then $B \subset B_1 \subset X$. Therefore replacing B by B_1, we have the required union. $\square$

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ON FIXED POINT DATA OF SMOOTH ACTIONS ON SPHERES

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We report results obtained jointly with K. Pawałowski.

There are two fundamental questions about **smooth actions** on manifolds. Let G be a finite group and M a manifold.

Question 1. Which manifolds F can be the G-fixed point sets of G-actions on M, i.e. $M^G = F$?

Question 2. Which G-vector bundles ν over F can be the G-tubular neighborhoods (i.e. G-normal bundles) of $F = M^G$ in M?

If ν can be realized as a subset of M in the way above, we say that (F, ν) occurs as the G**-fixed point data** in M. If for a real G-module W with $W^G = 0$, $(F, \nu \oplus \varepsilon_M^W)$ occurs as the G-fixed point data in M then we say that (F, ν) **stably** occurs as the G-fixed point data in M. These questions were studied by B. Oliver [O2] in the case where G is not of prime power order and M is a disk or a Euclidean space. The topic of the current talk is the case where G is an Oliver group and M is a sphere.

Let G be a finite group not of prime power order. A G-action on M is called $\mathcal{P}$**-proper** if $M^P \supsetneq M^G$ for any Sylow subgroup P of G. There are necessary conditions for (F, ν) to stably occur as the G-fixed point data of a $\mathcal{P}$-proper G-action on a sphere.

(F1) (Oliver Condition) $\chi(F) \equiv \chi(M) \mod n_G$ (where n_G is the integer called Oliver's number [O1]).

(B1) (Product Bundle Condition) $\tau_F \oplus \nu = 0$ in $\widetilde{KO}(F)$.

(B2) (Smith Condition) For each prime p and any Sylow p-subgroup P of G, $\tau_F \oplus \nu = 0$ in $\widetilde{KO}_P(F)_{(p)}$.

By Oliver [O2], Conditions (F1), (B1) and (B2) are also necessary–sufficient conditions for (F, ν) to stably occur as the G-fixed point data in a disk. By [O1], n_G is equal to 1 if and only if there are no normal series $P \triangleleft H \triangleleft G$ such that $|P| = p^s$, H/P is cyclic, and $|G/H| = q^t$ (s, $t \geq 0$). A group G with $n_G = 1$ is called an **Oliver group**. Clearly any nonsolvable group is an Oliver group. A nilpotent group is an Oliver group if and only if it has at least three noncyclic Sylow subgroups. In the case where G is an Oliver group, Condition (F1) provides no restriction.

We begin the preparation for our sufficient conditions. For a finite group G and a prime p, let G^p denote the minimal normal subgroup of G such that G/G^p is of

p-power order (possibly $G^p = G$). Let $\mathcal{L}(G)$ denote the set of all subgroups H of G such that $H \supseteq G^p$ for some prime p. Let $\mathcal{P}(G)$ denote the set of all subgroups P of G such that $|P|$ is a prime power (possibly $|P| = 1$). A G-action on M is said to be $(\mathcal{P}, \mathcal{L})$-**proper** if the action is $\mathcal{P}$-proper and if any connected component of X^H ($H \in \mathcal{L}(G)$) does not contain a connected component of M^G as a proper subset. If G is an Oliver group then the G-action on

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \backslash |G|} (\mathbb{R}[G/G^p] - \mathbb{R})$$

is $(\mathcal{P}, \mathcal{L})$-proper ([LM]). A finite group G is said to be **admissible** if there is a real G-module V such that $\dim V^P > 2 \dim V^H$ for any $P \in \mathcal{P}(G)$ and any $H \leq G$ with $H \supsetneq P$, and $\dim V^H = 0$ for any $H \in \mathcal{L}(G)$.

Theorem (M.M.–M. Yanagihara [MY1–2])**.** *Let G be an Oliver group. If $G^2 = G$ or $G^p \neq G$ for at least 2 distinct odd primes then G is admissible. In particular, an Oliver group G is admissible in each case: G is nilpotent; G is perfect.*

The symmetric group of degree 5 is not admissible.

K. H. Dovermann–M. Herzog recently proved that S_n ($n \geqq 6$) are admissible.

Our main result is:

Theorem A. *Let G be an admissible Oliver group* (*resp. an Oliver group*) *. Let F be a closed manifold* (*resp. a finite discrete space*) *and let ν be a real G-vector bundle over F such that* $\dim \nu^H = 0$ *whenever $H \in \mathcal{L}(G)$. Then the following* (1)–(3) *are equivalent:*

(1) *(F, ν) stably occurs as the G-fixed point data of a $(\mathcal{P}, \mathcal{L})$-proper G-action on a sphere.*

(3) *(F, ν) stably occurs as the G-fixed point data in a disk.*

(3) *$\tau_M \oplus \nu$ satisfies* (B1)–(B2).

A finite group not of prime power order belongs to exactly one of the following six classes ([O2]):

$\mathcal{A}$: G has a dihedral subquotient of order $2n$ for a composite integer n.

$\mathcal{B}$: $G \notin \mathcal{A}$ and G has a composite order element conjugate to its inverse.

$\mathcal{C}$: $G \notin \mathcal{A} \cup \mathcal{B}$, G has a composite order element and the Sylow 2-subgroups are not normal in G.

$\mathcal{C}_2$: G has a composite order element and the Sylow 2-subgroup is normal in G.

$\mathcal{D}$: G has no elements of composite order and the Sylow 2-subgroups are not normal in G.

$\mathcal{D}_2$: G has no elements of composite order and the Sylow 2-subgroup is normal in G.

Corollary B. *Let G be a nontrivial perfect group and F a closed manifold. Then F occurs as the G-fixed point set of a $\mathcal{P}$-proper G-action on a sphere if and only if F occurs as the G-fixed point set in a disk (in other words,*

$G \in \mathcal{A}$: *there is no restriction.*

$G \in \mathcal{B}$: $c_{\mathbb{R}}([\tau_F]) \in c_{\mathbb{H}}(\widetilde{KSp}(F)) + \mathrm{Tor}(\widetilde{K}(F))$

$G \in \mathcal{C}$: $[\tau_F] \in r_{\mathbb{C}}(\widetilde{K}(F)) + \mathrm{Tor}(\widetilde{KO}(F))$

$G \in \mathcal{D}$: $[\tau_F] \in \mathrm{Tor}(\widetilde{KO}(F))$.)

Theorem C. *Let G be a nilpotent Oliver group and F a closed manifold. Then the following* (1)–(3) *are equivalent.*

(1) *F occurs as the G-fixed point set of a $\mathcal{P}$-proper G-action on a sphere.*

(2) *τ_F is stably complex.*

(3) *F occurs as the G-fixed point set of a G-action on a disk.*

Our basic methods are:

(1) An extension of the method of equivariant bundles in [O2] (with modifications).

(2) The equivariant thickening of [P].

(3) The equivariant surgery results of [M1–2].

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SOME RESULTS ON KNOTS AND LINKS IN ALL DIMENSIONS

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An *(oriented) (ordered) m-component n-(dimensional) link* is a smooth, oriented submanifold $L = \{K_1, \ldots, K_m\}$ of S^{n+2}, which is the ordered disjoint union of m manifolds, each PL homeomorphic to the standard n-sphere. (If $m = 1$, then L is called a *knot*.) We say that m-component n-dimensional links, L_0 and L_1, are *(link-)concordant* or *(link-)cobordant* if there is a smooth oriented submanifold $\widetilde{C}=\{C_1, \ldots, C_m\}$ of $S^{n+2} \times [0,1]$, which meets the boundary transversely in $\partial\widetilde{C}$, is PL homeomorphic to $L_0 \times [0,1]$ and meets $S^{n+2} \times \{l\}$ in L_l $(l = 0,1)$.

We work in the smooth category.

§1

Let S^3_1 and S^3_2 be 3-spheres embedded in the 5-sphere S^5 and intersect transversely. Then the intersection C is a disjoint collection of circles. Thus we obtain a pair of 1-links C in S^3_i, and a pair of 3-knots S^3_i in S^5.

Conversely let (L_1, L_2) be a pair of 1-links and (X_1, X_2) be a pair of 3-knots. It is natural to ask whether (L_1, L_2) is obtained as the intersection of X_1 and X_2.

In this paper we give a complete answer to the above question.

Definition. (L_1, L_2, X_1, X_2) is called a *quadruple of links* if the following conditions (1), (2) and (3) hold.
(1) $L_i = (K_{i1}, \ldots, K_{im_i})$ is an oriented ordered m_i-component 1-dimensional link $(i = 1,2)$. (2) $m_1 = m_2$. (3) X_i is an oriented 3-knot.

Definition. A quadruple of links (L_1, L_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S^3_1 \coprod S^3_2 \looparrowright S^5$ satisfying the following conditions.
(1) $f|S^3_i$ is a smooth embedding and defines the 3-knot $X_i (i = 1,2)$ in S^5.
(2) For $C = f(S^3_1) \cap f(S^3_2)$, the inverse image $f^{-1}(C)$ in S^3_i defines the 1-link $L_i (i =$

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1, 2). Here, the orientation of C is induced naturally from the preferred orientations of S^3_1, S^3_2, and S^5, and an arbitrary order is given to the components of C.

The following theorem characterizes the realizable quadruples of links.

Theorem 1.1. *A quadruple of links (L_1, L_2, X_1, X_2) is realizable if and only if (L_1, L_2, X_1, X_2) satisfies one of the following conditions (i) and (ii).*

(i) Both L_1 and L_2 are proper links, and

$$\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2).$$

(ii) Neither L_1 nor L_2 is proper, and

$$lk(K_{1j}, L_1 - K_{1j}) \equiv lk(K_{2j}, L_2 - K_{2j}) \mod 2 \quad \text{for all} \;\; j.$$

Let $f : S^3 \looparrowright S^5$ be a smooth transverse immersion with a connected self-intersection C in S^5. Then the inverse image $f^{-1}(C)$ in S^3 is a knot or a 2-component link. For a similar realization problem, we have:

Theorem 1.2.
(1) All 2-component links are realizable as above.
(2) All knots are realizable as above.

Remark. By Theorem 1.1 a quadruple of links (L_1, L_2, X_1, X_2) with K_1 being the trivial knot and K_2 being the trefoil knot is not realizable. But by Theorem 1.2, the two component split link of the trivial knot and the trefoil knot is realizable as the self-intersection of an immersed 3-sphere.

§2

We discuss the high dimensional analogue of §1.

Definition. (K_1, K_2) is called a *pair of n-knots* if K_1 and K_2 are n-knots. (K_1, K_2, X_1, X_2) is called a *quadruple of n-knots and $(n+2)$-knots* or a *quadruple of $(n, n+2)$-knots* if K_1 and K_2 constitute a pair of n-knots (K_1, K_2) and X_1 and X_2 are diffeomorphic to the standard $(n+2)$-sphere.

Definition. A quadruple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S^{n+2}_1 \coprod S^{n+2}_2 \looparrowright S^{n+4}$ satisfying the following conditions.

(1) $f|S^{n+2}_i$ defines X_i (i=1,2).
(2) The intersection $\Sigma = f(S^{n+2}_1) \cap f(S^{n+2}_2)$ is PL homeomorphic to the standard sphere.
(3) $f^{-1}(\Sigma)$ in S^{n+2}_i defines an n-knot K_i $(i = 1, 2)$.

A pair of n-knots (K_1, K_2) is said to be *realizable* if there is a quadruple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) which is realizable.

The following theorem characterizes the realizable pairs of n-knots.

Theorem 2.1. *A pair of n-knots (K_1, K_2) is realizable if and only if (K_1, K_2) satisfies the condition that*

$$\begin{cases} (K_1, K_2) \text{ is arbitrary} & \text{if } n \text{ is even,} \\ \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \text{if } n = 4m+1, \quad (m \geqq 0, m \in \mathbb{Z}). \\ \sigma(K_1) = \sigma(K_2) & \text{if } n = 4m+3, \end{cases}$$

There exists a mod 4 periodicity in dimension similar to the periodicity of high-dimensional knot cobordism and surgery theory. ([CS1,2] and [L1,2].)

We have the following results on the realization of a quadruple of $(n, n+2)$-knots.

Theorem 2.2. *A quadruple of $(n, n+2)$-knots $T = (K_1, K_2, X_1, X_2)$ is realizable if K_1 and K_2 are slice.*

Kervaire proved that all even dimensional knots are slice ([Ke]). Hence we have:

Corollary 2.3. *If n is even, an arbitrary quadruple of $(n, n+2)$-knots $T = (K_1, K_2, X_1, X_2)$ is realizable.*

In order to prove Theorem 2.1, we introduce a new knotting operation for high dimensional knots, *high dimensional pass-moves.* The 1-dimensional case of Definition 2.1 is discussed on p.146 of [K1].

Definition. Let $(2k+1)$-knot K be defined by a smooth embedding $g : \Sigma^{2k+1} \hookrightarrow S^{2k+3}$, where Σ^{2k+1} is PL homeomorphic to the standard $(2k+1)$-sphere. ($k \geqq 0$.) Let $D_x^{k+1} = \{(x_1, \ldots, x_{k+1}) | \; \Sigma x_i^2 < 1\}$ and $D_y^{k+1} = \{(y_1, \ldots, y_{k+1}) | \; \Sigma y_i^2 < 1\}$. Let $D_x^{k+1}(r) = \{(x_1, \ldots, x_{k+1}) | \; \Sigma x_i^2 \leqq r^2\}$ and $D_y^{k+1}(r) = \{(y_1, \ldots, y_{k+1}) | \; \Sigma y_i^2 \leqq r^2\}$. A local chart (U, ϕ) of S^{2k+3} is called a *pass-move-chart of K* if it satisfies the following conditions.

(1) $\phi(U) \cong \mathbb{R}^{2k+3} = (0,1) \times D_x^{k+1} \times D_y^{k+1}$
(2) $\phi(g(S^{2k+1}) \cap U) = [\{\frac{1}{2}\} \times D_x^{k+1} \times \partial D_y^{k+1}(\frac{1}{3})] \amalg [\{\frac{2}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times D_y^{k+1}]$

Let $g_U : \Sigma^{2k+1} \hookrightarrow S^{2k+3}$ be an embedding such that:

(1) $g | \{\Sigma^{2k+1} - g^{-1}(U)\} = g_U | \{\Sigma^{2k+1} - g^{-1}(U)\}$, and
(2) $\phi(g_U(\Sigma^{2k+1}) \cap U) = [\{\frac{1}{2}\} \times D_x^{k+1} \times \partial D_y^{k+1}(\frac{1}{3})] \amalg$
$[\{\frac{2}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times (D_y^{k+1} - D_y^{k+1}(\frac{1}{2}))]$
$\cup [[\frac{1}{3}, \frac{2}{3}] \times \partial D_x^{k+1}(\frac{1}{3}) \times \partial D_y^{k+1}(\frac{1}{2})]$
$\cup [\{\frac{1}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times D_y^{k+1}(\frac{1}{2})\}]$

Let K_U be the $(2k+1)$-knot defined by g_U. Then we say that K_U is obtained from K by the *(high dimensional) pass-move in U.* We say that $(2k+1)$-knot K and K' are *(high dimensional) pass-move equivalent* if there exist $(2k+1)$-knots $K_1, \ldots, K_{q+1}$ and pass-move charts U_i ($i = 1, \ldots, q$) of K_1 such that (1) $K_1 = K$, $K_{q+1} = K'$, and (2) K_{i+1} is obtained from K_i by the high dimensional pass-move in U_i.

High dimensional pass-moves have the following relation with the Arf invariant and the signature of knots.

Theorem 2.4. *For simple $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. ($k \geqq 1$.)*

(1) *K_1 is pass-move equivalent to K_2.*

(2) *K_1 and K_2 satisfy the condition that* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

See [L2] for simple knots.

Theorem 2.5. *For $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. ($k \geqq 0$.)*

(1) *There exists a $(2k+1)$-knot K_3 which is pass-move equivalent to K_1 and cobordant to K_2.*

(2) *K_1 and K_2 satisfy the condition that* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

The case $k = 0$ of Theorem 2.5 follows from [K1],[K2].

§3

We discuss the case when three spheres intersect in a sphere.

Let F_i be closed surfaces ($i = 1, 2, \ldots, \mu$). A *surface-$(F_1, F_2, \ldots, F_\mu)$-link* is a smooth submanifold $L = (K_1, K_2, \ldots, K_\mu)$ of S^4, where K_i is diffeomorphic to F_i. If F_i is orientable we assume that F_i is oriented and K_i is an oriented submanifold which is orientation preserving diffeomorphic to F_i. If $\mu = 1$, we call L a *surface-F_1-knot.*

An (F_1, F_2)-link $L = (K_1, K_2)$ is called a *semi-boundary link* if

$$[K_i] = 0 \in H_2(S^4 - K_j; \mathbb{Z}) \ (i \neq j)$$

following [S].

An (F_1, F_2)-link $L = (K_1, K_2)$ is called a *boundary link* if there exist Seifert hypersurfaces V_i for K_i ($i = 1, 2$) such that $V_1 \cap V_2 = \phi$.

An (F_1, F_2)-link (K_1, K_2) is called a *split link* if there exist B^4_1 and B^4_2 in S^4 such that $B^4_1 \cap B^4_2 = \phi$ and $K_i \subset B^4_i$.

Definition. Let $L_1 = (K_{12}, K_{13})$, $L_2 = (K_{23}, K_{21})$, and $L_3 = (K_{31}, K_{32})$ be surface-links. (L_1, L_2, L_3) is called a *triple of surface-links* if K_{ij} is diffeomorphic to K_{ji}. ((i,j)=(1,2),(2,3),(3,1).) (Note that the knot type of K_{ij} is different from that of K_{ji}.)

Definition. Let $L_1 = (K_{12}, K_{13})$, $L_2 = (K_{23}, K_{21})$, and $L_3 = (K_{31}, K_{32})$ be surface-links. A triple of surface-links (L_1, L_2, L_3) is said to be *realizable* if there exists a transverse immersion $f : S^4_1 \amalg S^4_2 \amalg S^4_3 \looparrowright S^6$ such that (1)$f|S^4_i$ is an embedding(i=1,2,3), and (2) $(f^{-1}(f(S^4_i) \cap f(S^4_j)),\ f^{-1}(f(S^4_i) \cap f(S^4_k))\ )$ in S^4_i is L_i. ((i,j,k)=(1,2,3), (2,3,1), (3,1,2).)

Note. If (L_1, L_2, L_3) is realizable, then K_{ij} are orientable and are given an orientation naturally. From now on we assume that, when we say a triple of surface-links, the triple of surface-links consists of oriented surface-links.

We state the main theorem.

Theorem 3.1. *Let L_i $(i = 1, 2, 3)$ be semi-boundary surface-links. Suppose the triple of surface-links (L_1, L_2, L_3) is realizable. Then we have the equality*

$$\beta(L_1) + \beta(L_2) + \beta(L_3) = 0,$$

where $\beta(L_i)$ is the Sato-Levine invariant of L_i.

Refer to [S] for the Sato-Levine invariant. Since there exists a triple of surface-links (L_1, L_2, L_3) such that $\beta(L_1)$=0, $\beta(L_2)$=0 and $\beta(L_3)$=1 ([R] and [S]), we have:

Corollary 3.2. *Not all triples of oriented surface-links are realizable.*

We have sufficient conditions for the realization.

Theorem 3.3. *Let L_i $(i = 1, 2, 3)$ be split surface-links. Then the triple of surface-links (L_1, L_2, L_3) is realizable.*

Theorem 3.4. *Suppose L_i are (S^2, S^2)-links. If L_i are slice links$(i = 1, 2, 3)$, then the triple of surface-links (L_1, L_2, L_3) is realizable.*

It is well known that there exists a slice-link which is neither a boundary link nor a ribbon link. Hence we have:

Corollary 3.5. *There exists a realizable triple of surface-links (L_1, L_2, L_3) such that neither L_i are boundary links and all L_i are semi-boundary links.*

Besides the above results, we prove the following triple are realizable.

Theorem 3.6. *There exists a realizable triple of surface-links (L_1, L_2, L_3) such that neither L_i are semi-boundary links.*

Here we state:

Problem (1). *Suppose $\beta(L_1)+\beta(L_2)+\beta(L_3)$=0. Then is the triple of surface-links (L_1, L_2, L_3) realizable?*

Using a result of [O], we can make another problem from Problem (1).

Problem (2). *Is every triple of (S^2, S^2)-links realizable?*

Note. By Theorem 3.4, if the answer to Problem (2) is negative, then the answer to an outstanding problem: "Is every (S^2, S^2)-link slice?" is "no." (Refer [CO] to the slice problem.)

§4

An *(oriented) n-(dimensional) knot* K is a smooth oriented submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}$ which is PL homeomorphic to the standard n-sphere. We say that n-knots K_1 and K_2 are *equivalent* if there exists an orientation preserving diffeomorphism $f : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1} \times \mathbb{R}$ such that $f(K_1)$=K_2 and $f|_{K_1} : K_1 \to K_2$ is an orientation preserving diffeomorphism. Let π: $\mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ be the natural projection map. A subset P of $\mathbb{R}^{n+1}$ is called the *projection* of an n-knot K if there exists an orientation preserving diffeomorphism $\beta : X \to K$ and a smooth transverse immersion $\gamma : X \to \mathbb{R}^{n+1}$ such that $\pi|_K \circ \beta = \gamma$ and $\pi|_K \circ \beta(K) = \gamma(X) = P$. The *singular point set* of the projection of an n-knot is the set $\{ x \in \pi|_K (K) \mid \sharp (\pi|_K)^{-1}(x) \geqq 2. \}$.

It is well-known that the projection of any 1-dimensional knot is the projection of a 1-knot equivalent to the trivial knot. This fact is used in a way to define the Jones polynomial and in another way to define the Conway-Alexander polynomial.

We consider the following problem.

Problem. (1) Let K be an n-knot diffeomorphic to the standard sphere. Let P be the projection of K. Is P the projection of an n-knot equivalent to the trivial knot?

(2) Furthermore, suppose that the singular point set of P consists of double points. Is P the projection of an n-knot equivalent to the trivial knot?

The author proved that the answer to Problem (2) in the case of $n \geqq 3$ is negative and hence that the answer to Problem (1) in the case of $n \geqq 3$ is also negative. We prove:

Theorem 4.1. *Let n be any integer greater than two. There exists an n-knot K such that the projection P has the following properties.*

(1) *P is not the projection of any knot equivalent to the trivial knot.*
(2) *The singular point set of P consists of double points.*
(3) *K is diffeomorphic to the standard sphere.*

Note. Problem (1) in the case of $n = 2$ remains open. Dr. Taniyama has informed that the answer to Problem (2) in the case of $n = 2$ is positive, that the proof is easy, but that he has not published the proof.

§5

We have the following outstanding open problems.

Problem (1). *Classify n-links up to link concordance for $n \geqq 1$.*

Problem (2). *Is every even dimensional link slice?*

Problem (3). *Is every odd dimensional link concordant to (a sublink of) a homology boundary link?*

The author has modified Problem (2) to formulate the following Problem (4). We consider the case of 2-component links. Let $L = (K_1, K_2) \subset S^{n+2} \subset B^{n+3}$ be a $2m$-link ($2m \geq 2$). By Kervaire's theorem in [Ke] there exist D_i^{2m+1} $(i = 1, 2)$ embedded in B^{2m+3} such that $D_i^{2m+1} \cap S^{2m+2} = \partial D_i^{2m+1} = K_i$. Then D_1^{2m+1} and D_2^{2m+1} intersect mutually in general. Furthermore $D_1^{2m+1} \cap D_2^{2m+1}$ in D_i^{2m+1} defines $(2m-1)$-link.

Problem (4). *Can we remove the above intersection $D_1^{2m+1} \cap D_2^{2m+1}$ by modifying embedding of D_1^{2m+1} and D_2^{2m+1}?*

If the answer to Problem (4) is positive, the answer to Problem (2) is positive.

Here we make another problem from Problem (4).

Problem (5). *What is obtained as a pair of $(2m-1)$-links ($D_1^{2m+1} \cap D_2^{2m+1}$ in D_1^{2m+1}, $D_1^{2m+1} \cap D_2^{2m+1}$ in D_2^{2m+1}) by modifying embedding of D_1^{2m+1} and D_2^{2m+1}?*

We have the following theorem, which is an answer to Problem (5).

Theorem 5.1. *For all 2-component $2m$-link $L = (K_1, K_2)$ ($m \geq 0$), there exist D_1^{2m+1} and D_2^{2m+1} as above such that each of $(2m-1)$-links $D_1^{2m+1} \cap D_2^{2m+1}$ in D_1^{2m+1} and $D_1^{2m+1} \cap D_2^{2m+1}$ in D_2^{2m+1} is the trivial knot.*

We have the following Theorem 5.2. We say that n-dimensional knots, K and K', are *(link-)concordant* or *(link-)cobordant* if there is a smooth oriented submanifold C of $S^{n+2} \times [0,1]$, which meets the boundary transversely in ∂C, is PL homeomorphic to $L_0 \times [0,1]$, and meets $S^{n+2} \times \{l\}$ in L_l ($l = 0,1$). Then we call C a *concordance-cylinder of K and K'.*

Theorem 5.2. *For all 2-component n-link $L = (K_1, K_2)$ ($n > 1$), there exist a boundary link $L' = (K_1', K_2')$ satisfying that K_i' is concordant to K_i and a concordance-cylinder $\left\{\begin{array}{l} C_1 \\ C_2 \end{array}\right.$ of $\left\{\begin{array}{l} K_1 \\ K_1' \end{array}\right.$ and $\left\{\begin{array}{l} K_2 \\ K_2' \end{array}\right.$ such that each of $(n-1)$-links, $C_1 \cap C_2$ in C_1 and $C_1 \cap C_2$ in C_2, is the trivial knot.*

When n is even, Theorem 5.2 is Theorem 5.1. Because all even dimensional boundary links are slice.

By the following exciting theorem of Cochran and Orr, when n is odd, Theorem 5.2 is best possible from a viewpoint.

Theorem. *[CO] Not all 2-component odd dimensional links are concordant to boundary links.*

§6

Let D_1^n, D_2^n, D_3^n be submanifolds of S^{n+2} diffeomorphic to the n-disc such that Int $(D_i^n) \cap$ Int $(D_j^n)=\phi$ (for $i \neq j$) and $\partial D_1^n = \partial D_2^n = \partial D_3^n$. Then $D_1^n \cup D_2^n \cup D_3^n$ is called an *n-dimensional θ-curve in $\mathbb{R}^{n+2}$. The set of the constituent knots of an n-dimensional θ-curve θ in $\mathbb{R}^{n+2}$* is a set of three n-knots in S^{n+2}, which are made from $D_1^n \cup D_2^n$, $D_2^n \cup D_3^n$, and $D_1^n \cup D_3^n$.

The definitions in the case of the PL category are written in [Y].

Problem. Take any set of three n-knots. Is it the set of the constituent knots of an n-dimensional θ-curve?

In [Y] it is proved that if K_1, K_2, and K_3 are ribbon n-knots, then the set (K_1, K_2, K_3) is the set of the constituent knots of an n-dimensional θ-curve.

We discuss the case of non-ribbon knots. The following theorems hold both in the smooth category and in the PL category.

We have the following theorems.

Theorem 6.1. *Let n be any positive integer. Let K_1 and K_2 be trivial knots. There exist an n-dimensional θ-curve θ in $\mathbb{R}^{n+2}$ and a non-ribbon knot K_3 such that (K_1, K_2, K_3) is the set of the constituent knots of the n-dimensional θ-curve θ in $\mathbb{R}^{n+2}$.*

Furthermore we have the following.

Theorem 6.2. *Let m be any odd positive integer. Let K_1 and K_2 be trivial knots.*

(1) There exist m-dimensional θ-curves θ in $\mathbb{R}^{m+2}$ and a non-ribbon and non-slice knot K_3 such that (K_1, K_2, K_3) is the set of the constituent knots of the m-dimensional θ-curve θ in $\mathbb{R}^{m+2}$.

(2) There exist m-dimensional θ-curves θ in $\mathbb{R}^{m+2}$ and a non-ribbon and slice knot K_3 such that (K_1, K_2, K_3) is the set of the constituent knots of the m-dimensional θ-curve θ in $\mathbb{R}^{m+2}$.

We have the following.

Theorem 6.3. *When $n = 2m + 1 (m \geq 1)$, there exists a set of three n-knots which is never the set of the constituent knots of any θ-curve.*

The above problem remains open.

§7

We use Theorem 1.1 in §1 to give an answer to a problem of Fox.

In [F] Fox submitted the following problem about 1-links. Here, note that "slice link" in the following problem is now called "ordinary sense slice link," and "slice link in the strong sense" in the following problem is now called "slice link" by knot theorists.

Problem 26 of [F]. Find a necessary condition for L to be a slice link; a slice link in the strong sense.

Our purpose is to give some answers to the former part of this problem. The latter half is not discussed here. The latter half seems discussed much more often than the former half. See e.g. [CO], [L3], etc.

We review the definition of ordinary sense slice links and that of slice links, which we now use.

We suppose m-component 1-links are oriented and ordered.

Let $L = (K_1, \ldots, K_m)$ be an m-component 1-link in $S^3 = \partial B^4$. L is called a *slice 1-link*, which is "a slice link in the strong sense" in the sense of Fox, if there exist 2-discs $D_i^2 (i = 1, \ldots, m)$ in B^4 such that $D_i^2 \cap \partial B^4 = \partial D_i^2$, $D_i^2 \cap D_j^2 = \phi (i \neq j)$, and $(\partial D_1^2, \ldots, \partial D_m^2)$ in ∂B^4 defines L.

Take a 1-link L in S^3. Take S^4 and regard S^4 as $(\mathbb{R}^3 \times \mathbb{R}) \cup \{\infty\}$. Regard the 3-sphere S^3 as $\mathbb{R}^3 \cup \{\infty\}$ in S^4. L is called an *ordinary sense slice 1-link*, which is "a slice link" in the sense of Fox, if there exists an embedding $f : S^2 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}$ such that f is transverse to $\mathbb{R}^3 \times \{0\}$ and $f(S^2) \cap (\mathbb{R}^3 \times \{0\})$ in $\mathbb{R}^3 \times \{0\}$ defines L. Suppose f defines a 2-knot X. Then L is called a *cross-section* of the 2-knot X.

From now on we use the terms in the ordinary sense now current.

Ordinary sense slice 1-links have the following properties.

Theorem 7.1. *Let L be a 1-dimensional ordinary sense slice link. Then the followings hold.*

(1) *L is a proper link.*
(2) $\mathrm{Arf}(L) = 0$.

§8

Let K_1 and K_2 be smooth submanifolds of S^{n+2} diffeomorphic to an n-dimensional closed smooth manifold M. The notion of cobordism between K_1 and K_2 is defined naturally. A Seifert surface of K_i and a Seifert matrix of K_i are defined naturally. The notion of matrix cobordism between two Seifert matrices is defined naturally.

It is also natural to ask the following problem.

Problem. Are K_1 and K_2 as above cobordant?

The author thinks that there exists a kind of surgery exact sequence.

The author obtained the following results.

Theorem 8.1. *There exist a $(2n+1)$-dimensional closed oriented smooth manifold M and smooth submanifolds K_1 and K_2 of S^{2n+3} diffeomorphic to M such that (1) K_1 and K_2 are not cobordant, and (2) the Seifert matrices of K_1 and K_2 are matrix cobordant.*

Theorem 8.2. *There exist a $2n$-dimensional closed oriented smooth manifold M and smooth submanifolds K_1 and K_2 of S^{2n+2} diffeomorphic to M such that K_1 and K_2 are not cobordant*

In the both cases the obstructions live in certain homotopy groups.

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CONTROLLED ALGEBRA AND TOPOLOGY

ERIK KJÆR PEDERSEN

Let R be a ring and $(\overline{X}, \partial X)$ a pair of compact Hausdorff spaces. We assume $X = \overline{X} - \partial X$ is dense in $\overline{X}$.

Definition 1. The continuously controlled category $\mathcal{B}(\overline{X}, \partial X; R)$ has objects $A = \{A_x\}_{x \in X}$, A_x a finitely generated free R-module, satisfying that $\{x | A_x \neq 0\}$ is locally finite in X.

Given a subset U in $\overline{X}$ we define $A|U$ by

$$(A|U)_x = \begin{cases} A_x & \text{if } x \in U \cap X \\ 0 & \text{if } x \notin U \cap X \end{cases}$$

A morphism $\phi \in \mathcal{B}(\overline{X}, \partial X; R)$, is an R-module morphism $\phi : \oplus A_x \to \oplus B_y$ satisfying a continuously controlled condition:

$$\forall z \in \partial X, \forall U \text{ open in } \overline{X}, z \in U, \exists V \text{ open in } \overline{X}, z \in V$$
$$\text{such that } \phi(A|V) \subset A|U \text{ and } \phi(A|X - U) \subset A|X - V$$

Clearly $\mathcal{B}(\overline{X}, \partial X; R)$ is an additive category with $(A \oplus B)_x = A_x \oplus B_x$ as direct sum.

If A is an object of $\mathcal{B}(\overline{X}, \partial X; R)$, then $\{x | A_x \neq 0\}$ has no limit point in X, all limit points must be in ∂X. We denote the set of limit points by $\operatorname{supp}_\infty(A)$. The full subcategory of $\mathcal{B}(\overline{X}, \partial X; R)$ on objects A with

$$\operatorname{supp}_\infty(A) \subset Z \subset \partial X$$

is denoted by $\mathcal{B}(\overline{X}, \partial X; R)_Z$. Putting $\mathcal{U} = \mathcal{B}(\overline{X}, \partial X; R)$ and $\mathcal{A} = \mathcal{B}(\overline{X}, \partial X; R)_Z$, this is a typical example of an $\mathcal{A}$-filtered additive category $\mathcal{U}$ in the sense of Karoubi [6]. The quotient category $\mathcal{U}/\mathcal{A}$ has the same objects as $\mathcal{U}$, but two morphisms are identified if the difference factors through an object of $\mathcal{A}$. In the present example this means two morphisms are identified if they agree on the object restricted to a neighborhood of $\partial X - Z$. We denote $\mathcal{U}/\mathcal{A}$ in this case by $\mathcal{B}(\overline{X}, \partial X; R)^{\partial X - Z}$. Given an object A and a neighborhood W of $\partial X - Z$ we have $A \cong A|W$ in this category.

If R is a ring with involution these categories become additive categories with involution in the sense of Ranicki [7]. It was proved in [2] that

Theorem 2. *There is a fibration of spectra*

$$\mathbb{L}^k(\mathcal{A}) \to \mathbb{L}^h(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A})$$

where k consists of projectives, i.e. objects in the idempotent completion of $\mathcal{A}$, that become free in $\mathcal{U}$, i.e. stably, by adding objects in $\mathcal{U}$ become isomorphic to an object of $\mathcal{U}$.

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Indication of proof. Using the bordism definition of L-spectra of Quinn and Ranicki, it is immediate that we have a fibration of spectra

$$\mathbb{L}^h(\mathcal{A}) \to \mathbb{L}^h(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}, \mathcal{A})$$

An element in $L_n^h(\mathcal{U}, \mathcal{A})$, the n-th homotopy group of $\mathbb{L}^h(\mathcal{U}, \mathcal{A})$ is a pair of chain complexes with boundary in $\mathcal{A}$ and a quadratic Poincaré structure. The boundary is isomorphic to 0 in $\mathcal{U}/\mathcal{A}$ since all $\mathcal{A}$-objects are isomorphic to 0 in $\mathcal{U}/\mathcal{A}$. This produces a map

$$\mathbb{L}^h(\mathcal{U}, \mathcal{A}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A})$$

which we ideally would like to be a homotopy equivalence. Given a quadratic Poincaré complex in $\mathcal{U}/\mathcal{A}$, it is easy to lift the chain complex to a chain complex in $\mathcal{U}$, and to lift the quadratic structure, but it is no longer a Poincaré quadratic structure. We may use [8, Prop. 13.1] to add a boundary so that we lift to a Poincaré pair. It follows that the boundary is contractible in $\mathcal{U}/\mathcal{A}$. It turns out that a chain complex in $\mathcal{U}$ is contractible in $\mathcal{U}/\mathcal{A}$ if and only if the chain complex is dominated by a chain complex in $\mathcal{A}$, and such a chain complex is homotopy equivalent to a chain complex in the idempotent completion of $\mathcal{A}$. This is the reason for the variation in the decorations in this theorem. See [2] for more details. □

Lemma 3. *If $(\overline{X}, \partial X)$ is a compact pair then*

$$\mathcal{B}(\overline{X}, \partial X; R) \cong \mathcal{B}(C\partial X, \partial X; R)$$

Proof. The isomorphism is given by moving the modules A_x, $x \in X$ to point in $C\partial X$, the same module, and if two are put the same place we take the direct sum. On morphisms the isomorphism is induced by the identity, so we have to ensure the continuously controlled condition is not violated. Choose a metric on $\overline{X}$ so that all distances are ≤ 1. Given $z \in X$, let y be a point in ∂X closest to z, and send z to $(1 - d(z, y))y$. Clearly, as z approaches the boundary it is moved very little. In the other direction send $t \cdot y$ to a point in $B(y; 1-t)$, the ball with center y and radius $1-t$, which is furthest away from ∂X. Again moves become small as t approaches 1 or equivalently as the point approaches ∂X. □

Lemma 4. $\mathbb{L}^h(\mathcal{B}(\overline{X}, \partial X; R)_*) \simeq *$

Proof. The first $*$ denotes a point in ∂X and the second that the spectrum is contractible. The proof is an Eilenberg swindle towards the point. □

Theorem 5. [2] *The functor*

$$Y \to L_*^h(\mathcal{B}(CY, Y; \mathbb{Z}))$$

is a generalized homology theory on compact metric spaces

Proof. We have a fibration

$$\mathbb{L}^h(\mathcal{B}(CY, Y; \mathbb{Z}))_Z \to \mathbb{L}^h(\mathcal{B}(CY, Y; \mathbb{Z})) \to \mathbb{L}^h(\mathcal{B}(CY, Y; \mathbb{Z}))^{Y-Z}$$

But an argument similar to the one used in Lemma 3 shows

$$\mathcal{B}(CY, Y; \mathbb{Z})_Z \cong \mathcal{B}(CZ, Z; \mathbb{Z}).$$

When everything is away from Z it does not matter if we collapse Z so we have

$$\mathcal{B}(CY, Y; \mathbb{Z})^{Y-Z} \cong \mathcal{B}((CY)/Z, Y/Z; \mathbb{Z})^{Y/Z - Z/Z},$$

but $Y/Z - Z/Z$ is only one point from Y/Z so by Lemma 4 the $\mathbb{L}$-spectrum is homotopy equivalent to $\mathbb{L}((CY)/Z, Y/Z; \mathbb{Z})$. Finally Lemma 3 shows that

$$\mathcal{B}(CY/Z, Y/Z; \mathbb{Z}) \cong \mathcal{B}(C(Y/Z), Y/Z); \mathbb{Z}).$$

and we are done. □

Consider a compact pair (X, Y) so that $X - Y$ is a CW-complex. If we subdivide so that cells in $X - Y$ become small near Y, the cellular chain complex $C_\sharp(X - Y; \mathbb{Z})$ may be thought of as a chain complex in $\mathcal{B}(X, Y; \mathbb{Z})$ simply by choosing a point in each cell (a choice which is no worse than the choice of the cellular structure.) If we have a strict map

$$(f, 1_Y) : (W, Y) \to (X, Y)$$

(meaning $f^{-1}(X - Y) \subset W - Y$) it is easy to see that given appropriate local simple connectedness conditions, this map is a strict homotopy equivalence (homotopies through strict maps) if and only if the induced map is a homotopy equivalence of chain complexes in $\mathcal{B}(X, Y; \mathbb{Z})$. If the fundamental group of $X - Y$ is π and the universal cover satisfies the appropriate simply connectedness conditions, strict homotopy equivalence is measured by chain homotopy equivalences in $\mathcal{B}(X, Y; \mathbb{Z}\pi)$. We have the ingredients of a surgery theory which may be developed along the lines of [4] with a surgery exact sequence

$$\to L^h_{n+1}(\mathcal{B}(X, Y; \mathbb{Z}\pi)) \to \mathcal{S}^h_{cc}\begin{pmatrix} X-Y \\ \downarrow \\ X \end{pmatrix} \to [X - Y; F/\operatorname{Top}] \to$$

We will use this sequence to discuss a question originally considered in [1].

Suppose a finite group π acts freely on S^{n+k} fixing S^{k-1}, a standard $k-1$-dimensional subsphere. We may suspend this action to an action on S^{n+k+1} fixing S^k and the question arises whether a given action can be desuspended. Notice this question is only interesting in the topological category. In the PL or differentiable category it is clear that all such actions can be maximally desuspended, by taking a link or by an equivariant smooth normal bundle consideration.

Denoting $(S^{n+k} - S^{k-1})/\pi$ by X, X is the homotopy type of a Swan complex (a finitely dominated space with universal cover homotopy equivalent to a sphere). The strict homotopy type of $(S^{n+k}/\pi, S^{k-1})$ can be seen to be $(X * S^{k-1}, S^{k-1})$, [1], and if we have a strict homotopy equivalence from a manifold to $X * S^{k-1} - S^{k-1}$ it is easy to see that we may complete to get a semifree action on a sphere fixing a standard subsphere. This means that this kind of semifree action is classified by the surgery exact sequence

$$\to L_{n+1}(\mathcal{B}(D^k, S^{k-1}; \mathbb{Z}\pi)) \to \mathcal{S}^h_{cc}\begin{pmatrix} X*S^{k-1} - S^{k-1} \\ \downarrow \\ X*S^{k-1} \end{pmatrix} \to [X, F/\operatorname{Top}] \to$$

Now let $\mathcal{C}(\mathbb{R}^n; R)$ denote the subcategory of $\mathcal{B}(\mathbb{R}^n, \emptyset; R)$ where the morphisms are required to be bounded i. e. $\phi : A \to B$ has to satisfy that there exists $k = k(\phi)$ so that $\phi^y_x = 0$ if $|x - y| > k$. Radial shrinking defines a functor $\mathcal{C}(\mathbb{R}^n, R) \to \mathcal{B}(D^n, S^{n-1}; R)$, and it is easy to see by the kind of arguments developed above that this functor induces isomorphism in L-theory. We get a map from the bounded surgery exact sequence to the continuously controlled surgery exact sequence

$$\begin{tikzcd}
{} \arrow[r] & L^h_{n+1}(\mathcal{C}(\mathbb{R}^k;\mathbb{Z}\pi)) \arrow[r] \arrow[d] & \mathcal{S}^h_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \arrow[r] \arrow[d] & {[X, F/\operatorname{Top}]} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & L^h_{n+1}(\mathcal{B}(D^k, S^{k-1};\mathbb{Z}\pi)) \arrow[r] & \mathcal{S}^h_{cc}\begin{pmatrix} X*S^{k-1}-S^{k-1} \\ \downarrow \\ D^k \end{pmatrix} \arrow[r] & {[X, F/\operatorname{Top}]} \arrow[r] & {}
\end{tikzcd}$$

which is an isomorphism on two out of three terms, hence also on the structure set. This is useful because we can <u>not</u> define an operation corresponding to suspension of the action on the continuously controlled structure set. An attempt would be to cross with an open interval, but an open interval would have to have a specific cell structure to get a controlled algebraic Poincaré structure on the interval, but then we would lose control along the suspension lines. In the bounded context suspension corresponds precisely to crossing with the reals, and giving the reals a bounded triangulation we evidently have no trouble getting a map corresponding to crossing with $\mathbb{R}$. Since crossing with $\mathbb{R}$ kills torsion (think of crossing with $\mathbb{R}$ as crossing with S^1 and pass to the universal cover), we get a map from the h-structure set to the s-structure set. The desuspension problem is now determined by the diagram

$$\begin{tikzcd}
{} \arrow[r] & L^h_{n+1}(\mathcal{C}(\mathbb{R}^k;\mathbb{Z}\pi)) \arrow[r] \arrow[d] & \mathcal{S}^h_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \arrow[r] \arrow[d] & {[X, F/\operatorname{Top}]} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & L^s_{n+2}(\mathcal{C}(\mathbb{R}^{k+1};\mathbb{Z}\pi)) \arrow[r] & \mathcal{S}^s_b\begin{pmatrix} X\times\mathbb{R}^{k+1} \\ \downarrow \\ \mathbb{R}^{k+1} \end{pmatrix} \arrow[r] & {[X, F/\operatorname{Top}]} \arrow[r] & {}
\end{tikzcd}$$

with two out of three maps isomorphisms once again. This shows we may desuspend if and only if the element in the structure set can be thought of as a simple structure, i.e. if and only if an obstruction in

$$Wh(\mathcal{C}(\mathbb{R}^{k+1};\mathbb{Z}\pi)) = K_1(\mathcal{C}(\mathbb{R}^{k+1};\mathbb{Z}\pi))/\{\pm\pi\} = K_{-k}(\mathbb{Z}\pi)$$

vanishes. Since $K_{-k}(\mathbb{Z}\pi) = 0$ for $k \geq 2$ [3], this means we can always desuspend untill we have a fixed circle, but then we encounter a possible obstruction. The computations in [5] show these obstructions are realized.

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PROBLEMS IN LOW-DIMENSIONAL TOPOLOGY

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Introduction

Four-dimensional topology is in an unsettled state: a great deal is known, but the largest-scale patterns and basic unifying themes are not yet clear. Kirby has recently completed a massive review of low-dimensional problems [Kirby], and many of the results assembled there are complicated and incomplete. In this paper the focus is on a shorter list of "tool" questions, whose solution could unify and clarify the situation. However we warn that these formulations are implicitly biased toward positive solutions. In other dimensions tool questions are often directly settled one way or the other, and even a negative solution leads to a general conclusion (eg. surgery obstructions, Whitehead torsion, characteristic classes, etc). In contrast, failures in dimension four tend to be indirect inferences, and study of the failure leads nowhere. For instance the failure of the disk embedding conjecture in the smooth category was inferred from Donaldson's nonexistence theorems for smooth manifolds. And although some direct information about disks is now available, eg. [Kr], it does not particularly illuminate the situation.

Topics discussed are: in section 1, embeddings of 2-disks and 2-spheres needed for surgery and s-cobordisms of 4-manifolds. Section 2 describes uniqueness questions for these, arising from the study of isotopies. Section 3 concerns handlebody structures on 4-manifolds. Finally section 4 poses a triangulation problem for certain low-dimensional stratified spaces.

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1: 2-disks and spheres in 4-manifolds

The target results here are surgery and the s-cobordism theorem. In general these are reduced, via handlebody theory, to questions about disks and spheres in the middle dimension of the ambient manifold. The tool results, hence the targets, are known in the topological category for 4-manifolds when the fundamental group is "small", [FQ, FT1], but are unsettled in general.

Two n-dimensional submanifolds of a manifold of dimension $2n$ will usually intersect themselves and each other in isolated points. The "Whitney trick" uses an isotopy across an embedded 2-disk to simplify these intersections. Roughly speaking this reduces the study of n-dimensional embeddings to embeddings of 2-disks. But this is not a reduction when the dimension is 4: the 2-disks themselves are middle-dimensional, so trying to embed them encounters exactly the same problems they are supposed to solve. This is the phenomenon that separates dimension 4 from others. The central conjecture is that some embeddings exist in spite of this problem.

1.1 Disk conjecture. *Suppose A is an immersion of a 2-disk into a 4-manifold, boundary going to boundary, and there is a framed immersed 2-sphere B with trivial algebraic selfintersection and algebraic intersection 1 with A. Then there is an embedded 2-disk with the same framed boundary as A*

If this were true then the whole apparatus of high-dimensional topology would apply in dimension 4. There are very interesting generalizations, which for example ask about the minimal genus of an embedded surface with a given boundary, or in a given homology class (cf. [Kirby, Problem 4.36]). However the data in 1.1 is available in the Whitney disk applications, so its inclusion reflects the "tool" orientation of this paper.

The conjecture is very false for smooth embeddings, since it would imply existence and uniqueness results that are known to be false [Kirby Problems 4.1, 4.6]. It may be true for topological (locally flat) embeddings. The current best results are by Freedman and Teichner [FT1, FT2]. In [FT1] they show that the conjecture as stated holds if the fundamental group of the 4-manifold has "subexponential growth," while [FT2] gives a technical but useful statement about embeddings when the 4-manifold changes slightly. We briefly discuss the proofs.

For surfaces in 4-manifolds here is a correspondence between intersections and fundamental group of the image: adding an intersection point enlarges the fundamental group of the image by one free generator (if the image is connected). Freedman's work roughly gives a converse: in order to remove intersections in M, it is sufficient to kill the image of the fundamental group of the data, in the fundamental group of M. More precisely, if we add the hypothesis that $A \cap B$ is a single point, and π_1 of the image $A \cup B$ is trivial in $\pi_1 M$ then there is an embedded disk. However applications of this depend on the technology for reducing images in fundamental groups. Freedman's earlier work showed (essentially) how to change A and B so the fundamental group image becomes trivial under any $\phi\colon \pi_1 M \to G$, where G is poly-(finite or cyclic). [FT] improves this to allow G of subexponential growth. Quite a lot of effort is required for this rather minute advance, giving the impression that we are near the limits of validity of the theorem. In a nutshell, the new ingredient is the use of (Milnor) link homotopy. Reduction of fundamental group images is achieved by trading an intersection with a nontrivial loop for a great many intersections with trivial, or at least smaller, loops. The delicate point is to avoid reintroducing big loops through unwanted intersections. The earlier argument uses explicit moves. The approach in [FT1] uses a more efficient abstract existence theorem. The key is to think of a collection of disks as a nullhomotopy of a link. Selfintersections are harmless, while intersections between different components are deadly. Thus the nullhomotopies needed are exactly the ones studied by Milnor, and existence of the desired disks can be established using link homotopy invariants.

While the conjecture is expected to be false for arbitrary fundamental groups, no proof is in sight. Constructing an invariant to detect failure is a very delicate limit problem. The fundamental group of the image of the data can be compressed into arbitrarily far-out terms in the lower central series of the fundamental group of M. If it could be pushed into the intersection the general conjecture would follow. (This is because it is sufficient to prove the conjecture for M with free fundamental group, eg. a regular neighborhood of the data, and the intersection of the lower central series of a free group is trivial). One approach is to develop a notion of nesting of data so that the intersection of an infinite nest gives something useful. Then in order for the theorem to fail there must be data with no properly nested subdata, and maybe this can be detected.

There is a modification of the conjecture, in which we allow the ambient manifold to change by s-cobordism. This form implies that "surgery" works, but not the s-cobordism theorem. [FQ, 6] shows that if the fundamental group of the image of the data of 1.1 is trivial in the whole manifold, then there is an embedding up to s-cobordism. This differs from the hypothesis of the version above in that $A \cap B$ is not required to be one point, just algebraically 1. The improvement of [FT2] is roughly that infinitesimal holes are allowed in the data. A regular neighborhood of the data gives a 4-manifold with boundary, and carrying certain homology classes. In the regular neighborhood the homology class is represented by a sphere, since a sphere is given in the data. The improvement relaxes this: the homology class is required to be in a certain subgroup of H_2, but not necessarily in the image of π_2. Heuristically we can drill a hole in the sphere, as long as it is small enough not to move it too far out of π_2 (technically, still in the ω term of Dwyer's filtration on H_2).

The improved version has applications, but again falls short of the full conjecture. Again it is a limit problem: we can start with arbitrary data and drill very small holes to get the image π_1 trivial in M. The holes can be made "small" enough that the resulting homology classes are in an arbitrarily far-out term in the Dwyer filtration, but maybe not in the infinite intersection.

There is still room for hope that this form of the conjecture is true, but it may require a more elaborate construction or another infinite process. A "shell game" approach would begin with arbitrary data, introduce some $S^2 \times S^2$ summands, and use them as gently as possible to represent the original data as a π_1-trivial submanifold with homology in Dwyer's ω term. The $S^2 \times S^2$'s are now messed up, and to repair this we want to represent them also with π_1-trivial submanifolds with ω-filtration homology. The new advantage is that the data is no longer random, given by an abstract existence theorem, but is obtained from an embedding by carefully controlled damage done in the first step. An infinite swindle would involve introducing infinitely many copies of $S^2 \times S^2$ and moving the damage down the line. The objective would be to do this with control on sizes, so the construction will converge in an appropriate sense (see [BFMW]). The limit should be an ANR homology 4-manifold, but this can be resolved to regain a topological manifold [Q1].

2: Uniqueness

The uniqueness question we want to address is: when are two homeomorphisms of a 4-manifold topologically isotopic? This is known for compact 1-connected 4-manifolds [Q2], but not for nontrivial groups even in the good class for surgery.

Neither is there a controlled version, not even in the 1-connected case. The controlled version may be more important than general fundamental groups, since it is the main missing ingredient in a general topological isotopy extension theorem for stratified sets [Q4].

The study of isotopies is approached in two steps. First determine if two homeomorphisms are concordant (pseudoisotopic), then see if the concordance is an isotopy. The first step still works for 4-manifolds, since it uses 5-dimensional surgery. The high-dimensional approach to the second step [HW] reduces it to a "tool" question. However the uniqueness tool question is *not* simply the uniqueness analog of the existence question. In applications Conjecture 1.1 would be used to find Whitney disks to manipulate 2-spheres. The tool question needed to analyse isotopies directly concerns these Whitney disks.

Conjecture 2.1. *Suppose A and B, are framed embedded families of 2-spheres, and V, W are two sets of Whitney disks for eliminating AB intersections. Each set of Whitney disks reduces the intersections to make the families transverse: the spheres in A and B are paired, and the only intersections are a single point between each pair. Then the sets V, W equivalent up to isotopy and disjoint replacement.*

"Isotopic" means there is an ambient isotopy that preserves the spheres A, B setwise, and takes one set of disks to the other. Note that $A \cap B$ must be pointwise fixed under such an isotopy. "Disjoint replacement" means we declare two sets to be equivalent if the only intersections are the endpoints (in $A \cap B$). Actually there are further restrictions on framings and π_2 homotopy classes, related to Hatcher's secondary pseudoisotopy obstruction [HW]. In practice these do not bother us because the work is done in a relative setting that encodes a vanishing of the high-dimensional obstruction: we try to show that a 4-dimensional concordance is an isotopy if and only if the product with a disk is an isotopy. In [Q2] this program is reduced to conjecture 2.1. The conjecture itself is proved for simply connected manifolds and A, B each a single sphere.

Consider the boundary arcs of the disks V and W, on A and B. These fit together to form circles and arcs: each intersection point in $A \cap B$ is an endpoint of exactly one arc in each of $V \cap A$ and $W \cap A$ unless it is one of the special intersections left at the end of one of the deformations. Thus there is exactly one arc on each sphere. The proof of [Q1] works on the arcs. Focus on a single pair of spheres. The 1-connectedness is used to merge the circles into the arc. Intersections among Whitney disks strung out along the arc are then "pushed off the end" of the arc. This makes the two sets of disks equivalent in the sense of 2.1, and allows them to be cancelled from the picture. Finitely many pairs can be cancelled by iterating this, but this cannot be done with control since each cancellation will greatly rearrange the remaining spheres. To get either nontrivial fundamental groups or control will require dealing directly with the circles of Whitney arcs.

3: 4-DIMENSIONAL HANDLEBODIES

Handlebody structures on 4-manifolds correspond exactly to smooth structures. The targets in studying handlebody structures are therefore the detection and manipulation of smooth structures. However these are much more complicated than in other dimensions, and we are not yet in a position to identify tool questions that might unravel them. Consequently the questions in this section suggest useful directions rather than specific problems.

The first problem concerns detection of structures. The Donaldson and Seiberg-Witten invariants are defined using global differential geometry. But since a handlebody structure determines a smooth structure, these invariants are somehow encoded in the handle structure. There can be no direct topological understanding of these structures until we learn to decode this.

3.1: Problem. *Find a combinatorially-defined topological quantum field theory that detects exotic smooth structures.*

Three-dimensional combinatorial field theories were pioneered by Reshetikhin and Turaev [RT]. They attracted a lot of attention for a time but have not yet led to anything really substantial. Four-dimensional attempts have not gotten anywhere, cf. [CKY]. The Donaldson and Seiberg-Witten invariants do not satisfy the full set of axioms currently used to define a "topological quantum field theory", so there is no guarantee that working in this framework will ever lead anywhere. Nonetheless this is currently our best hope, and a careful exploration of it will probably be necessary before we can see something better.

4-dimensional handlebodies are described by their attaching maps, embeddings of circles and 2-spheres in 3-manifolds. The dimension is low enough to draw explicit pictures of many of these. Kirby developed notations and a "calculus" of such pictures for 1- and 2-handles, cf. [HKK]. This approach has been used to analyse specific manifolds; a good example is Gompf's identification of some homotopy spheres as standard [Gf]. However this approach has been limited even in the study of examples because:

(1) it only effectively tracks 1- and 2-handles, and Gompf's example shows one cannot afford to ignore 3-handles;
(2) it is a non-algorithmic "art form" that can hide mistakes from even skilled practitioners; and
(3) there is no clue how the pictures relate to effective (eg. Donaldson and Seiberg-Witten) invariants.

The most interesting possibility for manipulating handlebodies is suggested by the work of Poenaru on the 3-dimensional Poincaré conjecture. The following is suggested as a test problem to develop the technique:

3.2 Conjecture. *A 4-dimensional (smooth) s-cobordism without 1-handles is a product.*

Settling this would be an important advance, but a lot of work remains before it would have profound applications. To some extent it would show that the real problem is getting rid of 1-handles ([Kirby Problems 4.18, 4.88, 4.89]). It might have some application to this: if we can arrange that some subset of the 2-handles together with the 1-handles forms an s-cobordism, then the dual handlebody structure has no 1-handles and the conjecture would apply. Replacing these 1- and 2-handles with a product structure gives a new handlebody without 1-handles. The problem encountered here is control of the fundamental group of the boundary above the 2-handles. The classical manipulations produce a homology s-cobordism (with $\mathbb{Z}[\pi_1]$ coefficients), but to get a genuine s-cobordism we need for the new boundary to have the same π_1. Thus to make progress we would have to understand the relationship between things like Seiberg-Witten invariants and restrictions on fundamental groups of boundaries of sub-handlebodies.

To analyse the conjecture consider the level between the 2- and 3-handles in the s-cobordism. The attaching maps for the 3-handles are 2-spheres, and the dual spheres of the 2-handles are circles. The usual manipulations arrange the algebraic intersection matrix between these to be the identity. In other dimensions the next step is to realize this geometrically: find an isotopy of the circles so each has exactly one point of intersection with the family of spheres. But the usual methods fail miserably in this dimension. V. Poenaru has attacked this problem in the special case of $\Delta \times I$, where Δ is a homotopy 3-ball, [P, Gi]. The rough idea is an infinite process in which one repeatedly introduces new cancelling pairs of 2- and 3-handles, then damages these in order to fix the previous ones. The limit has an infinite collections of circles and spheres with good intersections. Unfortunately this limit is a real mess topologically, in terms of things converging to each other. The goal is to see that, by being incredibly clever and careful, one can arrange the spheres to converge to a singular lamination with control on the fundamental groups of the complementary components. As an outline this makes a lot of sense. Unfortunately Poenaru's manuscript is extremely long and complicated, and as a result of many years of work without feedback from the rest of the mathematical community, is quite idiosyncratic. It would probably take years of effort to extract clues from this on how to deal with the difficult parts.

4: Stratified spaces

A class of stratified spaces with a relatively weak relationship between the strata has emerged as the proper setting for purely topological stratified questions, see eg. [Q3, W]. The analysis of these sets, to obtain results like isotopy extension theorems, uses a great deal of handlebody theory, etc., so often requires the assumption that all strata have dimension 5 or greater. This restriction is acceptable in some applications, for example in group actions, but not in others like smooth singularity theory, algebraic varieties, and limit problems in differential geometry. The suggestion here is that many of the low-dimensional issues can be reduced to (much easier) PL and differential topology. The conjecture, as formulated, is a tool question for applications of stratified sets. After the statement we discuss it's dissection into topological tool questions.

4.1: Conjecture. *A three-dimensional homotopically stratified space with manifold strata is triangulable. A 4-dimensional space of this type is triangulable in the complement of a discrete set of points.*

As stated this implies the 3-dimensional Poincaré conjecture. To avoid this assume either that there are no fake 3-balls below a certain diameter, or change the statement to "obtained from a polyhedron by replacing sequences of balls converging to the 2-skeleton by fake 3-balls." The "Hauptvermutung" for 3-dimensional polyhedra [Papa] asserts that homeomorphisms are isotopic to PL homeomorphisms. This reduces the 3-dimensional version to showing that stratified spaces are locally triangulable. The 2-skeleton and its complement are both triangulable, so the problem concerns how the 3-dimensional part approaches neighborhoods of points in the 2-skeleton. Consider a manifold point in the skeleton; a neighborhood in the skeleton is isomorphic with $\mathbb{R}^n$ for $n = 0, 1$, or 2. Near this the 3-stratum looks locally homotopically like a fibration over $\mathbb{R}^n$ with fiber a Poincaré space of dimension $3 - n - 1$. We can reduce to the case where the fiber is connected by considering components of the 3-stratum one at a time. If $n = 2$ then the fiber is a point, and the union of the two strata is a homology 3-manifold with $\mathbb{R}^2$ as boundary. Thus the question: is this union a manifold, or equivalently, is the $\mathbb{R}^2$ collared in the union? If $n = 1$ then the fiber is S^1, and the union gives an arc homotopically tamely embedded in the interior of a homology 3-manifold. Is it locally flat? Finally if $n = 0$ then the fiber is a surface (2-dimensional Poincaré spaces are surfaces, [EL]). This is an end problem: if a 3-manifold has a tame end homotopic to $S \times \mathbb{R}$, S a surface, is the end collared? Answers to these are probably known. The next step is to consider a point in the closure of strata of three different dimensions. There are three cases: $(0, 1, 3)$, $(0, 2, 3)$ and $(1, 2, 3)$. Again each case can be described quite explicitly, and should either be known or accessible to standard 3-manifold techniques.

Now consider 4-dimensional spaces. 4-manifolds are triangulable in the complement of a discrete set, so again the question concerns neighborhoods of the 3-skeleton. In dimension 4 homeomorphism generally does not imply PL isomorphism, so this does not immediately reduce to a local question. However the objective is to construct bundle-like structures in a neighborhood of the skeleton, and homeomorphism of total spaces of bundles in most cases will imply isomorphism of bundles. So the question might be localized in this way, or just approached globally using relative versions of the local questions. As above we start with manifold points in the skeleton. If the point has a 2- or 3-disk neighborhood then the question reduces to local flatness of boundaries or 2-manifolds in a homology 4-manifold, see [Q2, FQ 9.3A]. If the point has a 1-disk neighborhood then a neighborhood looks homotopically like the mapping cylinder of a surface bundle over $\mathbb{R}$. This leads to the question: is it homeomorphic to such a mapping cylinder? If the surface fundamental group has subexponential growth then this probably can be settled by current techniques, but the general case may have to wait on solution of the conjectures of section 1. Finally neighborhoods of isolated points in the skeleton correspond exactly to tame ends of 4-manifolds. Some of these are known not to be triangulable, so these would have to be among the points that the statement allows to be deleted. From here the analysis progresses to points in the closure of strata of three or four different dimensions. Again there are a small number of cases, each of which has a detailed local homotopical description.

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45 SLIDES ON CHAIN DUALITY

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Abstract The texts of 45 slides[1] on the applications of chain duality to the homological analysis of the singularities of Poincaré complexes, the double points of maps of manifolds, and to surgery theory.

1. Introduction

- Poincaré duality
$$H^{n-*}(M) \;\cong\; H_*(M)$$
is the basic algebraic property of an n-dimensional manifold M.
- A chain complex C with n-dimensional Poincaré duality
$$H^{n-*}(C) \;\cong\; H_*(C)$$
is an algebraic model for an n-dimensional manifold, generalizing the intersection form.
- Spaces with Poincaré duality (such as manifolds) determine Poincaré duality chain complexes in additive categories with chain duality, giving rise to interesting invariants, old and new.

2. What is chain duality?

- $\mathbb{A}$ = additive category.
- $\mathbb{B}(\mathbb{A})$ = additive category of finite chain complexes in $\mathbb{A}$.
- A contravariant additive functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ extends to
$$T : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A}) \;;\; C \to T(C)$$
by the total double complex
$$T(C)_n = \sum_{p+q=n} T(C_{-p})_q \;.$$
- **Definition**: A chain duality (T, e) on $\mathbb{A}$ is a contravariant additive functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$, together with a natural transformation $e : T^2 \to 1 : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ such that for each object A in $\mathbb{A}$:
  - $e(T(A))\,.\,T(e(A)) = 1 : T(A) \to T(A)$,
  - $e(A) \colon T^2(A) \to A$ is a chain equivalence.

[1] The lecture at the conference on Surgery and Geometric Topology, Josai University, Japan on 17 September, 1996 used slides 1.–36.

3. Properties of chain duality

- The dual of an object A is a chain complex $T(A)$.
- The dual of a chain complex C is a chain complex $T(C)$.
- Motivated by Verdier duality in sheaf theory.
- A.Ranicki, Algebraic L-theory and topological manifolds, Tracts in Mathematics 102, Cambridge (1992)

4. Involutions

- An involution (T, e) on an additive category $\mathbb{A}$ is a chain duality such that $T(A)$ is a 0-dimensional chain complex (= object) for each object A in $\mathbb{A}$, with $e(A) : T^2(A) \to A$ an isomorphism.
- **Example**: An involution $R \to R; r \to \overline{r}$ on a ring R determines the involution (T, e) on the additive category $\mathbb{A}(R)$ of f.g. free left R-modules:
  - $T(A) = \mathrm{Hom}_R(A, R)$
  - $R \times T(A) \to T(A)\ ;\ (r, f) \to (x \to f(x)\overline{r})$
  - $e(A)^{-1} : A \to T^2(A)\ ;\ x \to (f \to \overline{f(x)})$.

5. Manifolds and homeomorphisms up to homotopy

- Traditional questions of surgery theory:
  - Is a space with Poincaré duality homotopy equivalent to a manifold?
  - Is a homotopy equivalence of manifolds homotopic to a homeomorphism?
- Answered for dimensions ≥ 5 by surgery exact sequence in terms of the assembly map

$$A : H_*(X; \mathbb{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi_1(X)])\ .$$

- L-theory of additive categories with involution suffices for surgery groups $L_*(\mathbb{Z}[\pi_1(X)])$.
- Need chain duality for the generalized homology groups $H_*(X; \mathbb{L}_\bullet(\mathbb{Z}))$ and A.

6. Manifolds and homeomorphisms

- Will use chain duality to answer questions of the type:
  - Is a space with Poincaré duality a manifold?
  - Is a homotopy equivalence of manifolds a homeomorphism?

7. Controlled topology

- Controlled topology (Chapman-Ferry-Quinn) considers:
  - the approximation of manifolds by Poincaré complexes,
  - the approximation of homeomorphisms of manifolds by homotopy equivalences.
- Philosophy of controlled topology, with control map $1 : X \to X$:
  - A Poincaré complex X is a homology manifold if and only if it is an ϵ-controlled Poincaré complex for all $\epsilon > 0$.
  - A map of homology manifolds $f : M \to X$ has contractible point inverses if and only if it is an ϵ-controlled homotopy equivalence for all $\epsilon > 0$.

8. Simplicial complexes

- In dealing with applications of chain duality to topology will only work with (connected, finite) simplicial complexes and (oriented) polyhedral homology manifolds and Poincaré complexes.
- Can also work with Δ-sets and topological spaces, using the methods of:
  - M.Weiss, Visible L-theory, Forum Math. 4, 465–498 (1992)
  - S.Hutt, Poincaré sheaves on topological spaces, Trans. A.M.S. (1996)

9. Simplicial control

- Additive category $\mathbb{A}(\mathbb{Z}, X)$ of X-controlled $\mathbb{Z}$-modules for a simplicial complex X.
  - A.Ranicki and M.Weiss, Chain complexes and assembly, Math. Z. 204, 157–186 (1990)
- Will use chain duality on $\mathbb{A}(\mathbb{Z}, X)$ to obtain homological obstructions for deciding:
  - Is a simplicial Poincaré complex X a homology manifold? (Singularities)
  - Does a degree 1 map $f : M \to X$ of polyhedral homology manifolds have acyclic point inverses? (Double points)
- Acyclic point inverses $\widetilde{H}_*(f^{-1}(x)) = 0$ is analogue of homeomorphism in the world of homology.

10. The X-controlled $\mathbb{Z}$-module category $\mathbb{A}(\mathbb{Z}, X)$

- X = simplicial complex.
- A $(\mathbb{Z}, X)$-module is a finitely generated free $\mathbb{Z}$-module A with direct sum decomposition

$$A = \sum_{\sigma \in X} A(\sigma) .$$

- A $(\mathbb{Z}, X)$-module morphism $f : A \to B$ is a $\mathbb{Z}$-module morphism such that

$$f(A(\sigma)) \subseteq \sum_{\tau \geq \sigma} B(\tau) .$$

- **Proposition**: A $(\mathbb{Z}, X)$-module chain map $f : C \to D$ is a chain equivalence if and only if the $\mathbb{Z}$-module chain maps

$$f(\sigma, \sigma) : C(\sigma) \to D(\sigma) \ (\sigma \in X)$$

  are chain equivalences.

11. Functorial formulation

- Regard simplicial complex X as the category with:
  - objects: simplexes $\sigma \in X$
  - morphisms: face inclusions $\sigma \leq \tau$.
- A $(\mathbb{Z}, X)$-module $A = \sum_{\sigma \in X} A(\sigma)$ determines a contravariant functor

$$[A] : X \to \mathbb{A}(\mathbb{Z}) = \{\text{f.g. free abelian groups}\} \ ; \ \sigma \to [A][\sigma] = \sum_{\tau \geq \sigma} A(\tau) .$$

- The $(\mathbb{Z},X)$-module category $\mathbb{A}(\mathbb{Z},X)$ is a full subcategory of the category of contravariant functors $X \to \mathbb{A}(\mathbb{Z})$.

12. Dual cells

- The <u>barycentric subdivision</u> X' of X is the simplicial complex with one n-simplex $\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n$ for each sequence of simplexes in X
$$\sigma_0 < \sigma_1 < \cdots < \sigma_n \ .$$
- The <u>dual cell</u> of a simplex $\sigma \in X$ is the contractible subcomplex
$$D(\sigma,X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n \,|\, \sigma \leq \sigma_0\} \subseteq X' \ ,$$
with boundary
$$\partial D(\sigma,X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n \,|\, \sigma < \sigma_0\} \subseteq D(\sigma,X) \ .$$
- Introduced by Poincaré to prove duality.
- A simplicial map $f : M \to X'$ has acyclic point inverses if and only if
$$(f|)_* : H_*(f^{-1}D(\sigma,X)) \cong H_*(D(\sigma,X)) \ (\sigma \in X) \ .$$

13. Where do $(\mathbb{Z},X)$-module chain complexes come from?

- For any simplicial map $f : M \to X'$ the simplicial chain complex $\Delta(M)$ is a $(\mathbb{Z},X)$-module chain complex:
$$\Delta(M)(\sigma) = \Delta(f^{-1}D(\sigma,X), f^{-1}\partial D(\sigma,X))$$
with a degreewise direct sum decomposition
$$[\Delta(M)][\sigma] = \sum_{\tau \geq \sigma} \Delta(M)(\tau) = \Delta(f^{-1}D(\sigma,X)) \ .$$
- The simplicial cochain complex $\Delta(X)^{-*}$ is a $(\mathbb{Z},X)$-module chain complex with:
$$\Delta(X)^{-*}(\sigma)_r = \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

14. The $(\mathbb{Z},X)$-module chain duality

- **Proposition**: The additive category $\mathbb{A}(\mathbb{Z},X)$ of $(\mathbb{Z},X)$-modules has a chain duality (T,e) with
$$\boxed{T(A) \ = \ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X)^{-*},A),\mathbb{Z})}$$
- $TA(\sigma) = [A][\sigma]^{|\sigma|-*}$
- $T(A)_r(\sigma) \ = \ \begin{cases} \sum\limits_{\tau \geq \sigma} \mathrm{Hom}_{\mathbb{Z}}(A(\tau),\mathbb{Z}) & \text{if } r = -|\sigma| \\ 0 & \text{if } r \neq -|\sigma| \end{cases}$
- $T(C) \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C,\Delta(X'))^{-*} \simeq_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z})^{-*}$
- $T(\Delta(X')) \simeq_{(\mathbb{Z},X)} \Delta(X)^{-*}$
- <u>Terminology</u> $T(C)^{n-*} \ = \ T(C_{*+n}) \ (n \geq 0)$

15. Products

- The product of $(\mathbb{Z},X)$-modules A,B is the $(\mathbb{Z},X)$-module
$$A \otimes_{(\mathbb{Z},X)} B = \sum_{\lambda,\mu\in X,\lambda\cap\mu\neq\emptyset} A(\lambda)\otimes_{\mathbb{Z}} B(\mu) \subseteq A\otimes_{\mathbb{Z}} B ,$$
$$(A \otimes_{(\mathbb{Z},X)} B)(\sigma) = \sum_{\lambda,\mu\in X,\lambda\cap\mu=\sigma} A(\lambda)\otimes_{\mathbb{Z}} B(\mu) .$$
- $C\otimes_{(\mathbb{Z},X)} \Delta(X') \simeq_{(\mathbb{Z},X)} C$.
- $T(C)\otimes_{(\mathbb{Z},X)} D \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C,D)$.
- For simplicial maps $f: M\to X'$, $g: N\to X'$
  - $\Delta(M)\otimes_{(\mathbb{Z},X)}\Delta(N) \simeq_{(\mathbb{Z},X)} \Delta((f\times g)^{-1}\Delta_X)$
  - $T\Delta(M)\otimes_{(\mathbb{Z},X)} T\Delta(N) \simeq_{\mathbb{Z}} \Delta(M\times N, M\times N\backslash(f\times g)^{-1}\Delta_X)^{-*}$.

16. Cap product

- The Alexander-Whitney diagonal chain approximation
$$\Delta : \Delta(X') \to \Delta(X')\otimes_{\mathbb{Z}}\Delta(X') ;$$
$$(\widehat{x}_0\dots\widehat{x}_n) \to \sum_{i=0}^{n}(\widehat{x}_0\dots\widehat{x}_i)\otimes(\widehat{x}_i\dots\widehat{x}_n)$$
is the composite of a chain equivalence
$$\Delta(X') \simeq_{(\mathbb{Z},X)} \Delta(X')\otimes_{(\mathbb{Z},X)}\Delta(X')$$
and the inclusion
$$\Delta(X')\otimes_{(\mathbb{Z},X)}\Delta(X') \subseteq \Delta(X')\otimes_{\mathbb{Z}}\Delta(X') .$$
- Homology classes $[X]\in H_n(X)$ are in one-one correspondence with the chain homotopy classes of $(\mathbb{Z},X)$-module chain maps
$$[X]\cap - \ :\ \Delta(X)^{n-*} \to \Delta(X') .$$

17. Homology manifolds

- **Definition**: A simplicial complex X is an n-dimensional homology manifold if
$$H_*(X, X\backslash\widehat{\sigma}) = \begin{cases}\mathbb{Z} & \text{if } * = n\\ 0 & \text{otherwise}\end{cases} \quad (\sigma\in X) .$$
- **Proposition**: A simplicial complex X is an n-dimensional homology manifold if and only if there exists a homology class $[X]\in H_n(X)$ such that the cap product
$$[X]\cap - \ :\ \Delta(X)^{n-*} \to \Delta(X')$$
is a $(\mathbb{Z},X)$-module chain equivalence.
- **Proof**: For any simplicial complex X
$$H_*(X,X\backslash\widehat{\sigma}) = H_{*-|\sigma|}(D(\sigma,X),\partial D(\sigma,X)) ,$$
$$H^{n-*}(D(\sigma,X)) = \begin{cases}\mathbb{Z} & \text{if } * = n\\ 0 & \text{otherwise}\end{cases} \quad (\sigma\in X) .$$

18. Poincaré complexes

- **Definition**: An n-dimensional Poincaré complex X is a simplicial complex with a homology class $[X] \in H_n(X)$ such that
$$[X] \cap - : H^{n-*}(X) \cong H_*(X) \ .$$
- **Poincaré duality theorem**: An n-dimensional homology manifold X is an n-dimensional Poincaré complex.
- **Proof**: A $(\mathbb{Z}, X)$-module chain equivalence
$$[X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$$
is a $\mathbb{Z}$-module chain equivalence.
- There is also a $\mathbb{Z}[\pi_1(X)]$-version.

19. McCrory's Theorem

- $X = n$-dimensional Poincaré complex
  - $X \times X$ is a $2n$-dimensional Poincaré complex.
  - Let $V \in H^n(X \times X)$ be the Poincaré dual of $\Delta_*[X] \in H_n(X \times X)$.
  - Exact sequence
$$H^n(X \times X, X \times X \backslash \Delta_X) \to H^n(X \times X) \to H^n(X \times X \backslash \Delta_X) \ .$$

Theorem (McCrory) X is an n-dimensional homology manifold if and only if V has image $0 \in H^n(X \times X \backslash \Delta_X)$.

- A characterization of homology manifolds, J. Lond. Math. Soc. 16 (2), 149–159 (1977)

20. Chain duality proof of McCrory's Theorem

- V has image $0 \in H^n(X \times X \backslash \Delta_X)$ if and only if there exists $U \in H^n(X \times X, X \times X \backslash \Delta_X)$ with image V.
- U is a chain homotopy class of $(\mathbb{Z}, X)$-module chain maps $\Delta(X') \to \Delta(X)^{n-*}$, since
$$\begin{aligned} H^n(X \times X, X \times X \backslash \Delta_X) &= H_n(T\Delta(X) \otimes_{(\mathbb{Z},X)} T\Delta(X)) \\ &= H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X'), \Delta(X)^{n-*})) \ . \end{aligned}$$
- U is a chain homotopy inverse of
$$\phi = [X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$$
with
$$\begin{aligned} &\phi U = 1 \in H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X'), \Delta(X'))) = H^0(X) \ , \\ &\phi = T\phi \ , \ (TU)\phi = (TU)(T\phi) = T(\phi U) = 1 \ . \end{aligned}$$

21. The homology tangent bundle

- The tangent bundle τ_X of a manifold X is the normal bundle of the diagonal embedding
$$\Delta : X \to X \times X \ ; \ x \to (x,x) \ .$$
- The homology tangent bundle τ_X of an n-dimensional homology manifold X is the fibration
$$(X, X\backslash\{*\}) \longrightarrow (X \times X, X \times X\backslash\Delta_X) \longrightarrow X$$
with $X \times X \to X; (x,y) \to x$.
- Thom space of τ_X
$$T(\tau_X) \ = \ (X \times X)/(X \times X\backslash\Delta_X) \ .$$
- Thom class of τ_X
$$U \in \widetilde{H}^n(T(\tau_X)) = H^n(X \times X, X \times X\backslash\Delta_X)$$
has image $V \in H^n(X \times X)$.

22. Euler

- The Euler characteristic of a simplicial complex X is
$$\chi(X) = \sum_{r=0}^{\infty}(-)^r \dim_{\mathbb{R}} H_r(X;\mathbb{R}) \in \mathbb{Z} \ .$$
- For an n-dimensional Poincaré complex X
$$\chi(X) = \Delta^*(V) \in H^n(X) = \mathbb{Z} \ .$$
- The Euler class of n-plane bundle η over X
$$e(\eta) = [U] \in \mathrm{im}(\widetilde{H}^n(T(\eta)) \to H^n(X)) \ .$$
- Reformulation of McCrory's Theorem:
an n-dimensional Poincaré complex X is a homology manifold if and only if $V \in H^n(X \times X)$ is the image of Thom class $U \in \widetilde{H}^n(T(\tau_X))$, in which case
$$\chi(X) = e(\tau_X) \in H^n(X) = \mathbb{Z} \ .$$

23. Degree 1 maps

- A map $f : M \to X$ of n-dimensional Poincaré complexes has degree 1 if
$$f_*[M] = [X] \in H_n(X) \ .$$
- A homology equivalence has degree 1.
- The Umkehr $\mathbb{Z}$-module chain map of a degree 1 map $f : M \to X$
$$f^! \ : \ \Delta(X) \simeq \Delta(X)^{n-*} \xrightarrow{f^*} \Delta(M)^{n-*} \simeq \Delta(M)$$
is such that $ff^! \simeq 1 : \Delta(X) \to \Delta(X)$.
- A degree 1 map f is a homology equivalence if and only if
$$f^!f \simeq 1 : \Delta(M) \to \Delta(M) \ ,$$
if and only if
$$(f^! \otimes f^!)\Delta_*[X] = \Delta_*[M] \in H_n(M \times M) \ .$$

24. The double point set

- Does a degree 1 map of n-dimensional homology manifolds $f : M \to X$ have acyclic point inverses?
- Obstruction in homology of double point set

$$(f \times f)^{-1}\Delta_X = \{(x,y) \in M \times M \,|\, f(x) = f(y) \in X\}.$$

- Define maps

$$\begin{aligned} i &: M \to (f \times f)^{-1}\Delta_X \ ;\ a \to (a,a)\ , \\ j &: (f \times f)^{-1}\Delta_X \to X \ ;\ (x,y) \to f(x) = f(y) \end{aligned}$$

such that $f = ji : M \to X$.
- The Umkehr map

$$\begin{aligned} j^! : H_n(X) &\cong H^n(X \times X, X \times X\backslash\Delta_X) \\ &\to H^n(M \times M, M \times M\backslash(f \times f)^{-1}\Delta_X) \\ &\cong H_n((f \times f)^{-1}\Delta_X) \text{ (Lefschetz duality)} \end{aligned}$$

is such that $j_* j^! = 1$.

25. Lefschetz

- Lefschetz duality: If W is an m-dimensional homology manifold and $A \subseteq W$ is a subcomplex then

$$H^*(W, W\backslash A) \cong H_{m-*}(A)\ .$$

- Proof: For any regular neighbourhood $(V, \partial V)$ of A in W there are defined isomorphisms

$$\begin{aligned} H^*(W, W\backslash A) &\cong H^*(W, W\backslash V) \text{ (homotopy invariance)} \\ &\cong H^*(W, \overline{W\backslash V}) \text{ (collaring)} \\ &\cong H^*(V, \partial V) \text{ (excision)} \\ &\cong H_{m-*}(V) \text{ (Poincaré-Lefschetz duality)} \\ &\cong H_{m-*}(A) \text{ (homotopy invariance).} \end{aligned}$$

- Alexander duality is the special case $W = S^m$.

26. Acyclic Point Inverse Theorem

Theorem A degree 1 map $f : M \to X$ of n-dimensional homology manifolds has acyclic point inverses if and only if

$$i_*[M] = j^![X] \in H_n((f \times f)^{-1}\Delta_X)\ .$$

- Equivalent conditions:
  - $i_* : H_n(M) \cong H_n((f \times f)^{-1}\Delta_X)$,
  - $i_* : H_*(M) \cong H_*((f \times f)^{-1}\Delta_X)$,
  - $H^{lf}_*((f \times f)^{-1}\Delta_X\backslash\Delta_M) = 0$.

- Conditions satisfied if $f : M \to X$ is injective, with
$$(f \times f)^{-1}\Delta_X = \Delta_M \ .$$
- In general, $i_* \neq j^! f_*$ and $i_*[M] \neq j^![X]$.

27. Proof of Theorem - Part I

- A simplicial map $f : M \to X'$ has acyclic point inverses if and only if $f : \Delta(M) \to \Delta(X')$ is a $(\mathbb{Z}, X)$-module chain equivalence.
- For degree 1 map $f : M \to X'$ of n-dimensional homology manifolds define the Umkehr $(\mathbb{Z}, X)$-module chain map
$$f^! \ : \ \Delta(X') \simeq \Delta(X')^{n-*} \xrightarrow{f^*} \Delta(M)^{n-*} \simeq \Delta(M) \ .$$
- $f^!$ is a chain homotopy right inverse for f
$$ff^! \simeq 1 : \Delta(X') \to \Delta(X') \ .$$
- $f^!$ is also a chain homotopy left inverse for f if and only if
$$f^!f = 1 \in H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(M), \Delta(M))) \ .$$

28. Proof of Theorem - Part II

- Use the $(\mathbb{Z}, X)$-Poincaré duality
$$\Delta(M)^{n-*} \simeq \Delta(M)$$
and the properties of chain duality in $\mathbb{A}(\mathbb{Z}, X)$ to identify
$$\begin{aligned} 1 \ = \ i_*[M] \ , \ f^!f \ = \ j^![X] &\in H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(M), \Delta(M))) \\ &= H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(M)^{n-*}, \Delta(M))) \\ &= H_n(\Delta(M) \otimes_{(\mathbb{Z},X)} \Delta(M)) \\ &= H_n((f \times f)^{-1}\Delta_X) \ . \end{aligned}$$

29. Cohomology version of Theorem

Theorem* A degree 1 map $f : M \to X$ of n-dimensional homology manifolds has acyclic point inverses if and only if the Thom classes $U_M \in H^n(M \times M, M \times M \backslash \Delta_M)$, $U_X \in H^n(X \times X, X \times X \backslash \Delta_X)$ have the same image in $H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X)$.

- Same proof as homology version, after Lefschetz duality identifications
$$\begin{aligned} &U_M = [M] \in H^n(M \times M, M \times M \backslash \Delta_M) = H_n(M) \ , \\ &U_X = [X] \in H^n(X \times X, X \times X \backslash \Delta_X) = H_n(X) \ , \\ &H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) = H_n((f \times f)^{-1}\Delta_X) \ . \end{aligned}$$

30. The double point obstruction

- The <u>double point obstruction</u> of a degree 1 map $f : M \to X$ of homology manifolds
$$i_*[M] - j^![X] \in H_n((f \times f)^{-1}\Delta_X)$$
is 0 if and only if f has acyclic point inverses.
- The obstruction has image
$$\chi(M) - \chi(X) \in H^n(M) = \mathbb{Z} .$$
- If f is covered by a map of homology tangent bundles
$$b : (M \times M, M \times M \backslash \Delta_M) \to (X \times X, X \times X \backslash \Delta_X)$$
then
  - $U_M = b^* U_X \in H^n(M \times M, M \times M \backslash \Delta_M)$,
  - the double point obstruction is 0, and f has acyclic point inverses.

31. Normal maps

- A degree 1 map $f : M \to X$ of n-dimensional homology manifolds is <u>normal</u> if it is covered by a map $b : \tau_M \oplus \epsilon^\infty \to \tau_X \oplus \epsilon^\infty$ of the stable tangent bundles.
- The stable map of Thom spaces
$$T(b) : \Sigma^\infty T(\tau_M) \to \Sigma^\infty T(\tau_X)$$
induces a map in cohomology
$$\begin{aligned} T(b)^* : \widetilde{H}^n(T(\tau_X)) &= H^n(X \times X, X \times X \backslash \Delta_X) \\ &\to \widetilde{H}^n(T(\tau_M)) = H^n(M \times M, M \times M \backslash \Delta_M) \end{aligned}$$
which sends the Thom class U_X to U_M.
- However, Theorem* may not apply to a normal map $(f, b) : M \to X$, since in general
$$\begin{aligned} (f \times f)^* \neq (\text{inclusion})^* T(b)^* : \widetilde{H}^n(T(\tau_X)) & \\ \to H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) & \end{aligned}$$
(dual of $i_* \neq j^! f_*$).

32. The surgery obstruction

- The Wall surgery obstruction of a degree 1 normal map $(f, b) : M \to X$ of n-dimensional homology manifolds
$$\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$$
is 0 if (and for $n \geq 5$ only if) (f, b) is <u>normal bordant</u> to a homotopy equivalence.
- A degree 1 map $f : M \to X$ with acyclic point inverses is a normal map with zero surgery obstruction.
- What is the relationship between the double point obstruction of a degree 1 normal map $(f, b) : M \to X$ and the surgery obstruction?
- Use chain level surgery obstruction theory:
A.Ranicki, <u>The algebraic theory of surgery</u>, Proc. Lond. Math. Soc. (3) 40, 87–283 (1980)

33. Quadratic Poincaré complexes

- The simply-connected surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z})$ is the cobordism class of the n-dimensional quadratic Poincaré complex

$$(C,\psi) = (C(f^!), (e \otimes e)\psi_b)$$

  where
  - $e : \Delta(M) \to C(f^!)$ is the inclusion in the algebraic mapping cone of the $\mathbb{Z}$-module chain map $f^! : \Delta(X) \to \Delta(M)$,
  - the quadratic structure ψ is the image of

$$\psi_b \in H_n(E\Sigma_2 \times_{\Sigma_2} (M \times M)) = H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{\mathbb{Z}} \Delta(M))) \ ,$$

  - $E\Sigma_2 = S^\infty$, a contractible space with a free Σ_2-action,
  - $W = \Delta(E\Sigma_2)$.
- There is also a $\mathbb{Z}[\pi_1(X)]$-version.

34. The double point and surgery obstructions - Part I

- For any degree 1 map $f : M \to X$ of n-dimensional homology manifolds the composite of

$$i_* f^! - j^! : H_*(X) \to H_*((f \times f)^{-1}\Delta_X)$$

  and $H_*((f \times f)^{-1}\Delta_X) \to H_*(M \times M)$ is

$$\Delta_* f^! - (f^! \otimes f^!)\Delta_* : H_*(X) \to H_*(M \times M) \ .$$

- For a degree 1 normal map $(f,b) : M \to X$

$$H_n((f \times f)^{-1}\Delta_X) \to H_n(M \times M)$$

  sends the double point obstruction $i_*[M] - j^![X]$ to

$$(1+T)\psi_b = \Delta_*[M] - (f^! \otimes f^!)\Delta_*[X] \in H_n(M \times M) \ .$$

- $(1+T)\psi_b = 0$ if and only if f is a homology equivalence.

35. The double point and surgery obstructions - Part II

- A degree 1 normal map $(f,b) : M \to X$ of n-dimensional homology manifolds determines the X-controlled quadratic structure

$$\begin{aligned}\psi_{b,X} &\in H_n(E\Sigma_2 \times_{\Sigma_2} (f \times f)^{-1}\Delta_X) \\ &= H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M) \otimes_{(\mathbb{Z},X)} \Delta(M))) \ .\end{aligned}$$

- $\psi_{b,X}$ has images
  - the quadratic structure

$$[\psi_{b,X}] = \psi_b \in H_n(E\Sigma_2 \times_{\Sigma_2} (M \times M)) \ ,$$

  - the double point obstruction

$$(1+T)\psi_{b,X} = i_*[M] - j^![X] \in H_n((f \times f)^{-1}\Delta_X) \ .$$

36. The normal invariant

- The X-controlled quadratic Poincaré cobordism class
$$\sigma_*^X(f,b) = (C(f^!),(e\otimes e)\psi_{b,X}) \in L_n(\mathbb{A}(\mathbb{Z},X)) = H_n(X;\mathbb{L}_\bullet(\mathbb{Z}))$$
is the normal invariant of an n-dimensional degree 1 normal map $(f,b) : M \to X$.
- $\sigma_*^X(f,b) = 0$ if (and for $n \geq 5$ only if) (f,b) is normal bordant to a map with acyclic point inverses.
- The non-simply-connected surgery obstruction of (f,b) is the assembly of the normal invariant
$$\sigma_*(f,b) = A\sigma_*^X(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]) \ .$$

37. Hom and Derived Hom

- For $(\mathbb{Z},X)$-modules A,B the additive group $\mathrm{Hom}_{(\mathbb{Z},X)}(A,B)$ does not have a natural $(\mathbb{Z},X)$-module structure, but the chain duality determines a natural $(\mathbb{Z},X)$-module resolution.
- Derived Hom of $(\mathbb{Z},X)$-module chain complexes C,D
$$\mathrm{RHom}_{(\mathbb{Z},X)}(C,D) \ = \ T(C)\otimes_{(\mathbb{Z},X)} D \ .$$
- Adjoint properties:
$$\begin{aligned}&\mathrm{RHom}_{(\mathbb{Z},X)}(C,D) \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C,D)\\ &\mathrm{RHom}_{(\mathbb{Z},X)}(T(C),D) \simeq_{(\mathbb{Z},X)} C\otimes_{(\mathbb{Z},X)} D \ .\end{aligned}$$
- $D = \Delta(X')$ is the dualizing complex for chain duality
$$T(C) \simeq_{(\mathbb{Z},X)} \mathrm{RHom}_{(\mathbb{Z},X)}(C,\Delta(X'))$$
as for Verdier duality in sheaf theory.

38. When is a Poincaré complex homotopy equivalent to a manifold?

- Every n-dimensional topological manifold is homotopy equivalent to an n-dimensional Poincaré complex
- Is every n-dimensional Poincaré complex homotopy equivalent to an n-dimensional topological manifold?
- From now on $n \geq 5$
- Browder-Novikov-Sullivan-Wall obstruction theory has been reformulated in terms of chain duality
  - the total surgery obstruction.

39. Browder-Novikov-Sullivan-Wall theory

- An n-dimensional Poincaré complex X is homotopy equivalent to an n-dimensional topological manifold if and only if
  1. the Spivak normal fibration of X admits a topological reduction,
  2. there exists a reduction such that the corresponding normal map $(f,b) : M \to X$ has surgery obstruction
$$\sigma_*(f,b) = 0 \in L_n(\mathbb{Z}[\pi_1(X)]) \ .$$

40. Algebraic Poincaré cobordism

- Λ = ring with involution.
- $L_n(\Lambda)$ = Wall surgery obstruction group
  = the cobordism group of n-dimensional quadratic Poincaré complexes over Λ
  - n-dimensional f.g. free Λ-module chain complexes C with
$$H^{n-*}(C) \cong H_*(C) \ ,$$
  - uses ordinary duality
$$C^{n-*} \ = \ \mathrm{Hom}_\Lambda(C,\Lambda)_{*-n} \ .$$

41. Assembly

- X = connected simplicial complex
  - $\widetilde{X}$ = universal cover
  - $p : \widetilde{X} \to X$ covering projection.
- Assembly functor
$$A \ : \ \mathbb{A}(\mathbb{Z},X) = \{(\mathbb{Z},X)\text{-modules}\} \to \mathbb{A}(\mathbb{Z}[\pi_1(X)]) = \{\mathbb{Z}[\pi_1(X)]\text{-modules}\} \ ;$$
$$M = \sum_{\sigma\in X} M(\sigma) \to M(\widetilde{X}) = \sum_{\widetilde{\sigma}\in\widetilde{X}} M(p\widetilde{\sigma}) \ .$$
- The assembly $A(T(M))$ of dual $(\mathbb{Z},X)$-module chain complex
$$T(M) \ \simeq_{\mathbb{Z}} \ \mathrm{Hom}_{(\mathbb{Z},X)}(M,\Delta(X'))$$
  is chain equivalent to dual $\mathbb{Z}[\pi_1(X)]$-module
$$M(\widetilde{X})^* \ = \ \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(M(\widetilde{X}),\mathbb{Z}[\pi_1(X)]) \ .$$

42. The algebraic surgery exact sequence

- For any simplicial complex X exact sequence
$$\cdots \to H_n(X;\mathbb{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet(\mathbb{Z})) \to \dots$$
  with
- A = assembly,
- $\mathbb{L}_\bullet(\mathbb{Z})$ = the 1-connective simply-connected surgery spectrum
  - $\pi_*(\mathbb{L}_\bullet(\mathbb{Z})) = L_*(\mathbb{Z})$,
- $H_n(X;\mathbb{L}_\bullet(\mathbb{Z}))$ = generalized homology group
  - cobordism group of n-dimensional quadratic Poincaré $(\mathbb{Z},X)$-module complexes $C \simeq T(C)^{n-*}$
  - uses chain duality
$$T(C)^{n-*} \ \simeq_{(\mathbb{Z},X)} \ \mathrm{RHom}_{(\mathbb{Z},X)}(C,\Delta(X'))_{*-n} \ .$$

43. The structure group

- X = simplicial complex.
- $\mathbb{S}_n(X)$ = structure group.
- $\mathbb{S}_n(X)$ = cobordism group of
  - $(n-1)$-dimensional quadratic Poincaré $(\mathbb{Z},X)$-module complexes
  - with contractible $\mathbb{Z}[\pi_1(X)]$-module assembly.

44. Local and global Poincaré duality

- $X = n$-dimensional Poincaré complex.
- The cap product $[X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$:
  - is a $(\mathbb{Z}, X)$-module chain map,
  - assembles to $\mathbb{Z}[\pi_1(X)]$-module chain equivalence

$$[X] \cap - : \Delta(\widetilde{X})^{n-*} \to \Delta(\widetilde{X}') \ .$$

- The algebraic mapping cone

$$C \ = \ \mathcal{C}([X] \cap - : \Delta(X)^{n-*} \to \Delta(X'))_{*-1}$$

  - is an $(n-1)$-dimensional quadratic Poincaré $(\mathbb{Z}, X)$-module complex,
  - with contractible $\mathbb{Z}[\pi_1(X)]$-assembly.
- X is a homology manifold if and only if C is $(\mathbb{Z}, X)$-contractible.

45. The total surgery obstruction

- $X = n$-dimensional Poincaré complex.
- The <u>total surgery obstruction</u> of X is the cobordism class

$$s(X) \ = \ \mathcal{C}([X] \cap -)_{*-1} \in \mathbb{S}_n(X) \ .$$

- **Theorem 1**: X is homotopy equivalent to an n-dimensional topological manifold if and only if $s(X) = 0 \in \mathbb{S}_n(X)$.
- **Theorem 2**: A homotopy equivalence $f : M \to N$ of n-dimensional topological manifolds has a total surgery obstruction $s(f) \in \mathbb{S}_{n+1}(N)$ such that f is homotopic to a homeomorphism if and only if $s(f) = 0$.
  - Should also consider Whitehead torsion.

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Controlled L-Theory
(Preliminary announcement)

A. Ranicki and M. Yamasaki

Introduction.

This is a preliminary announcement of a controlled algebraic surgery theory, of the type first proposed by Quinn [1]. We define and study the ϵ-controlled L-groups $L_n(X, p_X, \epsilon)$, extending to L-theory the controlled K-theory of Ranicki and Yamasaki [4].

The most immediate application of the algebra to controlled geometric surgery is the controlled surgery obstruction: a normal map $(f, b) : K \to L$ from a closed n-dimensional manifold to a δ-controlled Poincaré complex determines an element

$$\sigma_*^{\delta}(f, b) \in L_n(X, 1_X, 100\delta) \ .$$

(The construction in Ranicki and Yamasaki [3] can be used to produce a 6ϵ n-dimensional quadratic Poincaré structure on an $(n+1)$-dimensional chain complex. There is a chain equivalence from this to an n-dimensional chain complex with a 100δ n-dimensional quadratic Poincaré structure, and $\sigma_*^{\delta}(f, b)$ is the cobordism class of this complex in $L_n(X, 1_X, 100\delta)$.) A relative construction shows that if (f, b) can be made into a δ-controlled homotopy equivalence by δ-controlled surgery then

$$\sigma_*^{\delta}(f, b) = 0 \in L_n(X, 1_X, 100\delta) \ .$$

Conversely, if $n \geq 5$ and (f, b) is such that

$$\sigma_*^{\delta}(f, b) = 0 \in L_n(X, 1_X, 100\delta)$$

then (f, b) can be made into an ϵ-controlled homotopy equivalence by ϵ-controlled surgery, where $\epsilon = C \times 100\delta$ for a certain constant $C > 1$ that depends on n. Proofs of difficult results and the applications of the algebra to topology are deferred to the final account.

The algebraic properties required to obtain these applications include the controlled L-theory analogues of the homology exact sequence of a pair (3.1, 3.2) and the Mayer-Vietoris sequence (3.3, 3.4).

The limit of the controlled L-groups

$$L_n^c(X; 1_X) \ = \ \varprojlim_{\epsilon} \varprojlim_{\delta} \mathrm{im}\{L_n(X, 1_X, \delta) \longrightarrow L_n(X, 1_X, \epsilon)\}$$

is the obstruction group for controlled surgery to ϵ-controlled homotopy equivalence for all $\epsilon > 0$.

Theorem. (5.4.) *Fix a compact polyhedron X and an integer $n(\geq 0)$. There exist numbers $\epsilon_0 > 0$ and $0 < \mu_0 \leq 1$ such that*

$$L_n^c(X; 1_X) = \mathrm{im}\{L_n(X, 1_X, \delta) \longrightarrow L_n(X, 1_X, \epsilon)\}$$

for every $\epsilon \leq \epsilon_0$ and every $\delta \leq \mu_0 \epsilon$.

Throughout this paper all the modules are assumed to be finitely generated unless otherwise stated explicitly. But note that all the definitions and the constructions are valid also for possibly-infinitely-generated modules and chain complexes. Actually we heavily use finite dimensional but infinitely generated chain complexes in the later part of the paper. (That is where the bounded-control over $\mathbb{R}$ comes into the game.) So we first pretend that everything is finitely generated, and later we introduce a possibly-infinitely-generated analogue without any details.

1. Epsilon-controlled L-groups.

In this section we introduce ϵ-controlled L-groups $L_n(X, p_X, \epsilon)$ and $L_n(X, Y, p_X, \epsilon)$ for $p_X : M \to X$, $Y \subset X$, $n \geq 0$, $\epsilon > 0$. These are defined using geometric module chain complexes with quadratic Poincaré structures, which were discussed in Yamasaki [5].

We use the convention in Ranicki and Yamasaki [4] for radii of geometric morphisms, etc. The dual of a geometric module is the geometric module itself, and the dual of a geometric morphism is defined by reversing the orientation of paths. Note that if f has radius ϵ then so does its dual f^* and that $f \sim_\epsilon g$ implies $f^* \sim_\epsilon g^*$, by our convention. For a geometric module chain complex C, its dual C^{n-*} is defined using the sign convention used in Ranicki [2].

For a subset S of a metric space X, S^ϵ will denote the closed ϵ neighborhood of S in X when $\epsilon \geq 0$. When $\epsilon < 0$, S^ϵ will denote the set $X - (X - S)^{-\epsilon}$.

Let C be a free chain complex on $p_X : M \to X$. An *n-dimensional ϵ quadratic structure ψ on C* is a collection $\{\psi_s | s \geq 0\}$ of geometric morphisms

$$\psi_s \;:\; C^{n-r-s} \;=\; (C_{n-r-s})^* \;\to\; C_r \quad (r \in \mathbb{Z})$$

of radius ϵ such that

$$(*) \qquad d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) \sim_{3\epsilon} 0 : C^{n-r-s-1} \to C_r,$$

for $s \geq 0$. An n-dimensional free ϵ chain complex C on p_X equipped with an n-dimensional ϵ quadratic structure is called an *n-dimensional ϵ quadratic complex on p_X*. (Here, a complex C is n-dimensional if $C_i = 0$ for $i < 0$ and $i > n$.)

Next let $f : C \to D$ be a chain map between free chain complexes on p_X. An $(n+1)$-*dimensional* ϵ *quadratic structure* $(\delta\psi, \psi)$ *on* f is a collection $\{\delta\psi_s, \psi_s | s \geq 0\}$ of geometric morphisms

$$\delta\psi_s : D^{n+1-r-s} \to D_r \quad , \quad \psi_s : C^{n-r-s} \to C_r \quad (r \in \mathbb{Z})$$

of radius ϵ such that the following holds in addition to $(*)$:

$$\begin{aligned} d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}) + (-)^n f\psi_s f^* \sim_{3\epsilon} 0 \\ : D^{n-r-s} \to D_r \quad (s \geq 0) \ . \end{aligned}$$

An ϵ chain map $f : C \to D$ between an n-dimensional free ϵ chain complex C on p_X and an $(n+1)$-dimensional free ϵ chain complex D on p_X equipped with an $(n+1)$-dimensional ϵ quadratic structure is called an $(n+1)$-*dimensional* ϵ *quadratic pair on* p_X. Obviously its *boundary* (C, ψ) is an n-dimensional ϵ quadratic complex on p_X.

An ϵ *cobordism* of n-dimensional ϵ quadratic structures ψ on C and ψ' on C' is an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi \oplus -\psi')$ on some chain map $C \oplus C' \to D$. An ϵ *cobordism* of n-dimensional ϵ quadratic complexes (C, ψ), (C', ψ') on p_X is an $(n+1)$-dimensional ϵ quadratic pair on p_X

$$((f \quad f') : C \oplus C' \to D, \quad (\delta\psi, \psi \oplus -\psi'))$$

with boundary $(C \oplus C', \psi \oplus -\psi')$. The union of adjoining cobordisms are defined using the formula in Chapter 1.7 of Ranicki [2]. The union of adjoining ϵ cobordisms is a 2ϵ cobordism.

ΣC and ΩC will denote the suspension and the desuspension of C respectively, and $\mathcal{C}(f)$ will denote the algebraic mapping cone of a chain map f.

Definition. Let W be a subset of X. An n-dimensional ϵ quadratic structure ψ on C is ϵ *Poincaré (over* W*)* if the algebraic mapping cone of the duality 3ϵ chain map

$$\mathcal{D}_\psi = (1+T)\psi_0 : C^{n-*} \longrightarrow C$$

is 4ϵ contractible (over W). A quadratic complex (C, ψ) is ϵ *Poincaré (over* W*)* if ψ is ϵ Poincaré (over W). Similarly, an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : C \to D$ is ϵ *Poincaré (over* W*)* if the algebraic mapping cone of the duality 4ϵ chain map

$$\mathcal{D}_{(\delta\psi,\psi)} = ((1+T)\delta\psi_0 \quad f(1+T)\psi_0) \ : \ \mathcal{C}(f)^{n+1-*} \longrightarrow D$$

is 4ϵ contractible (over W) (or equivalently the algebraic mapping cone of the 4ϵ chain map

$$\bar{\mathcal{D}}_{(\delta\psi,\psi)} = \begin{pmatrix} (1+T)\delta\psi_0 \\ (-)^{n+1-r}(1+T)\psi_0 f^* \end{pmatrix} : D^{n+1-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is 4ϵ contractible (over W)) and ψ is ϵ Poincaré (over W). A quadratic pair $(f, (\delta\psi, \psi))$ is ϵ *Poincaré (over* W*)* if $(\delta\psi, \psi)$ is ϵ Poincaré (over W). We will also use the notation $\mathcal{D}_{\delta\psi} = (1+T)\delta\psi_0$, although it does not define a chain map from D^{n+1-*} to D in general.

Definition. (1) A positive geometric chain complex C ($C_i = 0$ for $i < 0$) is ϵ *connected* if there exists a 4ϵ morphism $h : C_0 \to C_1$ such that $dh \sim_{8\epsilon} 1_{C_0}$.
(2) A chain map $f : C \to D$ of positive chain complexes is ϵ *connected* if $\mathcal{C}(f)$ is ϵ connected.
(3) A quadratic complex (C, ψ) is ϵ *connected* if $\mathcal{D}_\psi$ is ϵ connected.
(4) A quadratic pair $(f : C \to D, (\delta\psi, \psi))$ is ϵ *connected* if $\mathcal{D}_\psi$ and $\mathcal{D}_{(\delta\psi,\psi)}$ are ϵ connected.

Now we define the ϵ-controlled L-groups. Let Y be a subset of X.

Definition. For $n \geq 0$ and $\epsilon \geq 0$, $L_n(X, Y, p_X, \epsilon)$ is defined to be the equivalence classes of n-dimensional ϵ connected ϵ quadratic complexes on p_X that are ϵ Poincaré over $X - Y$. The equivalence relation is generated by ϵ connected ϵ cobordisms that are ϵ Poincaré over $X - Y$. For $Y = \emptyset$ write

$$L_n(X, p_X, \epsilon) \ = \ L_n(X, \emptyset, p_X, \epsilon) \ .$$

Remarks. (1) We use only n-dimensional complexes and not the complexes chain equivalent to n-dimensional ones in order to make sure we have size control on some constructions.
(2) The ϵ connectedness condition is automatic for complexes that are ϵ Poincaré over X. Connectedness condition is used to insure that the *boundary* $\partial C = \Omega\mathcal{C}(\mathcal{D}_\psi)$ is chain equivalent to a positive one. There is a quadratic structure $\partial\psi$ for ∂C so that $(\partial C, \partial\psi)$ is Poincaré (Ranicki [2]).
(3) Using locally-finitely generated chain complexes on M, one can similarly define ϵ-controlled locally-finite L-groups $L_n^{lf}(X, Y, p_X, \epsilon)$. All the results in sections 1 – 3 are valid for locally-finite L-groups.

Proposition 1.1. *The direct sum*

$$(C, \psi) \oplus (C', \psi') \ = \ (C \oplus C', \psi \oplus \psi')$$

induces an abelian group structure on $L_n(X, Y, p_X, \epsilon)$. *Furthermore, if*

$$[C, \psi] \ = \ [C', \psi'] \in L_n(X, Y, p_X, \epsilon) \ ,$$

then there is a 100ϵ *connected* 2ϵ *cobordism between* (C, ψ) *and* (C', ψ') *that is* 100ϵ *Poincaré over* $X - Y^{100\epsilon}$.

Next we study the functoriality. A *map between control maps* $p_X : M \to X$ and $p_Y : N \to Y$ means a pair of continuous maps $(f : M \to N,\ \bar{f} : X \to Y)$ which makes the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p_X"'] & N \arrow[d, "p_Y"] \\ X \arrow[r, "\bar{f}"'] & Y. \end{tikzcd}$$

For example, given a control map $p_Y : N \to Y$ and a subset $X \subset Y$, let us denote the control map $p_Y|p_Y^{-1}(X) : p_Y^{-1}(X) \to X$ by $p_X : M \to X$. Then the inclusion maps $j : M \to N$, $\bar{j} : X \to Y$ form a map form p_X to p_Y.

Epsilon controlled L-groups are functorial with respect to maps and relaxation of control in the following sense.

Proposition 1.2. *Let $F = (f, \bar{f})$ be a map from $p_X : M \to X$ to $p_Y : N \to Y$, and suppose that $\bar{f}$ is Lipschitz continuous with Lipschitz constant λ, i.e., there exists a constant $\lambda > 0$ such that*

$$d(\bar{f}(x_1), \bar{f}(x_2)) \le \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

Then F induces a homomorphism

$$F_* \ : \ L_n(X, X', p_X, \delta) \longrightarrow L_n(Y, Y', p_Y, \epsilon)$$

if $\epsilon \ge \lambda\delta$ and $\bar{f}(X') \subset Y'$. If two maps $F = (f, \bar{f})$ and $G = (g, \bar{g})$ are homotopic through maps $H_t = (h_t, \bar{h}_t)$ such that each $\bar{h}_t$ is Lipschitz continuous with Lipschitz constant λ, $\epsilon \ge \lambda\delta$, $\epsilon' > \epsilon$, and $\bar{h}_t(X') \subset Y'$, then the following two compositions are the same:

$$L_n(X, X', p_X, \delta) \xrightarrow{F_*} L_n(Y, Y', p_Y, \epsilon) \longrightarrow L_n(Y, Y', p_Y, \epsilon')$$
$$L_n(X, X', p_X, \delta) \xrightarrow{G_*} L_n(Y, Y', p_Y, \epsilon) \longrightarrow L_n(Y, Y', p_Y, \epsilon')$$

Proof: The direct image construction for geometric modules and morphisms [4, p.7] can be used to define the direct images $f_\#(C, \psi)$ of quadratic complexes and the direct images of cobordism. And this induces the desired F_*. The first part is obvious. For the second part, split the homotopy in small pieces to construct small cobordisms. The size of the cobordism may be slightly bigger than the size of the object itself. □

Remark. The above is stated for Lipschitz continuous maps to simplify the statement. For a specific δ and a specific ϵ, the following condition, instead of the Lipschitz condition above, is sufficient for the existence of F_* :

$$d(\bar{f}(x_1), \bar{f}(x_2)) \le k\epsilon \quad \text{whenever} \quad d(x_1, x_2) \le k\delta,$$

for a certain finite set of integers k (more precisely, for $k = 1, 3, 4, 8$)

and similarly for the isomorphism in the second part. When X is compact and ϵ is given, the continuity of $\bar{f}$ implies that this condition is satisfied for sufficiently small δ's. [Use the continuity of the distance function $d : X \times X \to \mathbb{R}$ and the compactness of the diagonal set $\Delta \subset X \times X$.] And, in the second half of the proposition, there are cases when the equality $F_* = G_*$ holds without composing with the relax-control map; *e.g.,* see 4.1.

We are interested in the "limit" of ϵ-controlled L-groups.

Definition. Let $p_X : M \to X$ be a control map.

(1) Let ϵ, δ be positive numbers such that $\delta \leq \epsilon$. We define:

$$L_n^\epsilon(X, p_X, \delta) = \mathrm{im}\{L_n(X, p_X, \delta) \longrightarrow L_n(X, p_X, \epsilon)\}.$$

(2) For $\epsilon > 0$, we define the *stable ϵ-controlled L-group of X with coefficient p_X* by:

$$L_n^\epsilon(X; p_X) \ := \bigcap_{0<\delta<\epsilon} L_n^\epsilon(X, p_X, \delta).$$

(3) The *controlled L-group with coefficient p_X* is defined by:

$$L_n^c(X; p_X) \ := \ \varprojlim_{\epsilon} L_n^\epsilon(X; p_X),$$

where the limit is taken with respect to the obvious relax-control maps:

$$L_n^{\epsilon'}(X; p_X) \longrightarrow L_n^\epsilon(X; p_X), \qquad (\epsilon' < \epsilon).$$

In section 5, we study a certain stability result for the controlled L-groups in some special case.

2. Epsilon-controlled projective L-groups.

Fix a subset Y of X, and let $\mathcal{F}$ be a family of subsets of X such that $Z \supset Y$ for each $Z \in \mathcal{F}$. In this section we introduce intermediate ϵ-controlled L-groups $L_n^{\mathcal{F}}(Y, p_X, \epsilon)$, which will appear in the stable-exact sequence of a pair and also in the Mayer-Vietoris sequence. Roughly speaking, these are defined using "controlled projective quadratic chain complexes" $((C, p), \psi)$ with vanishing ϵ-controlled reduced projective class $[C, p] = 0 \in \widetilde{K}_0(Z, p_Z, n, \epsilon)$ (Ranicki and Yamasaki [4]) for each $Z \in \mathcal{F}$. Here p_Z denotes the restriction $p_X|p_X^{-1}(Z) : p_X^{-1}(Z) \to Z$ of p_X as in the previous section.

For a projective module (A, p) on p_X, its dual $(A, p)^*$ is the projective module (A^*, p^*) on p_X. If $f : (A, p) \to (B, q)$ is an ϵ morphism ([4]), then $f^* : (B, q)^* \to (A, p)^*$ is also an ϵ morphism. For an ϵ projective chain complex on p_X

$$(C, p) \ : \ \dots \longrightarrow (C_r, p_r) \xrightarrow{d_r} (C_{r-1}, p_{r-1}) \xrightarrow{d_{r-1}} \dots$$

in the sense of [4], $(C, p)^{n-*}$ will denote the ϵ projective chain complex on p_X defined by:

$$\dots \longrightarrow (C^{n-r}, p^*_{n-r}) \xrightarrow{(-)^r d^*_{n-r+1}} (C^{n-r+1}, p^*_{n-r+1}) \longrightarrow \dots .$$

An *n-dimensional ϵ quadratic structure* on a projective chain complex (C,p) on p_X is an n-dimensional ϵ quadratic structure ψ on C (in the sense of §1) such that $\psi_s : (C^{n-r-s}, p^*) \to (C_r, p)$ is an ϵ morphism for every $s \geq 0$ and $r \in \mathbb{Z}$. Similarly, an $(n+1)$*-dimensional ϵ quadratic structure* on a chain map $f : (C,p) \to (D,q)$ is an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : C \to D$ such that $\delta\psi_s : (D^{n+1-r-s}, q^*) \to (D_r, q)$ and $\psi_s : (C^{n-r-s}, p^*) \to (C_r, p)$ are ϵ morphisms for every $s \geq 0$ and $r \in \mathbb{Z}$. An n-dimensional ϵ projective chain complex (C,p) on p_X equipped with an n-dimensional ϵ quadratic structure is called an *n-dimensional ϵ projective quadratic complex on p_X*, and an ϵ chain map $f : (C,p) \to (D,q)$ between an n-dimensional ϵ projective chain complex (C,p) on p_X and an $(n+1)$-dimensional ϵ projective chain complex (D,q) on p_X equipped with an $(n+1)$-dimensional ϵ quadratic structure is called an $(n+1)$*-dimensional ϵ projective quadratic pair on p_X*.

An *ϵ cobordism* of n-dimensional ϵ projective quadratic complexes $((C,p),\psi)$, $((C',p'),\psi')$ on p_X is an $(n+1)$-dimensional ϵ projective quadratic pair on p_X

$$((\,f \quad f'\,) : (C,p) \oplus (C',p') \longrightarrow (D,q), (\delta\psi, \psi \oplus -\psi'))$$

with boundary $((C,p) \oplus (C',p'), \psi \oplus -\psi')$.

An n-dimensional ϵ quadratic structure ψ on (C,p) is *ϵ Poincaré* if

$$\partial(C,p) = \Omega\mathcal{C}((1+T)\psi_0 : (C^{n-*}, p^*) \longrightarrow (C,p))$$

is 4ϵ contractible. $((C,p),\psi)$ is *ϵ Poincaré* if ψ is ϵ Poincaré. Similarly, an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : (C,p) \to (D,q)$ is *ϵ Poincaré* if $\partial(C,p)$ and

$$\partial(D,q) = \Omega\mathcal{C}(((1+T)\delta\psi_0 \quad f(1+T)\psi_0) : \mathcal{C}(f)^{n+1-*} \longrightarrow (D,q))$$

are both 4ϵ contractible. A pair $(f, (\delta\psi, \psi))$ is *ϵ Poincaré* if $(\delta\psi, \psi)$ is ϵ Poincaré.

Let Y and be a subset of X and $\mathcal{F}$ be a family of subsets of X such that $Z \supset Y$ for every $Z \in \mathcal{F}$.

Definition. Let $n \geq 0$ and $\epsilon \geq 0$. $L_n^{\mathcal{F}}(Y, p_X, \epsilon)$ is the equivalence classes of n-dimensional ϵ Poincaré ϵ projective quadratic complexes $((C,p),\psi)$ on p_Y such that $[C,p] = 0$ in $\widetilde{K}_0(Z, p_Z, n, \epsilon)$ for each $Z \in \mathcal{F}$. The equivalence relation is generated by ϵ Poincaré ϵ cobordisms $((\,f \quad f'\,) : (C,p) \oplus (C',p') \to (D,q), (\delta\psi, \psi \oplus -\psi'))$ on p_Y such that $[D,q] = 0$ in $\widetilde{K}_0(Z, p_Z, n+1, \epsilon)$ for each $Z \in \mathcal{F}$. When $\mathcal{F} = \{X\}$, we omit the braces and write $L_n^X(Y, p_X, \epsilon)$ instead of $L_n^{\{X\}}(Y, p_X, \epsilon)$. When $\mathcal{F} = \{\ \}$, then we use the notation $L_n^p(Y, p_Y, \epsilon)$, since it depends only on p_Y.

Proposition 2.1. *Direct sum induces an abelian group structure on* $L_n^{\mathcal{F}}(Y, p_X, \epsilon)$. *Furthermore, if*

$$[(C,p),\psi] = [(C',p'),\psi'] \in L_n^{\mathcal{F}}(Y, p_X, \epsilon) \ ,$$

then there is a 100ϵ *Poincaré* 2ϵ *cobordism on* p_Y

$$((f \quad f') : (C,p) \oplus (C',p') \to (D,q), (\delta\psi, \psi \oplus -\psi'))$$

such that $[D,q] = 0$ *in* $\widetilde{K}_0(Z, p_Z, n+1, 9\epsilon)$ *for each* $Z \in \mathcal{F}$.

A functoriality with respect to maps and relaxation of control similar to 1.2 holds for epsilon controlled projective L-groups.

Proposition 2.2. *Let* $F = (f, \bar{f})$ *be a map from* $p_X : M \to X$ *to* $p_Y : N \to Y$, *and suppose that* $\bar{f}$ *is Lipschitz continuous with Lipschitz constant* λ, *i.e., there exists a constant* $\lambda > 0$ *such that*

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

If $\epsilon \geq \lambda\delta$, $\bar{f}(A) \subset B$, *and there exists a* $Z \in \mathcal{F}$ *satisfying* $\bar{f}(Z) \subset Z'$ *for each* $Z' \in \mathcal{F}'$, *then* F *induces a homomorphism*

$$F_* \ : \ L_n^{\mathcal{F}}(A, p_X, \delta) \ \longrightarrow \ L_n^{\mathcal{F}'}(B, p_Y, \epsilon).$$

Remark. As in the remark to 1.2, for a specific δ and a ϵ, we do not need the full Lipschitz condition to guarantee the existence of F_*.

There is an obvious homomorphism

$$\iota_\epsilon \ : \ L_n(Y, p_Y, \epsilon) \longrightarrow L_n^{\mathcal{F}}(Y, p_X, \epsilon); \qquad [C,\psi] \mapsto [(C,1),\psi].$$

On the other hand, the controlled K-theoretic condition posed in the definition can be used to construct a homomorphism from a projective L-group to a free L-group:

Proposition 2.3. *There exist a constant* $\alpha > 1$ *such that the following holds true: for any control map* $p_X : M \to X$, *any subset* $Y \subset X$, *any family of subsets* $\mathcal{F}$ *of* X *containing* Y , *any element* Z *of* $\mathcal{F}$, *any number* $n \geq 0$, *and any positive numbers* δ, ϵ *such that* $\epsilon \geq \alpha\delta$, *there is a well-defined homomorphism functorial with respect to relaxation of control:*

$$(i_Z)_* \ : \ L_n^{\mathcal{F}}(Y, p_X, \delta) \longrightarrow L_n(Z, p_Z, \epsilon)$$

such that the following compositions are equal to the maps induced from inclusion maps:

$$L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{(i_Z)_*} L_n(Z, p_Z, \epsilon) \xrightarrow{\iota_\epsilon} L_n^{\{Z\}}(Z, p_Z, \epsilon) \ ,$$

$$L_n(Y, p_Y, \delta) \xrightarrow{\iota_\delta} L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{(i_Z)_*} L_n(Z, p_Z, \epsilon) \ .$$

Remark. Actually $\alpha = 30000$ works.

3. Stably-exact sequences.

In this section we describe two 'stably-exact' sequences. The first is the stably-exact sequence of a pair:

$$\overset{\partial}{\cdots\triangleright} L_n^X(Y_n, p_X, \epsilon) \overset{i_*}{\cdots\triangleright} L_n(X, p_X, \epsilon) \xrightarrow{j_*} L_n(X, Y, p_X, \epsilon) \overset{\partial}{\cdots\triangleright} L_{n-1}^X(Y, p_X, \epsilon) \overset{i_*}{\cdots\triangleright}$$

where the dotted arrows are only 'stably' defined. The precise meaning will be explained below. The second is the Mayer-Vietoris-type stably-exact sequence:

$$\begin{aligned}\overset{\partial}{\cdots\triangleright} L_n^{\mathcal{F}}(C, p_X, \epsilon) \overset{i_*}{\cdots\triangleright} L_n(A, p_A, \epsilon) \oplus L_n(B, p_B, \epsilon) &\xrightarrow{j_*} L_n(X, p_X, \epsilon) \\ &\overset{\partial}{\cdots\triangleright} L_{n-1}^{\mathcal{F}}(C, p_X, \epsilon) \overset{i_*}{\cdots\triangleright}\end{aligned}$$

where $X = A \cup B$, $C = A \cap B$, and $\mathcal{F} = \{A, B\}$.

Fix an integer $n \geq 0$, let Y_n, Z_n be subsets of X, and let γ_n, δ_n, ϵ_n be three positive numbers satisfying

$$\epsilon_n \geq \delta_n, \qquad \delta_n \geq \alpha\gamma_n$$

where α is the number (> 1) posited in 2.3. Then there is a sequence

$$L_n^X(Y_n, p_X, \gamma_n) \xrightarrow{i_* = (i_X)_*} L_n(X, p_X, \delta_n) \xrightarrow{j_*} L_n(X, Z_n, p_X, \epsilon_n),$$

where i_* is the homomorphism given in 2.3 and j_* is the homomorphism induced by the inclusion map and relaxation of control. (The subscripts are there just to remind the reader of the degrees of the relevant L-groups.)

Theorem 3.1. *There exist constants κ_0, κ_1, κ_2, $\ldots (> 1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$, $Z_n \supset Y_n^{\kappa_n\delta_n}$, and $\epsilon_n \geq \kappa_n\delta_n$, then the following composition j_*i_* is zero:*

$$j_*i_* = 0 \ : \ L_n^X(Y_n, p_X, \gamma_n) \xrightarrow{i_*} L_n(X, p_X, \delta_n) \xrightarrow{j_*} L_n(X, Z_n, p_X, \epsilon_n),$$

(2) *if $n \geq 1$, $Y_{n-1} \supset Z_n^{\kappa_n\epsilon_n}$ and $\gamma_{n-1} \geq \kappa_n\epsilon_n$, then there is a connecting homomorphism*

$$\partial \ : \ L_n(X, Z_n, p_X, \epsilon_n) \longrightarrow L_{n-1}^X(Y_{n-1}, p_X, \gamma_{n-1}),$$

such that the following composition ∂j_* is zero:

$$\partial j_* = 0 \ : \ L_n(X, p_X, \delta_n) \xrightarrow{j_*} L_n(X, Z_n, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^X(Y_{n-1}, p_X, \gamma_{n-1}),$$

and, if $\delta_{n-1} \geq \alpha\gamma_{n-1}$ (so that the homomorphism i_* is well-defined), the following composition $i_*\partial$ is zero:

$$i_*\partial = 0 \ : \ L_n(X, Z_n, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^X(Y_{n-1}, p_X, \gamma_{n-1}) \xrightarrow{i_*} L_{n-1}(X, p_X, \delta_{n-1}).$$

Theorem 3.2. *There exist constants λ_0, λ_1, λ_2, $\ldots (> 1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$, $\delta_n \geq \alpha\gamma_n$ (so that i_* is well-defined), $\epsilon'_{n+1} \geq \lambda_n\delta_n$, $Z'_{n+1} \supset Y_n^{\lambda_n\delta_n}$, $Y'_n \supset {Z'_{n+1}}^{\kappa_{n+1}\epsilon'_{n+1}}$ and $\gamma'_n \geq \kappa_{n+1}\epsilon'_{n+1}$ (so that ∂' is well-defined), then the image of the kernel of i_* in $L_n^X(Y'_n, p_X, \gamma'_n)$ is in the image of ∂':*

$$\begin{tikzcd} & L_n^X(Y_n, p_X, \gamma_n) \arrow[r, "i_*"] \arrow[d] & L_n(X, p_X, \delta_n) \\ L_{n+1}(X, Z'_{n+1}, p_X, \epsilon'_{n+1}) \arrow[r, "\partial'"] & L_n^X(Y'_n, p_X, \gamma'_n) & \end{tikzcd}$$

(2) *if $n \geq 0$, $\epsilon_n \geq \delta_n$ (so that j_* is well-defined), $Y'_n \supset Z_n^{\lambda_n\epsilon_n}$, $\gamma'_n \geq \lambda_n\epsilon_n$, and $\delta'_n \geq \alpha\gamma'_n$ (so that i'_* is well-defined), then the image of the kernel of j_* in $L_n(X, p_X, \delta'_n)$ is in the image of i'_*:*

$$\begin{tikzcd} & L_n(X, p_X, \delta_n) \arrow[r, "j_*"] \arrow[d] & L_n(X, Z_n, p_X, \epsilon_n) \\ L_n^X(Y'_n, p_X, \gamma'_n) \arrow[r, "i'_*"] & L_n(X, p_X, \delta'_n) & \end{tikzcd}$$

(3) *if $n \geq 1$, $\gamma_{n-1} \geq \kappa_n\epsilon_n$ (so that ∂ is well-defined), $\epsilon'_n \geq \lambda_n\gamma_{n-1}$, and $Z'_n \supset Y_{n-1}^{\lambda_n\gamma_{n-1}}$, then the image of the kernel of ∂ in $L_n(X, Z'_n, p_X, \epsilon'_n)$ is in the image of j'_*:*

$$\begin{tikzcd} & L_n(X, Z_n, p_X, \epsilon_n) \arrow[r, "\partial"] \arrow[d] & L_{n-1}^X(Y_{n-1}, p_X, \gamma_{n-1}) \\ L_n(X, p_X, \epsilon'_n) \arrow[r, "j'_*"] & L_n(X, Z'_n, p_X, \epsilon'_n) & \end{tikzcd}$$

Here the vertical maps are the homomorphisms induced by inclusion maps and relaxation of control.

Next we investigate the Mayer-Vietoris-type stably-exact sequence. Fix an integer $n \geq 0$, and assume that X is the union of two closed subsets A_n and B_n with intersection $C_n = A_n \cap B_n$. Suppose three positive numbers γ_n, δ_n, ϵ_n satisfy

$$\delta_n \geq \alpha\gamma_n, \qquad \epsilon_n \geq \delta_n,$$

and define a family $\mathcal{F}_n$ to be $\{A_n, B_n\}$. Then we have a sequence

$$L_n^{\mathcal{F}_n}(C_n, p_X, \gamma_n) \xrightarrow{i_*} L_n(A_n, p_{A_n}, \delta_n) \oplus L_n(B_n, p_{B_n}, \delta_n) \xrightarrow{j_*} L_n(X, p_X, \epsilon_n).$$

Theorem 3.3. *There exist constants κ_0, κ_1, κ_2, $\ldots(>1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$ and $\epsilon_n \geq \kappa_n\delta_n$, then the following composition j_*i_* is zero:*

$$L_n^{\mathcal{F}_n}(C_n, p_X, \gamma_n) \xrightarrow{i_*} L_n(A_n, p_{A_n}, \delta_n) \oplus L_n(B_n, p_{B_n}, \delta_n) \xrightarrow{j_*} L_n(X, p_X, \epsilon_n).$$

(2) *if $n \geq 1$, $C_{n-1} \supset C_n^{\kappa_n\epsilon_n}$, $\gamma_{n-1} \geq \kappa_n\epsilon_n$, and if we set*

$$\mathcal{F}_{n-1} = \{A_{n-1} = A_n \cup C_{n-1}, B_{n-1} = B_n \cup C_{n-1}\},$$

then there is a connecting homomorphism

$$\partial \; : \; L_n(X, p_X, \epsilon_n) \longrightarrow L_{n-1}^{\mathcal{F}_{n-1}}(C_{n-1}, p_X, \gamma_{n-1}),$$

such that the following composition ∂j_* is zero:

$$L_n(A_n, p, \delta_n) \oplus L_n(B_n, p, \delta_n) \xrightarrow{j_*} L_n(X, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^{\mathcal{F}_{n-1}}(C_{n-1}, p_X, \gamma_{n-1}),$$

and, if $\delta_{n-1} \geq \alpha\gamma_{n-1}$ (so that the homomorphism i_* is well-defined), the following composition $i_*\partial$ is zero:

$$\begin{aligned} L_n(X, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^{\mathcal{F}_{n-1}}&(C_{n-1}, p_X, \gamma_{n-1}) \\ &\xrightarrow{i_*} L_{n-1}(A_{n-1}, p, \delta_{n-1}) \oplus L_{n-1}(B_{n-1}, p, \delta_{n-1}). \end{aligned}$$

Theorem 3.4. *There exist constants λ_0, λ_1, λ_2, $\ldots(>1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$, $\delta_n \geq \alpha\gamma_n$ (so that i_* is well-defined), $\epsilon'_{n+1} \geq \lambda_n\delta_n$, $C'_n \supset C_n^{\lambda_n\delta_n}$, $\gamma'_n \geq \kappa_{n+1}\epsilon'_{n+1}$ (so that ∂' is well-defined), then the image of the kernel of i_* in $L_{n-1}^{\mathcal{F}'_{n-1}}(C'_{n-1}, p_X, \gamma'_{n-1})$ is in the image of ∂':*

$$\begin{tikzcd} & L_n^{\mathcal{F}_n}(C_n, p_X, \gamma_n) \arrow[r, "i_*"] \arrow[d] & L_n(A_n, p, \delta_n) \oplus L_n(B_n, p, \delta_n) \\ L_{n+1}(X, p_X, \epsilon'_{n+1}) \arrow[r, "\partial'"] & L_n^{\mathcal{F}'_n}(C'_n, p_X, \gamma'_n) & \end{tikzcd}$$

(2) *if $n \geq 0$, $\epsilon_n \geq \delta_n$ (so that j_* is well-defined), $C'_n \supset C_n^{\lambda_n\epsilon_n}$, $\gamma'_n \geq \lambda_n\epsilon_n$, $\delta'_n \geq \alpha\gamma'_n$ (so that i'_* is well-defined), and $\mathcal{F}'_n = \{A'_n = A_n \cup C'_n, B'_n = B_n \cup C'_n\}$, then the image of the kernel of j_* in $L_n(A'_n, p, \delta'_n) \oplus L_n(B'_n, p, \delta'_n)$ is in the image of i'_*:*

$$\begin{tikzcd} & L_n(A_n, p, \delta_n) \oplus L_n(B_n, p, \delta_n) \arrow[r, "j_*"] \arrow[d] & L_n(X, p_X, \epsilon_n) \\ L_n^{\mathcal{F}'_n}(C'_n, p_X, \gamma'_n) \arrow[r, "i'_*"] & L_n(A'_n, p, \delta'_n) \oplus L_n(B'_n, p, \delta'_n) & \end{tikzcd}$$

(3) *if* $n \geq 1$, $C_{n-1} \supset C_n^{\kappa_n \epsilon_n}$, $\gamma_{n-1} \geq \kappa_n \epsilon_n$ *(so that* ∂ *is well-defined),* $\epsilon'_n \geq \lambda_n \gamma_{n-1}$, $C'_n \supset C_{n-1}^{\lambda_n \gamma_{n-1}}$, $A'_n = A_n \cup C'_n$, *and* $B'_n = B_n \cup C'_n$, *then the image of the kernel of* ∂ *in* $L_n(X, p_X, \epsilon'_n)$ *is in the image of* j'_**:*

$$\begin{array}{ccc} & L_n(X, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^{\mathcal{F}_{n-1}}(C_{n-1}, p_X, \gamma_{n-1}) \\ & \downarrow \\ L_n(A'_n, p, \epsilon'_n) \oplus L_n(B'_n, p, \epsilon'_n) \xrightarrow{j'_*} & L_n(X, p_X, \epsilon'_n) \end{array}$$

Here the vertical maps are the homomorphisms induced by inclusion maps and relaxation of control.

Theorems 3.1 – 3.4 are all straightforward to prove.

4. Locally-finite analogues.

Up to this point, we considered only finitely generated modules and chain complexes. To study the behaviour of controlled L-groups, we need to use infinitely generated objects; such objects arise naturally when we take the pullback of a finitely generated object via an infinite-sheeted covering map.

Consider a control map $p_X : M \to X$, and take the product with another metric space N:

$$p_X \times 1_N \ : \ M \times N \longrightarrow X \times N.$$

Here we use the maximum metric on the product $X \times N$.

Definition. (Ranicki and Yamasaki [4, p.14]) A geometric module on the product space $M \times N$ is said to be *M-finite* if, for any $y \in N$, there is a neighbourhood U of y in N such that $M \times U$ contains only finitely many basis elements; a projective module (A, p) on $M \times N$ is said to be *M-finite* if A is M-finite; a projective chain complex (C, p) on $M \times N$ is *M-finite* if each (C_r, p_r) is M-finite. [In [4], we used the terminology "*M-locally finite*", but this does not sound right and we decided to use "*M-finite*" instead. "*N-locally M-finite*" may be describing the meaning better, but it is too long.] When M is compact, M-finiteness is equivalent to the ordinary locally-finiteness.

Definition. Using this notion, one can define *M-finite ϵ-controlled L-groups* $L_n^M(X \times N, Y \times N, p_X \times 1_N, \epsilon)$, and *$M$-finite ϵ-controlled projective L-groups* $L_n^{M,\mathcal{F}}(Y \times N, p_X \times 1_N, \epsilon)$ by requiring that every chain complexes concerned are M-finite.

Consider the case when $N = \mathbb{R}$. We would like to apply the M-finite version of the Mayer-Vietoris-type stable exact sequence with respect to the splitting $\mathbb{R} = (-\infty, 0] \cup [1, \infty)$. The following says that one of the three terms in the sequence vanishes.

Proposition 4.1. Let $p_X : M \to X$ be a control map. For any $\epsilon > 0$ and $r \in \mathbb{R}$,

$$L_n^M(X \times (\infty, r], p_X \times 1, \epsilon) = L_n^M(X \times [r, \infty), p_X \times 1, \epsilon) = 0.$$
$$\widetilde{K}_0^M(X \times (\infty, r], p_X \times 1, n, \epsilon) = \widetilde{K}_0^M(X \times [r, \infty), p_X \times 1, n, \epsilon) = 0.$$

Proof: This is done using repeated shifts towards infinity and the 'Eilenberg Swindle'. Let us consider the case of $L_n^M(X \times [r, \infty), p_X \times 1, \epsilon)$. Let $J = [r, \infty)$ and define $T : M \times J \to M \times J$ by $T(x,t) = (x, t+\epsilon)$. Take an element $[c] \in L_n^M(X \times J, p_X \times 1, \epsilon)$. It is zero, because there exist M-finite ϵ Poincaré cobordisms:

$$\begin{aligned} c \ &\sim\ c \oplus (T_\#(-c) \oplus T_\#^2(c)) \oplus (T_\#^3(-c) \oplus T_\#^4(c)) \oplus \dots \\ &=\ (c \oplus T_\#(-c)) \oplus (T_\#^2(c) \oplus T_\#^3(-c)) \oplus \dots \ \sim\ 0\ . \end{aligned}$$

□

Thus, the Mayer-Vietoris stably-exact sequence reduces to:

$$0 \longrightarrow L_n^M(X \times \mathbb{R}, p_X \times 1_\mathbb{R}, \epsilon) \xrightarrow{\partial} L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \longrightarrow 0,$$

where $\gamma = \kappa_n \epsilon$, $I = [-\delta, \delta]$, for some $\delta > 0$. A diagram chase shows that there exists a well-defined homomorphism:

$$\beta : L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \longrightarrow L_n^M(X \times \mathbb{R}, p_X \times 1_\mathbb{R}, \epsilon'),$$

where $\epsilon'' = \lambda_n \kappa_n \lambda_{n-1} \alpha \gamma$. The homomorphisms ∂ and β are stable inverses of each other; the compositions

$$\begin{aligned} \beta\partial\ &:\ L_n^M(X \times \mathbb{R}, p_X \times 1_\mathbb{R}, \epsilon) \longrightarrow L_n^M(X \times \mathbb{R}, p_X \times 1_\mathbb{R}, \epsilon') \\ \partial\beta\ &:\ L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \longrightarrow L_{n-1}^p(X \times I, p_X \times 1_I, \kappa_n \epsilon') \end{aligned}$$

are both relax-control maps.

Note that, for any γ, a projective L-group analogue of 1.2 gives an isomorphism:

$$L_{n-1}^p(X \times I, p_X \times 1_I, \gamma)\ \cong\ L_{n-1}^p(X \times \{0\}, p_X, \gamma).$$

In this case, no composition with relax-control map is necessary, because $X \times I$ is given the maximum metric. Thus, we have obtained:

Theorem 4.2. *There is a stable isomorphism:*

$$L_n^M(X \times \mathbb{R}, p_X \times 1_\mathbb{R}, \epsilon) \longrightarrow L_{n-1}^p(X, p_X, \gamma).$$

Similarly, we have:

Theorem 4.3. *There is a stable isomorphism:*

$$L_n^{lf}(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon) \longrightarrow L_{n-1}^{p,lf}(X, p_X, \gamma).$$

5. Stability in a special case.

In this section we treat the special case when the control map is the identity map. The following can be used to replace the controlled projective L-group terms in the previous section by controlled L-groups.

Proposition 5.1. *Suppose that $Y(\subset X)$ is a compact polyhedron or a compact metric ANR embedded in the Hilbert cube and that p_Y is the identity map 1_Y on Y. Then for any $\epsilon > 0$ and n, there exists a $\delta_0 > 0$ such that for any positive number δ satisfying $\delta \leq \delta_0$ there is a well-defined homomorphism functorial with respect to relaxation of control:*

$$\tau_{\epsilon,\delta} \ : \ L_n^{\mathcal{F}}(Y, p_X, \delta) \ \longrightarrow \ L_n(Y, 1_Y, \epsilon)$$

such that the compositions

$$L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{\tau_{\epsilon,\delta}} L_n(Y, 1_Y, \epsilon) \xrightarrow{\iota_\epsilon} L_n^{\mathcal{F}}(Y, p_X, \epsilon)$$
$$L_n(Y, 1_Y, \delta) \xrightarrow{\iota_\delta} L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{\tau_{\epsilon,\delta}} L_n(Y, 1_Y, \epsilon)$$

are both relax-control maps. In particular $L_n^p(Y, 1_Y, \delta)$ and $L_n(Y, 1_Y, \epsilon)$ are stably isomorphic.

Proof: Let $\delta_1 = \epsilon/\alpha$, where α is the positive number posited in 2.3. By 8.2 and 8.3 of [4], there exists a $\delta_0 > 0$ such that the following map is a zero map:

$$\widetilde{K}_0(Y, 1_Y, n, \delta_0) \longrightarrow \widetilde{K}_0(Y, 1_Y, n, \delta_1); \qquad [C, p] \mapsto [C, p].$$

Therefore, if $\delta \leq \delta_0$, there is a homomorphism

$$L_n^{\mathcal{F}}(Y, p_X, \delta) \longrightarrow L_n^{\mathcal{F}\cup\{Y\}}(Y, p_X, \delta_1); \qquad [(C, p), \psi] \mapsto [(C, p), \psi].$$

The desired map $\tau_{\epsilon,\delta}$ is obtained by composing this with the map

$$(i_Y)_* \ : \ L_n^{\mathcal{F}\cup\{Y\}}(Y, p_X, \delta) \longrightarrow L_n(Y, 1_X, \epsilon)$$

corresponding to the subspace Y. □

Remark. If Y is a compact polyhedron, then there is a constant $\kappa_n^Y > 1$ which depends on n and Y such that δ_0 above can be taken to be δ_1/κ_n^Y. For this we need to change the statement and the proof of 8.1 of [4] like those of 5.4 below.

Recall that in our Mayer-Vietoris-type stably-exact sequence, each piece of space tends to get bigger in the process. The following can be used to remedy this in certain cases. (It is stated here for the identity control map, but there is an obvious extension to general control maps.)

Proposition 5.2. Let $r : X \to A$ be a strong deformation retraction, with a Lipschitz continuous strong deformation of Lipschitz constant λ, and $i : A \to X$ be the inclusion map. Then r and i induce "stable" isomorphisms of controlled L-groups in the following sense: if $\epsilon > 0$, then for any δ ($0 < \delta \leq \epsilon/\lambda$) the compositions

$$L_n(X, 1_X, \delta) \xrightarrow{r_*} L_n(A, 1_A, \epsilon) \xrightarrow{i_*} L_n(X, 1_X, \epsilon)$$
$$L_n(A, 1_A, \delta) \xrightarrow{i_*} L_n(X, 1_X, \delta) \xrightarrow{r_*} L_n(A, 1_A, \epsilon)$$

are relax-control maps.

Proof: Obvious from 1.2. ☐

Theorem 5.3. *Fix a compact polyhedron X and an integer $n \geq 0$. Then there exist numbers $\epsilon_1 > 0$, $\kappa \geq 1$ and $\lambda \geq 1$ (which depend on n, X, and the triangulation) such that, for any subpolyhedrons A and B of X, any integer $k \geq 0$, and any number $0 < \epsilon \leq \epsilon_1$, there exists a ladder:*

$$\begin{tikzcd}
L_n^{lf}(C \times \mathbb{R}^k, 1, \epsilon) \arrow[r, "i_*"] \arrow[d] & L_n^{lf}(A \times \mathbb{R}^k, 1, \epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^k, 1, \epsilon) \arrow[r, "j_*"] \arrow[d] & L_n^{lf}(K \times \mathbb{R}^k, 1, \epsilon) \arrow[d] \\
L_n^{lf}(C \times \mathbb{R}^k, 1, \lambda\epsilon) \arrow[r, "i_*"'] & L_n^{lf}(A \times \mathbb{R}^k, 1, \lambda\epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^k, 1, \lambda\epsilon) \arrow[r, "j_*"'] & L_n^{lf}(K \times \mathbb{R}^k, 1, \lambda\epsilon)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\partial"] & L_n^{lf}(C \times \mathbb{R}^{k+1}, 1, \kappa\epsilon) \arrow[r, "i_*"] \arrow[d] & L_n^{lf}(A \times \mathbb{R}^{k+1}, 1, \kappa\epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^{k+1}, 1, \kappa\epsilon) \arrow[d] \\
{} \arrow[r, "\partial"'] & L_n^{lf}(C \times \mathbb{R}^{k+1}, 1, \kappa\lambda\epsilon) \arrow[r, "i_*"'] & L_n^{lf}(A \times \mathbb{R}^{k+1}, 1, \kappa\lambda\epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^{k+1}, 1, \kappa\lambda\epsilon)
\end{tikzcd}$$

which is stably-exact in the sense that

1. *the image of a horizontal map is contained in the kernel of the next map, and*
2. *the relax-control image in the second row of the kernel of a map in the first row is contained in the image of a horizontal map from the left,*

where $C = A \cap B$ and $K = A \cup B$, and the vertical maps are relax-control maps.

Proof: This is obtained from the locally-finite versions of 3.3, 3.4 combined with 4.3, 5.1, and 5.2 (the strong deformations of the neighbourhoods of A and B in K can be chosen to be PL and hence Lipschitz). Since there are only finitely many subpolyhedrons of X (with a fixed triangulation), we may choose constants κ and λ independent of A and B. ☐

Theorem 5.4. *Suppose X is a compact polyhedron and $n \geq 0$ is an integer. Then there exist numbers $\epsilon_0 > 0$ and $0 < \mu_0 \leq 1$ which depend on X and n such that*

$$L_n^\epsilon(X, 1_X, \delta) = L_n^c(X; 1_X)$$

for every $\epsilon \leq \epsilon_0$ and every $\delta \leq \mu_0\epsilon$.

Proof: We inductively construct sequences of positive numbers

$$\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \ldots \ (> 0)$$
$$(1 \geq)\ \mu_1 \ \geq \ \mu_2 \ \geq \ \mu_3 \ \geq \ \ldots \ (> 0)$$

such that for any subcomplex K of X with the number of simplices $\leq l$,

(1) if $0 < \epsilon \leq \epsilon_l$, $0 < \delta \leq \mu_l\epsilon$, and $k \geq 0$, then

$$L_n^{lf,\epsilon}(K \times \mathbb{R}^k, 1_K \times 1, \delta) \ = \ L_n^{lf,\epsilon}(K \times \mathbb{R}^k, 1_K \times 1, \mu_l\epsilon),$$

and

(2) if $0 < \epsilon \leq \epsilon_l$, then the homomorphism

$$L_n^{lf,\epsilon}(K \times \mathbb{R}^k; 1_K \times 1) \longrightarrow L_n^{lf,\epsilon_l}(K \times \mathbb{R}^k; 1_K \times 1)$$

is injective.

Here $\mathbb{R}^k$ is given the maximum metric.

First suppose $l = 1$ (*i.e.* K is a single point). Any object with bounded control on $\mathbb{R}^k$ can be squeezed to obtain an arbitrarily small control; therefore,

$$\epsilon_1 \ = \ \text{the number posited in 5.2}, \qquad \mu_1 \ = \ 1$$

works.

Next assume we have constructed ϵ_i and μ_i for $i \leq l$. We claim that

$$\epsilon_{l+1} \ = \ \min\{\frac{\mu_l}{\lambda}, \frac{1}{\kappa}\}\epsilon_l, \qquad \mu_{l+1} \ = \ \frac{\mu_l^2}{\lambda\kappa}$$

satisfy the required condition. Suppose the number of simplices of K is less than or equal to $l+1$. Choose a simplex of K of the highest dimension, and call the simplex (viewed as a subpolyhedron) A, and let $B = K - \mathrm{int}A$. Suppose $0 < \epsilon \leq \epsilon_{l+1}$ and $0 < \delta \leq \mu_l\epsilon$. A diagram chase starting from an element of

$$L_n^{lf}(K \times \mathbb{R}^k, 1, \mu_{l+1}\epsilon)$$

in the following diagram establishes the property (1). Here the entries in each of the columns are

$$L_n^{lf}(A \times \mathbb{R}^k, 1, \gamma) \oplus L_n^{lf}(B \times \mathbb{R}^k, 1, \gamma)\ , \quad L_n^{lf}(C \times \mathbb{R}^k, 1, \gamma)$$
$$L_n^{lf}(K \times \mathbb{R}^k, 1, \gamma)\ , \quad \text{and } L_n^{lf}(A \times \mathbb{R}^{k+1}, 1, \gamma) \oplus L_n^{lf}(B \times \mathbb{R}^{k+1}, 1, \gamma)\ ,$$

for various γ's specified in the diagram.

$$\begin{tikzcd}
L_n^{lf}(A\ldots)\oplus L_n^{lf}(B\ldots) & L_n^{lf}(K\ldots) & L_n^{lf}(C\ldots) & L_n^{lf}(A\ldots)\oplus L_n^{lf}(B\ldots) \\
 & & \mu_l\frac{\delta}{\lambda} \arrow[r] \arrow[d] & \mu_l\frac{\delta}{\lambda} \arrow[d] \\
 & & \frac{\delta}{\lambda} \arrow[r] \arrow[d] & \frac{\delta}{\lambda} \arrow[ddd] \\
\delta \arrow[r] \arrow[ddd] & \delta \arrow[r] \arrow[d] & \kappa\delta \arrow[d] & \\
 & \mu_{l+1}\epsilon \arrow[r] \arrow[d] & \kappa\mu_{l+1}\epsilon \arrow[d] & \\
 & \frac{\mu_{l+1}}{\mu_l}\epsilon \arrow[r] \arrow[d] & \kappa\frac{\mu_{l+1}}{\mu_l}\epsilon \arrow[r] & \kappa\frac{\mu_{l+1}}{\mu_l}\epsilon \\
\lambda\kappa\frac{\mu_{l+1}}{\mu_l}\epsilon = \mu_l\epsilon \arrow[r] \arrow[d] & \mu_l\epsilon \arrow[d] & & \\
\epsilon \arrow[r] & \epsilon & &
\end{tikzcd}$$

Next suppose $0 < \epsilon \leq \epsilon_{l+1}$. A diagram chase starting from an element of

$$L_n^{lf}(K \times \mathbb{R}^k, 1, \mu_{l+1}\epsilon)$$

representing an element of

$$\ker\{L_n^{lf,\epsilon}(K \times \mathbb{R}^k; 1) \longrightarrow L_n^{lf,\epsilon_l}(K \times \mathbb{R}^k; 1)\}$$

establishes (2).

$$\begin{tikzcd}
L_n^{lf}(C\ldots) & L_n^{lf}(A\ldots)\oplus L_n^{lf}(B\ldots) & L_n^{lf}(K\ldots) & L_n^{lf}(C\ldots) \\
 & & \mu_{l+1}\epsilon = \frac{\mu_l^2}{\lambda\kappa}\epsilon \arrow[r] \arrow[d] & \frac{\mu_l^2}{\lambda}\epsilon \arrow[d] \\
 & & \frac{\mu_l}{\lambda\kappa}\epsilon \arrow[r] \arrow[d] & \frac{\mu_l}{\lambda}\epsilon \arrow[ddd] \\
\mu_l\epsilon \arrow[r] \arrow[d] & \mu_l\epsilon \arrow[r] \arrow[d] & \mu_l\epsilon \arrow[d] & \\
\epsilon \arrow[r] \arrow[dd] & \epsilon \arrow[r] \arrow[d] & \epsilon \arrow[d] & \\
 & \epsilon_{l+1} \arrow[r] \arrow[d] & \epsilon_{l+1} \arrow[r] & \kappa\epsilon_{l+1} \\
\lambda\epsilon_{l+1} \arrow[r] \arrow[d] & \lambda\epsilon_{l+1} \arrow[d] & & \\
\frac{1}{\mu_l}\lambda\epsilon_{l+1} \arrow[r] & \frac{1}{\mu_l}\lambda\epsilon_{l+1} & &
\end{tikzcd}$$

$\square$

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NOTES ON SURGERY AND C^*-ALGEBRAS

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1. Introduction

A C^**-algebra* is a complex Banach algebra A with an involution $*$, which satisfies the identity

$$\|x^*x\| = \|x\|^2 \quad \forall x \in A.$$

The study of C^*-algebras seems to belong entirely within the realm of functional analysis, but in the past twenty years they have played an increasing rôle in geometric topology. The reason for this is that C^*-algebra K-theory is a natural receptacle for 'higher indices' of elliptic operators, including the 'higher signatures' which feature as surgery obstructions. The 'big picture' was originated by Atiyah [1, 2] and Connes [5, 6]; in these notes, based on my talk at the Josai conference, I want to explain part of the connection with particular reference to surgery theory. For more details one could consult [24].

2. About C^*-algebras

The following are key examples of C^*-algebras

- The algebra $C(X)$ of continuous complex-valued functions on a compact Hausdorff space X.
- The algebra $\mathfrak{B}(H)$ of bounded linear operators on a Hilbert space H.

Gelfand and Naimark (about 1950) proved: Any commutative C^*-algebra with unit is of the form $C(X)$; any C^*-algebra is a subalgebra of some $\mathfrak{B}(H)$.

Let A be a unital C^*-algebra. Let $x \in A$ be *normal*, that is $xx^* = x^*x$. Then x generates a commutative C^*-subalgebra of A which must be of the form $C(X)$. In fact we can identify X as the *spectrum*

$$X = \sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ has no inverse}\}$$

with x itself corresponding to the canonical $X \to \mathbb{C}$.

Hence we get the *Spectral Theorem*: for any $\varphi \in C(\sigma(x))$ we can define $\varphi(x) \in A$ so that the assignment $\varphi \mapsto \varphi(x)$ is a ring homomorphism.

If x is *self adjoint* ($x = x^*$), then $\sigma(x) \subseteq \mathbb{R}$.

One can define K-theory groups for C^*-algebras. For A unital

- $K_0(A)$ = Grothendieck group of f.g. projective A-modules
- $K_1(A) = \pi_0 GL_\infty(A)$

with a simple modification for non-unital A. These groups agree with the ordinary topological K-theory groups of the space X in case A is the commutative C^*-algebra $C(X)$.

For any integer i define $K_i = K_{i\pm 2}$. Then to any short exact sequence of C^*-algebras

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$$0 \to J \to A \to A/J \to 0$$

there is a long exact K-theory sequence

$$\dots K_i(J) \to K_i(A) \to K_i(A/J) \to K_{i-1}(J) \dots$$

The 2-periodicity is a version of the Bott periodicity theorem. Notice that *algebraic* K-theory does not satisfy Bott periodicity; analysis is essential here.

A good reference for this material is [26].

Classical Fredholm theory provides a useful example of C^*-algebra K-theory at work. Recall that an operator T on a Hilbert space H is called *Fredholm* if it has finite-dimensional kernel and cokernel. Then the *index* of T is the difference of the dimensions of the kernel and cokernel.

(2.1) Definition: *The algebra of* compact operators, $\mathfrak{K}(H)$, *is the* C^**-algebra generated by the operators with finite-dimensional range.*

Compact and Fredholm operators are related by *Atkinson's Theorem*, which states that $T \in \mathfrak{B}(H)$ is Fredholm if and only if its image in $\mathfrak{B}(H)/\mathfrak{K}(H)$ is invertible.

Thus a Fredholm operator T defines a class $[T]$ in $K_1(\mathfrak{B}/\mathfrak{K})$. Under the connecting map this passes to $\partial[T] \in K_0(\mathfrak{K}) = \mathbb{Z}$; this is the index.

3. Abstract signatures

Recall that in symmetric L-theory we have isomorphisms $L^0(\mathbb{Z}) \to L^0(\mathbb{R}) \to \mathbb{Z}$. The second map associates to a nonsingular real symmetric matrix its *signature* = (Number of positive eigenvalues) − (Number of negative eigenvalues).

Can we generalize this to other rings?

If M is a nonsingular symmetric matrix over a C^*-algebra A we can use the spectral theorem to define projections p_+ and p_- corresponding to the positive and negative parts of the spectrum. Their difference is a class in $K_0(A)$.

This procedure defines a map $L_0^p(A) \to K_0(A)$ for every C^*-algebra A, and it can be shown that this map is an isomorphism [25]. There is a similar isomorphism on the level of L_1.

Now let Γ be a discrete group. The group ring $\mathbb{Z}\Gamma$ acts faithfully by convolution on the Hilbert space $\ell^2\Gamma$. The C^*-subalgebra of $\mathfrak{B}(\ell^2\Gamma)$ generated by $\mathbb{Z}\Gamma$ acting in this way is called the *group* C^**-algebra*, $C_r^*\Gamma$.

We have a map $L_0(\mathbb{Z}\Gamma) \to K_0(C_r^*\Gamma)$.

Gelfand and Mishchenko [10] observed that this map is a *rational isomorphism* for Γ free abelian. (Then $C_r^*\Gamma = C(T^k)$ by Fourier analysis.)

Remark: Our map from L_0 to K_0 is special to C^*-algebras; if it extended naturally to a map on all rings, we would have for a free abelian group Γ a diagram

$$\begin{array}{ccc} L_0(\mathbb{Z}\Gamma) & \longrightarrow & L_0(C_r^*\Gamma) \\ \downarrow & & \downarrow \\ K_0(\mathbb{Z}\Gamma) & \longrightarrow & K_0(C_r^*\Gamma) \end{array}$$

Going round the diagram via the top right we get Gelfand and Mishchenko's map, a rational isomorphism. But the bottom left-hand group is of rank one, by the Bass-Heller-Swan theorem [4, Chapter XII]. This contradiction shows that the left-hand vertical map cannot exist.

4. The signature operator

Let M be a complete oriented Riemannian manifold of even dimension (for simplicity). Define the operator $F = D(1+D^2)^{-\frac{1}{2}}$ on L^2 differential forms, where $D = d + d^*$, $d =$exterior derivative, $d^* =$ its adjoint.

F is *graded* by an involution $\varepsilon = i^? *$ (here $i = \sqrt{-1}$ and the power ? depends on the dimension and the degree of forms, see [3] for the correct formula). Thus graded it is called the *signature operator.*

If M is compact, then F is Fredholm. Moreover the index of F is the signature of M. This is a simple consequence of Hodge theory [3].

Remark: The choice of normalizing function $\varphi(x) = x(1+x^2)^{-\frac{1}{2}}$ in $F = \varphi(D)$ does not matter as long as it has the right asymptotic behaviour.

Consider now the signature operator on the universal cover $\widetilde{M}$ of a compact manifold M. F belongs to the algebra A of $\Gamma = \pi_1 M$ equivariant operators. Moreover it is invertible modulo the ideal J of Γ equivariant *locally compact* operators. This follows from the theory of elliptic operators.

Thus via the connecting map $\partial\colon K_1(A/J) \to K_0(J)$ we get an 'index' in $K_0(J)$.

(4.1) Lemma: *$J \equiv C_r^*\Gamma \otimes \mathfrak{K}$. Consequently $K_0(J) = K_0(C_r^*\Gamma)$.*

We have defined the *analytic signature* of M as an element of $K_0(C_r^*\Gamma)$. In general it can be defined in $K_i(M)$ where i is the dimension of M mod 2.

(4.2) Proposition: *The analytic signature is the image of the Mishchenko-Ranicki symmetric signature under the map $L^0 \to K_0$.*

(4.3) Corollary: *The analytic signature is invariant under orientation preserving homotopy equivalence.*

Direct proofs of this can be given [13].

We can now define an 'analytic surgery obstruction' (= difference of analytic signatures) for a degree one normal map.

Can we mimic the rest of the surgery exact sequence?

5. K-homology

Let A be a C^*-algebra. A *Fredholm module* for A is made up of the following things.

- A representation $\rho\colon A \to \mathfrak{B}(H)$ of A on a Hilbert space
- An operator $F \in \mathfrak{B}(H)$ such that for all $a \in A$ the operators

$$F\rho(a) - \rho(a)F, \quad (F^2-1)\rho(a), \quad (F-F^*)\rho(a)$$

  belong to $\mathfrak{K}(H)$.

The signature operator is an example with $A = C_0(M)$.

One can define both 'graded' and 'ungraded' Fredholm modules. These objects can be organized into Grothendieck groups to obtain Kasparov's *K-homology groups* $K^i(A)$ [15]. ($i = 0$ for graded and $i = 1$ for ungraded modules). They are contravariant functors of A.

REMARK: The critical condition in the definition is that $[F, \rho(a)] \in \mathfrak{K}$ for all a. One should regard this as a continuous control condition. In fact, if A is commutative it was shown by Kasparov that the condition is equivalent to $\rho(f)F\rho(g) \in \mathfrak{K}$ whenever f and g have disjoint supports — which is to say that F has 'only finite rank propagation' between open sets with disjoint closures.

Kasparov proved that the name 'K-homology' is justified.

(5.1) THEOREM: [15, 16] *Let $A = C(X)$ be a commutative C^*-algebra. Then $K^i(A)$ is naturally isomorphic to $H_i(X; \mathbb{K}(\mathbb{C}))$, the topological K-homology of X.*

We assume that X is metrizable here. If X is a 'bad' space (not a finite complex) then H refers to the Steenrod extension of K-homology [14, 9]; if X is only *locally* compact and we take $A = C_0(X)$ (the continuous functions vanishing at infinity), then we get locally finite K-homology.

Kasparov's definition was reformulated in the language of 'duality' by Paschke [20] and Higson. For a C^*-algebra A and ideal J define the algebra $\Psi(A//J)$ to consist of those $T \in \mathfrak{B}(H)$ such that

- $[T, \rho(a)] \in \mathfrak{K}\ \forall a \in A$, and
- $T\rho(j) \in \mathfrak{K}\ \forall j \in J$

where ρ is a good (i.e. sufficiently large) representation of A on H.

(5.2) PROPOSITION: (PASCHKE DUALITY THEOREM) *There is an isomorphism*

$$K^i(A) = K_{i+1}(\Psi(A//0)/\Psi(A//A))$$

for all separable C^*-algebras A.

Let us introduce some notation. For a locally compact space X, write $\Psi^0(X)$ for $\Psi(C_0(X)//0)$ (we call this the algebra of *pseudolocal* operators), and $\Psi^{-1}(X)$ for $\Psi(C_0(X)//C_0(X))$ (the algebra of locally compact operators).

Now let $X = \widetilde{M}$, the universal cover of a compact manifold M as above, and consider the exact sequence

$$0 \to \Psi^{-1}(\widetilde{M})^\Gamma \to \Psi^0(\widetilde{M})^\Gamma \to \Psi^0(M)/\Psi^{-1}(M) \to 0.$$

The superscript Γ denotes the Γ-equivariant part of the algebra. We have incorporated into the sequence the fundamental isomorphism

$$\Psi^0(\widetilde{M})^\Gamma/\Psi^{-1}(\widetilde{M})^\Gamma = \Psi^0(M)/\Psi^{-1}(M)$$

which exists because both sides consist of *local* objects — 'formal symbols' in some sense — and there is no difficulty in lifting a local object from a manifold to its universal cover.

Note that $\Psi^{-1}(\widetilde{M})$ = locally compact operators. Thus, applying the K-theory functor, we get a boundary map

$$A\colon K^i(M) = K_{i+1}(\Psi^0(M)/\Psi^{-1}(M)) \to K_i(C_r^*\Gamma).$$

This *analytic assembly map* takes the homology class of the signature operator F to the analytic signature.

6. On the Novikov conjecture

We use the above machinery to make a standard reduction of the Novikov conjecture. Assume $B\Gamma$ is compact and let $f\colon M \to B\Gamma$. Consider the diagram

$$\begin{tikzcd} H_*(M;\mathbb{Q}) \arrow[d, "f_*"'] & K_*(M) \arrow[l, "ch"'] \arrow[r] \arrow[d, "f_*"'] & K_*(C_r^*\Gamma) \\ H_*(B\Gamma;\mathbb{Q}) & K_*(B\Gamma) \arrow[l, "ch"'] \arrow[ur] & \end{tikzcd}$$

By the Atiyah-Singer index theorem $f_*(ch[F])$ is Novikov's higher signature (the push forward of the Poincaré dual of the L-class). So, if $A\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is injective, the Novikov conjecture is true for Γ.

This has led to a number of partial solutions to the Novikov conjecture using analysis. Methods used have included

- *Cyclic cohomology* [8, 7, 19] — pair $K_*(C_r^*\Gamma)$ with $H^*(B\Gamma;\mathbb{R})$. Need suitable dense subalgebras — very delicate.
- *Kasparov KK-theory* [16, 17] — sometimes allows one to construct an inverse of the assembly map as an 'analytic generalized transfer'.
- *Controlled C^*-algebra theory* [11] — parallel development to controlled topology, see later.

7. The analytic structure set

Recall the exact sequence

$$K_{i+1}(C_r^*\Gamma) \to K_i(\Psi^0(\widetilde{M})^\Gamma) \to K^i(M) \to K_i(C_r^*\Gamma)$$

The analogy with the surgery exact sequence suggests that we should think of $K_*(\Psi^0(\widetilde{M})^\Gamma)$ as the 'analytic structure set' of M.

Example: Suppose M is spin. Then one has the Dirac operator D and one can normalize as before to get a homology class

$[F]$, $F = \varphi(D)$.

If M has a metric of positive scalar curvature, then by Lichnerowicz there is a gap in the spectrum of D near zero. Thus we can choose the normalizing function φ so that $F^2 = 1$ *exactly.* Then $[F] \in K_*(\Psi^0(\widetilde{M})^\Gamma)$ gives the *structure invariant* of the positive scalar curvature metric.

Notice that Lichnerowicz' vanishing theorem [18] now follows from exactness in the analytic surgery sequence.

It is harder to give a map from the usual structure set to the analytic one! In the same way that the positive scalar curvature invariant gives a 'reason' for the Lichnerowicz vanishing theorem, we want an invariant which gives a 'reason' for the homotopy invariance of the symmetric signature.

Here is one possibility. Recall Pedersen's description (in these proceedings) of the structure set $\mathcal{S}^{TOP}(M)$, as the L-theory of the category

$$\mathcal{B}(\widetilde{M} \times I, \widetilde{M} \times 1; \mathbb{Z})^\Gamma.$$

Replacing $\mathbb{Z}$ by $\mathbb{C}$ we have a category

- whose objects can be completed to Hilbert spaces with $C_0(\widetilde{M})$-action
- whose morphisms are pseudolocal

Using Voiculescu's theorem (which says that all the objects can be embedded more or less canonically in a single 'sufficiently large' representation of $C_0(\widetilde{M})$) we should get a map from the structure set to $K^*(\Psi^0(\widetilde{M})^\Gamma)$. However, there is a significant problem: Are the morphisms bounded operators? Similar questions seem to come up elsewhere if one tries to use analysis to study homeomorphisms, and one needs some kind of torus trick to resolve them (compare [21]).

8. Controlled C^*-algebras

A more direct approach can be given [12] to obtaining a map from $\mathcal{S}^{DIFF}(M)$.

Let W be a metric space (noncompact) and suppose $\rho\colon C_0(W) \to \mathfrak{B}(H)$ as usual.

An operator T on H is *boundedly controlled* if there is $R = R(T)$ such that $\rho(\varphi)T\rho(\psi) = 0$ whenever distance from Support φ to Support ψ is greater than R.

Example: If D is a Dirac-type operator on complete Riemannian M, and φ has compactly supported Fourier transform, then $\varphi(D)$ is boundedly controlled [23].

Define $\Psi^j_{bc}(W)$, $j = 0, -1$, to be the C^* subalgebras generated by boundedly controlled elements. Then from the above one has that a Dirac type operator on a complete Riemannian manifold W has a 'boundedly controlled index' in $K_*(\Psi^{-1}_{bc}(W))$.

In fact *all* elliptic operators have boundedly controlled indices: in full generality one has a bounded assembly map $A\colon K^{lf}_*(W) \to K_*(\Psi^{-1}_{bc}(W))$, and

the assembly of the signature operator is the *bounded analytic signature*.

This bounded analytic signature can also be defined for suitable ('bounded, bounded') Poincaré complexes (bounded in both the analytic and geometric senses).

If W has a compactification $X = W \cup Y$ which is 'small at infinity', then there is a close relation between bounded and continuously controlled C^*-algebra theory [11].

In fact, consider a metrizable pair (X, Y), let $W = X \setminus Y$. We can define continuously controlled C^*-algebras, $\Psi^j_{cc}(W)$. Then one has

(8.1) Proposition: [11] *We have*

- $\Psi^0_{cc}(W) = \Psi^0(X) = \Psi(C(X)//0)$
- $\Psi^{-1}_{cc}(W) = \Psi(C(X)//C_0(W))$

The result for $\Psi^{-1}_{cc}(W)$ is an analytic counterpart to the theorem 'control means homology at infinity' (compare [22]).

Now we can define our map from the structure set; for simplicity we work in the simply connected case. Given a homotopy equivalence $f\colon M' \to M$, form the 'double trumpet space' W, consisting of open cones on M and M' joined by the mapping cylinder of f (there is a picture in [24]). This is a 'bounded, bounded' Poincaré space with a map to $M \times \mathbb{R}$, continuously controlled by $M \times S^0$.

Thus we have the analytic signature in $K_*(\Psi^{-1}_{cc}(X \times \mathbb{R}))$. Map this by the composite

$$\Psi^{-1}_{cc}(X \times \mathbb{R}) \to \Psi^0_{cc}(X \times \mathbb{R}) = \Psi^0(X \times I) \to \Psi^0(X)$$

using the preceding proposition. The image is the desired structure invariant.

The various maps we have defined fit into a diagram relating the geometric and C^* surgery exact sequences [12]. The diagram commutes up to some factors of 2, arising from the difference between the Dirac and signature operators.

9. Final remarks

- C^*-surgery can produce some information in a wide range of problems.
- Surjectivity of C^*-assembly maps is related to representation theory.
- Some techniques for Novikov are only available in the C^*-world.
- *But* We don't understand well how to do analysis on topological manifolds.
- Topologists construct; analysts only obstruct.

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CHARACTERIZATIONS OF INFINITE-DIMENSIONAL MANIFOLD TUPLES AND THEIR APPLICATIONS TO HOMEOMORPHISM GROUPS

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Chapter 1. Characterization of $(s, S_1, \cdots, S_l)$-manifolds

The purpose of this article is to survey tuples of infinite-dimensional topological manifolds and their application to the homeomorphism groups of manifolds.

1. $(s, S_1, \cdots, S_l)$-MANIFOLDS

A topological E-manifold is a space which is locally homeomorphic to a space E. In this article all spaces are assumed to be separable and metrizable. In infinite-dimensional topological manifold theory, we are mainly concerned with the following model spaces E:

(i) (the compact model) the Hilbert cube: $Q = [-\infty, \infty]^\infty$.

(ii) (the complete linear model) the Hilbert space: ℓ^2.

The Hilbert space ℓ^2, or more generally any separable Frechet space is homeomorphic to $s \equiv (-\infty, \infty)^\infty$ ([1]). If we regard s as a linear space of sequences of real numbers, then it contains several natural (incomplete) linear subspaces:

(iii) the big sigma: $\Sigma = \{(x_n) \in s \,:\, \sup_n |x_n| < \infty\}$ (the subspace of bounded sequences).

(iv) the small sigma: $\sigma = \{(x_n) \in s \,:\, x_n = 0$ for almost all $n\}$ (the subspace of finite sequences).

The main sources of infinite-dimensional manifolds are various spaces of functions, embeddings and homeomorphisms. In Chapter 2 we shall consider the group of homeomorphisms of a manifold. When M is a PL-manifold, the homeomorphism group $\mathcal{H}(M)$ contains the subgroup $\mathcal{H}^{\mathrm{PL}}(M)$ consisting of PL-homeomorphisms of M, and we can ask the natural question: How is $\mathcal{H}^{\mathrm{PL}}(M)$ sited in the ambient group $\mathcal{H}(M)$? This sort of question leads to the following general definition. An $(l+1)$-tuple of spaces means a tuple $(X, X_1, \cdots, X_l)$ consisting of an ambient space X and l subspaces $X_1 \supset \cdots \supset X_l$.

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Definition. A tuple $(X, X_1, \cdots, X_l)$ is said to be an $(E, E_1, \cdots, E_l)$-manifold if for every point $x \in X$ there exist an open neighborhood U of x in X and an open set V of E such that $(U, U \cap X_1, \cdots, U \cap X_l) \cong (V, V \cap E_1, \cdots, V \cap E_l)$.

In this article we shall consider the general model tuple of the form: $(s, S_1, \cdots, S_l)$, where $S_1 \supset \cdots \supset S_l$ are linear subspaces of s. Some typical examples are:
(v) the pairs: (s, Σ), (s, σ),
(vi) the triples: (s, Σ, σ), $(s^2, s \times \sigma, \sigma^2)$, $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$, and $(s^\infty, \sigma^\infty, \sigma_f^\infty)$.
where (a) s^∞, Σ^∞, σ^∞ are the countable product of s, Σ and σ respectively,
(b) $\Sigma_f^\infty = \{(x_n) \in \Sigma^\infty : x_n = 0 \text{ for almost all } n\}$, $\sigma_f^\infty = \{(x_n) \in \sigma^\infty : x_n = 0 \text{ for almost all } n\}$.
Note that (1) $(s^\infty, \sigma_f^\infty) \cong (s, \sigma)$, (2) $(s^\infty, \Sigma_f^\infty, \sigma_f^\infty) \cong (s, \Sigma, \sigma)$ and (3) $(s^\infty, \Sigma^\infty, \sigma_f^\infty) \cong (s^\infty, \sigma^\infty, \sigma_f^\infty)$. The statements (2) and (3) follow from the characterizations of manifolds modeled on these triples (§4.2.2, Theorems 3.11 - 3.14). In Section 4.2.1 we shall give a general characterization of $(s, S_1, \cdots, S_l)$-manifold under some natural conditions on the model $(s, S_1, \cdots, S_l)$.

2. Basic properties of infinite-dimensional manifolds

In this section we will list up some fundamental properties of infinite-dimensional manifolds. We refer to [10, 11, 24] for general references in infinite-dimensional manifold theory.

2.1. Stability.

Since s is a countable product of the interval $(-\infty, \infty)$, it is directly seen that $s^2 \cong s$. Applying this argument locally, it follows that $X \times s \cong X$ for every s-manifold X (cf.[25]). More generally, it has been shown that if (X, X_1, X_2) is an (s, Σ, σ)-manifold, then $(X \times s, X_1 \times \Sigma, X_2 \times \sigma) \cong (X, X_1, X_2)$ [27]. This property is one of characteristic properties of infinite-dimensional manifolds. To simplify the notation we shall use the following terminology:

Definition. We say that $(X, X_1, \cdots, X_l)$ is $(E, E_1, \cdots, E_l)$-stable if $(X \times E, X_1 \times E_1, \cdots, X_l \times E_l) \cong (X, X_1, \cdots, X_l)$.

2.2. Homotopy negligibility.

Definition. A subset B of Y is said to be homotopy negligible (h.n.) in Y if there exists a homotopy $\phi_t : Y \to Y$ such that $\phi_0 = id$ and $\phi_t(Y) \subset Y \setminus B$ $(0 < t \leq 1)$. In this case, we say that $Y \setminus B$ has the homotopy negligible (h.n.) complement in Y.

When Y is an ANR, B is homotopy negligible in Y iff for every open set U of Y, the inclusion $U \setminus B \subset U$ is a weak homotopy equivalence. Again using the infinite coordinates of s, we can easily verify that σ has the h.n. complement in

s. Therefore, it follows that if (X, X_1) is an (s, σ)-manifold, then X_1 has the h.n. complement in X.

2.3. **General position property – Strong universality.**

2.3.1. *Z-embedding approximation in s-manifolds.*

The most basic notion in infinite-dimensional manifolds is the notion of Z-sets:

Definition. A closed set Z of X is said to be a Z-set (a strong Z-set) of X if for every open cover $\mathcal{U}$ of X there is a map $f : X \to X$ such that $f(X) \cap Z = \emptyset$ ($cl\, f(X) \cap Z = \emptyset$) and $(f, id_X) \leq \mathcal{U}$.

Here, for an open cover $\mathcal{V}$ of Y, two map $f,\ g : X \to Y$ are said to be $\mathcal{V}$-close and written as $(f, g) \leq \mathcal{V}$ if for every $x \in X$ there exists a $V \in \mathcal{V}$ with $f(x)$, $g(x) \in V$. Using the infinite coordinates of s, we can show the following general position property of s-manifolds:

Facts 2.1. Suppose Y is an s-manifold. Then for every map $f : X \to Y$ from a separable completely metrizable space X and for every open cover $\mathcal{U}$ of Y, there exists a Z-embedding $g : X \to Y$ with $(f, g) \leq \mathcal{U}$. Furthermore, if K is a closed subset of X and $f|_K : K \to Y$ is a Z-embedding, then we can take g so that $g|_K = f|_K$.

2.3.2. *Strong universality.*

To treat various incomplete submanifolds of s-manifolds (σ-manifolds, Σ-manifolds, etc.), we need to restrict the class of domain X in the above statement. Let $\mathcal{C}$ be a class of spaces.

Definition. (M. Bestvina - J. Mogilski [5], et. al.)
A space Y is said to be strongly $\mathcal{C}$-universal if for every $X \in \mathcal{C}$, every closed subset K of X, every map $f : X \to Y$ such that $f|_K : K \to Y$ is a Z-embedding and for every open cover $\mathcal{U}$ of Y, there exists a Z-embedding $g : X \to Y$ such that $g|_K = f|_K$ and $(f, g) \leq \mathcal{U}$.

In some cases, the above embedding approximation conditions can be replaced by the following disjoint approximation conditions.

Definition. We say that a space X has the strong discrete approximation property (or the disjoint discrete cells property) if for every map $f : \oplus_{i \geq 1} Q_i \to X$ of a countable disjoint union of Hilbert cubes into X and for every open cover $\mathcal{U}$ of X there exists a map $g : \oplus_{i \geq 1} Q_i \to X$ such that $(f, g) \leq \mathcal{U}$ and $\{g(Q_i)\}_i$ is discrete in X.

2.3.3. *Strong universality of tuples.* (R. Cauty [6], J. Baars-H. Gladdines-J. van Mill [3], et. al.)

A map of tuples $f : (X, X_1, \cdots, X_l) \to (Y, Y_1, \cdots, Y_l)$ is said to be layer preserving if $f(X_{i-1} \setminus X_i) \subset Y_{i-1} \setminus Y_i$ for every $i = 1, \cdots, l+1$, where $X_0 = X$, $X_{l+1} = \emptyset$. Let $\mathcal{M}$ be a class of $(l+1)$-tuples of spaces.

Definition. An $(l+1)$-tuple $(Y, Y_1, \cdots, Y_l)$ is said to be strongly $\mathcal{M}$-universal if it satisfies the following condition:

$(*)$ for every tuple $(X, X_1, \cdots X_l) \in \mathcal{M}$, every closed subset K of X, every map $f : X \to Y$ such that $f|_K : (K, K \cap X_1, \cdots K \cap X_l) \to (Y, Y_1, \cdots, Y_l)$ is a layer preserving Z-embedding, and every open cover $\mathcal{U}$ of Y, there exists a layer preserving Z-embedding $g : (X, X_1, \cdots X_l) \to (Y, Y_1, \cdots, Y_l)$ such that $g|_K = f|_K$ and $(f, g) \leq \mathcal{U}$.

In Section 4.2.1 we shall see that the stability + h.n. complement implies the strong universality.

2.4. Uniqueness properties of absorbing sets.

The notion of h.n. complement can be regarded as a homotopical absorbing property of a subspace in an ambient space. The notion of strong universality of tuples also can be regarded as a sort of absorption property combined with the general position property. Roughly speaking, for a class $\mathcal{M}$, an $\mathcal{M}$-absorbing set of an s-manifold X is a subspace A of X such that (i) A has an absorption property in X for the class $\mathcal{M}$, (ii) A has a general position property for $\mathcal{M}$ and (iii) A "belongs" to the class $\mathcal{M}$. The notion of strong universality of tuples realizes the conditions (i) and (ii) simultaneously. The condition (iii) usually appears in the form: A is a countable union of Z-sets of A which belong to $\mathcal{M}$. The most important property of absorbing sets is the uniqueness property. This property will play a key role in the characterizations of tuples of infinite-dimensional manifolds.

2.4.1. *Capsets and fd capsets.* (R.D. Anderson and T.A. Chapman [9])

The most basic absorbing sets are capsets and fd capsets. A space is said to be σ-compact (σ-fd-compact) if it is a countable union of compact (finite-dimensional compact) subsets.

Definition. Suppose X is a Q-manifold or an s-manifold. A subset A of X is said to be a (fd) capset of X if A is a union of (fd) compact Z-sets A_n $(n \geq 1)$ which satisfy the following condition: for every $\varepsilon > 0$, every (fd) compact subset K of X and every $n \geq 1$ there exist an $m \geq n$ and an embedding $h : K \to A_m$ such that (i) $d(h, id_K) < \varepsilon$ and (ii) $h = id$ on $A_n \cap K$.

For example Σ is a capset of s and σ is fd capset of s. The (fd) capsets have the following uniqueness property:

Theorem 2.1. *If A and B are (fd) capsets of X, then for every open cover $\mathcal{U}$ of X there exists a homeomorphism $f : (X, A) \to (X, B)$ with $(f, id_X) \leq \mathcal{U}$.*

2.4.2. *Absorbing sets in s-manifolds.*

The notion of (fd) capsets works only for the class of σ-(fd-)compact subsets. To treat other classes of subsets we need to extend this notion.

Definition. A class $\mathcal{C}$ of spaces is said to be
(i) topological if $D \cong C \in \mathcal{C}$ implies $D \in \mathcal{C}$.
(ii) additive if $C \in \mathcal{C}$ whenever $C = A \cup B$, A and B are closed subsets of C, and A, $B \in \mathcal{C}$.
(iii) closed hereditary if $D \in \mathcal{C}$ whenever D is a closed subset of a space $C \in \mathcal{C}$.

[**1**] The non-ambient case: (M. Bestvina - J. Mogilski [5])

Let $\mathcal{C}$ be a class of spaces.

Definition. A subset A of an s-manifold X is said to be a $\mathcal{C}$-absorbing set of X if
(i) A has the h.n. complement in X,
(ii) A is strongly $\mathcal{C}$-universal,
(iii) $A = \cup_{n=1}^{\infty} A_n$, where each A_n is a Z-set of A and $A_n \in \mathcal{C}$.

Theorem 2.2. *Suppose a class $\mathcal{C}$ is topological, additive and closed hereditary. If A and B are two $\mathcal{C}$-absorbing sets in an s-manifold X, then every open cover $\mathcal{U}$ of X there exists a homeomorphism $h : X \to Y$ which is $\mathcal{U}$-close to the inclusion $A \subset X$.*

In general, h cannot be extended to any ambient homeomorphism of X.

[**2**] The ambient case: (J. Baars-H. Gladdines-J. van Mill [3], R. Cauty [6], T. Yagasaki [32], et.al.)

Let $\mathcal{M}$ be a class of $(l+1)$-tuples. We assume that $\mathcal{M}$ is topological, additive and closed hereditary. We consider the following condition (I):

The condition (I)
(I-1) $(X, X_1, \cdots, X_l)$ is strongly $\mathcal{M}$-universal,
(I-2) there exist Z-sets Z_n $(n \geq 1)$ of X such that
  (i) $X_1 \subset \cup_n Z_n$ and (ii) $(Z_n, Z_n \cap X_1, \cdots, Z_n \cap X_l) \in \mathcal{M}$ $(n \geq 1)$.

In this case we have ambient homeomorphisms:

Theorem 2.3. ([6, 32]) *Suppose E is an s-manifold and $(l+1)$-tuples $(E, X_1, \cdots, X_l)$ and $(E, Y_1, \cdots, Y_l)$ satisfy the condition* (I). *Then for any open cover $\mathcal{U}$ of E there exists a homeomorphism $f : (E, X_1, \cdots, X_l) \to (E, Y_1, \cdots, Y_l)$ with $(f, id_E) \leq \mathcal{U}$.*

2.5. **Homotopy invariance.**

Classification of infinite-dimensional manifolds is rather simple. Q-manifolds are classified by simple homotopy equivalence (T.A. Chapman [10]) and s-manifolds are classified by homotopy equivalence (D. W. Henderson and R. M. Schori [18]).

Theorem 2.4. *Suppose X and Y are s-manifolds. Then $X \cong Y$ iff $X \simeq Y$ (homotopy equivalence).*

3. Characterization of infinite-dimensional manifolds in term of general position property and stability

3.1. **Edwards' program.**

There is a general method, called as Edwards' program, of detecting topological E-manifolds. For infinite-dimensional topological manifolds, it takes the following form: Let X be an ANR.

(i) Construct a fine homotopy equivalence from an E-manifold to the target X.

(ii) Show that f can be approximated by homeomorphisms under some general position property of X.

This program yields basic characterizations of Q-manifolds, s-manifolds and other incomplete manifolds.

3.2. **The complete cases:**

(1) Q-manifolds:

Theorem 3.1. ([10]) *A space X is an Q-manifold iff*
(i) *X is a locally compact separable metrizable ANR*
(ii) *X has the disjoint cells property.*

(2) s-manifolds:

Theorem 3.2. ([30]) *A space X is an s-manifold iff*
(i) *X is a separable completely metrizable ANR*
(ii) *X has the strong discrete approximation property.*

Since the Q-stability implies the disjoint cells property and the s-stability implies the strong discrete approximation property, we can replace the condition (ii) by
(ii$'$) X is Q-stable (respectively s-stable)

3.3. **The incomplete cases:**

M. Bestvina-J. Mogilski [5] has shown that in the incomplete case the above program is formulated in the following form:

Theorem 3.3. (M. Bestvina-J. Mogilski [5])
Suppose $\mathcal{C}$ is a class of spaces which is topological, additive and closed hereditary.
(i) *For every ANR X there exists an s-manifold M such that for every $\mathcal{C}$-absorbing set Ω in M there exists a fine homotopy equivalence $f : \Omega \to X$.*
(ii) *Suppose* (a) *X is a strongly $\mathcal{C}$-universal ANR and* (b) *$X = \cup_{i=1}^{\infty} X_i$, where each X_i is a strong Z-set in X and $X_i \in \mathcal{C}$. Then every fine homotopy equivalence $f : \Omega \to X$ from any $\mathcal{C}$-absorbing set Ω in an s-manifold can be approximated by homeomorphisms.*

Example: Σ-manifolds and σ-manifolds

Let $\mathcal{C}_c$ ($\mathcal{C}_{fdc}$) denote the class of all (finite dimensional) compacta.

Theorem 3.4. (M. Bestvina-J. Mogilski [5, 23])
A space X is a Σ-manifold (σ-manifold) iff
(i) *X is a separable ANR and σ-compact (σ-fd compact),*
(ii) *X is strongly $\mathcal{C}_c$-universal (strongly $\mathcal{C}_{fdc}$-universal),*
(iii) *$X = \cup_{n=1}^{\infty} X_n$, where each X_n is a strong Z-set in X.*

The condition (iii) can be replaced by

(iii′) X satisfies the strong discrete approximation property.

In [28] H. Toruńczyk has obtained a characterization of σ-manifolds in term of stability.

Theorem 3.5. ([29])

X is a σ-manifold iff X is (i) *a separable ANR,* (ii) *σ-fd-compact and* (iii) *σ-stable.*

4. Characterizations of $(s, S_1, \cdots, S_l)$-manifolds

In this section we will investigate the problem of detecting $(s, S_1, \cdots, S_l)$-manifolds. Since we have obtained a characterization of s-manifolds (Theorem 3.2), the remaining problem is how to compare a tuple $(X, X_1, \cdots, X_l)$ locally with $(s, S_1, \cdots, S_l)$ when X is an s-manifold. For this purpose we will use the uniqueness property of absorbing sets in s-manifolds (§2.4). Since s-manifolds are homotopy invariant (Theorem 2.4), at the same time we can show the homotopy invariance of $(s, S_1, \cdots, S_l)$-manifolds.

4.1. Characterizations of manifold tuples in term of the absorbing sets.

4.1.1. *Characterizations in term of capsets and fd-capsets.*

Theorem 4.1. (T.A. Chapman [9])
(1) *(X, X_1) is an (s, Σ)-manifold ((s, σ)-manifold) iff*
(i) *X is an s-manifold,*
(ii) *X_1 is a capset (a fd capset).*

(2) *Suppose* (X, X_1) *and* (Y, Y_1) *are* (s, Σ)*-manifolds* ((s, σ)*-manifolds*). *Then* $(X, X_1) \cong (Y, Y_1)$ *iff* $X \simeq Y$.

Theorem 4.2. (K. Sakai-R.Y. Wong [27])
(1) (X, X_1, X_2) *is an* (s, Σ, σ)*-manifold iff*
(i) X *is an* s*-manifold,*
(ii) (X_1, X_2) *is a (cap, fd cap)-pair in* X.
(2) *Suppose* (X, X_1, X_2) *and* (Y, Y_1, Y_2) *are* (s, Σ, σ)*-manifolds. Then* $(X, X_1, X_2) \cong (Y, Y_1, Y_2)$ *iff* $X \simeq Y$.

4.1.2. *Characterizations in term of strong universality.*

We assume that $\underline{(s, S_1, \cdots, S_l) \text{ satisfies the condition (I)}}$ in Section 2.4.2.[2].

Theorem 4.3. (1) $(X, X_1, \cdots, X_l)$ *is an* $(s, S_1, \cdots, S_l)$*-manifold iff*
(i) X *is an* s*-manifold,*
(ii) $(X, X_1, \cdots, X_l)$ *satisfies the condition* (I).
(2) *Suppose* $(X, X_1, \cdots, X_l)$ *and* $(Y, Y_1, \cdots, Y_l)$ *are* $(s, S_1, \cdots, S_l)$*-manifolds. Then* $(X, X_1, \cdots, X_l) \cong (Y, Y_1, \cdots, Y_l)$ *iff* $X \simeq Y$.

4.2. **Characterization in term of stability and homotopy negligible complement.**

4.2.1. *General characterization theorem.*

We can show that the stability + h.n. complement implies the strong universality. This leads to a characterization based upon the stability condition. We consider the following condition (II).

The condition (II):
(II-1) S_1 is contained in a countable union of Z-sets of s,
(II-2) S_l has the h.n. complement in s,
(II-3) (Infinite coordinates) There exists a sequence of disjoint infinite subsets $A_n \subset \mathbb{N}$ $(n \geq 1)$ such that for each $i = 1, \cdots, l$ and $n \geq 1$, (a) $S_i = \pi_{A_n}(S_i) \times \pi_{\mathbb{N} \setminus A_n}(S_i)$ and (b) $(\pi_{A_n}(s), \pi_{A_n}(S_1), \cdots, \pi_{A_n}(S_l)) \cong (s, S_1, \cdots, S_l)$.

Here for a subset A of $\mathbb{N}$, $\pi_A : s \to \prod_{k \in A} (-\infty, \infty)$ denotes the projection onto the A-factor of s.

Assumption. We assume that $\underline{(s, S_1, \cdots, S_l) \text{ satisfies the condition (II)}}$:

Notation. Let $\mathcal{M} \equiv \mathcal{M}(s, S_1, \cdots, S_l)$ denote the class of $(l+1)$-tuples $(X, X_1, \cdots, X_l)$ which admits a layer preserving closed embedding $h : (X, X_1, \cdots, X_l) \to (s, S_1, \cdots, S_l)$

Theorem 4.4. (T.Yagasaki [32], R.Cauty, et. al.)
Suppose $(Y, Y_1, \cdots, Y_l)$ *satisfies the following conditions:*
(i) Y *is a completely metrizable ANR,*

(ii) *Y_l has the h.n. complement in Y*
(iii) *$(Y, Y_1, \cdots, Y_l)$ is $(s, S_1, \cdots, S_l)$-stable.*
Then $(Y, Y_1, \cdots, Y_l)$ is strongly $\mathcal{M}(s, S_1, \cdots, S_l)$-universal.

From Theorems 4.3, 4.4 we have:

Theorem 4.5. (1) *$(X, X_1, \cdots, X_l)$ is an $(s, S_1, \cdots, S_l)$-manifold iff*

(i) *X is a completely metrizable ANR,*
(ii) *$\underline{(X, X_1, \cdots, X_l) \in \mathcal{M}(s, S_1, \cdots, S_l)}$,*
(iii) *X_l has the h.n. complement in X,*
(iv) *$(X, X_1, \cdots, X_l)$ is $(s, S_1, \cdots, S_l)$-stable.*

(2) *Suppose $(X, X_1, \cdots, X_l)$ and $(Y, Y_1, \cdots, Y_l)$ are $(s, S_1, \cdots, S_l)$-manifolds. Then $(X, X_1, \cdots, X_l) \cong (Y, Y_1, \cdots, Y_l)$ iff $X \simeq Y$.*

4.2.2. *Examples.*

To apply Theorem 4.5 we must distinguish the class $\mathcal{M}(s, S_1, \cdots, S_l)$. This can be done for the triples: (s, Σ, σ), $(s^2, s \times \sigma, \sigma^2)$, $(s^\infty, \sigma^\infty, \sigma_f^\infty)$, and $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$. This leads to the practical characterizations of manifolds modeled on these triples.

[1] (s, Σ, σ):

$\mathcal{M}(s, \Sigma, \sigma)$ = the class of triples (X, X_1, X_2) such that

(a) X is completely metrizable, (b) X_1 is σ-compact, and (c) X_2 is σ-fd-compact.

Theorem 4.6.
(X, X_1, X_2) is an (s, Σ, σ)-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is $\underline{\sigma\text{-compact}}$, X_2 is $\underline{\sigma\text{-fd-compact}}$,*
(iii) *X_2 has the h.n. complement in X,*
(iv) *(X, X_1, X_2) is $\underline{(s, \Sigma, \sigma)}$-stable.*

[2] $(s^2, s \times \sigma, \sigma^2)$:

$\mathcal{M}(s^2, s \times \sigma, \sigma^2)$ = the class of triples (X, X_1, X_2) such that

(a) X is completely metrizable, (b) X_1 is F_σ in X, (c) X_2 is σ-fd-compact.

Theorem 4.7.
(X, X_1, X_2) is an $(s^2, s \times \sigma, \sigma^2)$-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is an $\underline{F_\sigma}$-subset of X, X_2 is $\underline{\sigma\text{-fd-compact}}$,*
(iii) *X_2 has the h.n. complement in X,*
(iv) *(X, X_1, X_2) is $\underline{(s^2, s \times \sigma, \sigma^2)}$-stable.*

[3] $(s^\infty, \sigma^\infty, \sigma_f^\infty)$:

$\mathcal{M}(s^\infty, \sigma^\infty, \sigma_f^\infty)$ = the class of triples (X, X_1, X_2) such that

(a) X is completely metrizable, (b) X_1 is $F_{\sigma\delta}$ in X, (c) X_2 is σ-fd-compact.

Theorem 4.8.
(X, X_1, X_2) is an $(s^\infty, \sigma^\infty, \sigma_f^\infty)$-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is an $\underline{F_{\sigma\delta}}$-subset of X, X_2 is $\underline{\sigma\text{-fd-compact}}$,*
(iii) *X_2 has the h.n. complement in X*
(iv) *(X, X_1, X_2) is $\underline{(s^\infty, \sigma^\infty, \sigma_f^\infty)}$-stable.*

[**4**] $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$:

$\mathcal{M}(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$ = the class of triples (X, X_1, X_2) such that
(a) X is completely metrizable, (b) X_1 is $F_{\sigma\delta}$ in X, (c) X_2 is σ-compact.

Theorem 4.9.
(X, X_1, X_2) is an $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$-manifold iff

(i) *X is a separable completely metrizable ANR,*
(ii) *X_1 is an $\underline{F_{\sigma\delta}}$-subset of X, X_2 is $\underline{\sigma\text{-compact}}$,*
(iii) *X_2 has the h.n. complement in X,*
(iv) *(X, X_1, X_2) is $\underline{(s^\infty, \Sigma^\infty, \Sigma_f^\infty)}$-stable.*

In the next chapter these characterizations will be applied to determine the local topological types of some triples of homeomorphism groups of manifolds.

Chapter 2. Applications to homeomorphism groups of manifolds

5. Main problems

Notation.
(i) $\mathcal{H}(X)$ denotes the homeomorphism group of a space of X with the compact-open topology.
(ii) When X has a fixed metric, $\mathcal{H}^{\mathrm{LIP}}(X)$ denotes the subgroup of locally LIP-homeomorphisms of X.
(iii) When X is a polyhedron, $\mathcal{H}^{\mathrm{PL}}(X)$ denotes the subgroup of PL-homeomorphisms of X.

We shall consider the following problem:

Problem.
Determine the local and global topological types of groups: $\mathcal{H}(M)$, $\mathcal{H}^{\mathrm{LIP}}(M)$, $\mathcal{H}^{\mathrm{PL}}(M)$, etc. and tuples: $(\mathcal{H}(M), \mathcal{H}^{\mathrm{PL}}(M))$, $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M), \mathcal{H}^{\mathrm{PL}}(M))$, etc.

In the analogy with diffeomorphism groups, when X is a topological manifold, we can expect that these groups are topological manifold modeled on some typical infinite-dimensional spaces. In fact, R.D. Anderson showed that:

Facts 5.1. ([2]) :
(i) $\mathcal{H}_+(\mathbb{R}) \cong s$.
(ii) If G is a finite graph, then $\mathcal{H}(G)$ is an s-manifold.

After this result it was conjectured that

Conjecture. $\mathcal{H}(M)$ is an s-manifold for any compact manifold M.

This basic conjecture is still open for $n \geq 3$ and this imposes a large restriction to our work since most results in Chapter 1 works only when ambient spaces are s-manifolds. Thus in the present situation, in order to obtain some results in dimension $n \geq 3$, we must assume that $\mathcal{H}(M)$ is an s-manifold. On the other hand in dimension 1 or 2 we can obtain concrete results due to the following fact:

Theorem 5.1. (R. Luke - W.K. Mason [22], W. Jakobsche [19])
If X is a 1 or 2-dimensional compact polyhedron, then $\mathcal{H}(X)$ is an s-manifold.

Below we shall follow the next conventions: For a pair (X, A), let $\mathcal{H}(X, A) = \{f \in \mathcal{H}(X) : f(A) = A\}$. When (X, A) is a polyhedral pair, let $\mathcal{H}^{\mathrm{PL}}(X, A) = \mathcal{H}(X, A) \cap \mathcal{H}^{\mathrm{PL}}(X)$ and $\mathcal{H}(X; \mathrm{PL}(A)) = \{f \in \mathcal{H}(X, A) : f \text{ is PL on } A\}$. The superscript "c" denotes "compact supports", the subscript "+" means "orientation preserving", and "0" denotes "the identity connected components" of the corresponding groups. An Euclidean PL-manifold means a PL-manifold which is a subpolyhedron of some Euclidean space $\mathbb{R}^n$ and has the standard metric induced from $\mathbb{R}^n$.

6. Stability properties of homeomorphism groups of polyhedra

First we shall summarize the stability properties of various triples of homeomorphism groups of polyhedra. These properties will be used to determine the corresponding model spaces.

(1) Basic cases: (R. Geoghegan [14, 15], J. Keesling-D. Wilson [20, 21], K. Sakai-R.Y. Wong [26])
(i) If X is a topological manifold, then $\mathcal{H}(X)$ is s-stable.
(ii) If X is a locally compact polyhedron, then the pair $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, σ)-stable.
(iii) If X is a Euclidean polyhedron with the standard metric, then the triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s, Σ, σ)-stable.
(iv) (T. Yagasaki [32]) If (X, K) is a locally compact polyhedral pair such that $\dim K \geq 1$ and $\dim (X \setminus K) \geq 1$, then $(\mathcal{H}(X, K), \mathcal{H}(X; \mathrm{PL}(K)), \mathcal{H}^{\mathrm{PL}}(X, K))$ is $(s^2, s \times \sigma, \sigma^2)$-stable.

(2) Noncompact cases: (T. Yagasaki [32])
(i) If X is a noncompact, locally compact polyhedron, then the triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X), \mathcal{H}^{\mathrm{PL,c}}(X))$ is $(s^\infty, \sigma^\infty, \sigma_f^\infty)$-stable.
(ii) If X is a noncompact Euclidean polyhedron with the standard metric, then the triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{LIP,c}}(X))$ is $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$-stable.

We can also consider the spaces of embeddings. Suppose X and Y are Euclidean polyhedra. Let $\mathcal{E}(X, Y)$ denote the spaces of embeddings of X into Y with the compact-open topology, and let $\mathcal{E}^{\mathrm{LIP}}(X, Y)$ and $\mathcal{E}^{\mathrm{PL}}(X, Y)$ denote the subspaces of locally Lipschitz embeddings and PL-embeddings respectively.

(3) Embedding case: (K. Sakai-R.Y. Wong [26], cf. [32])
The triple $(\mathcal{E}(X, Y), \mathcal{E}^{\mathrm{LIP}}(X, Y), \mathcal{E}^{\mathrm{PL}}(X, Y))$ is (s, Σ, σ)-stable.

These stability property are verified by using the Morse length of the image of a fixed segment under the homeomorphisms.

7. The triple $(\mathcal{H}(M)$, $\mathcal{H}^{\mathrm{LIP}}(X)$, $\mathcal{H}^{\mathrm{PL}}(X))$

[**1**] $\mathcal{H}(M)$

Suppose M^n is a compact n-dimensional manifold. Since $\mathcal{H}(M)$ is s-stable, by the characterization of s-manifold (Theorem 3.2), $\mathcal{H}(M)$ is an s-manifold iff it is an ANR. Here we face with the difficulty of detecting infinite-dimensional ANR's. A.V. Černavskiĭ [8] and R.D. Edwards - R.C. Kirby [12] have shown:

Theorem 7.1. (Local contractibility): *$\mathcal{H}(M)$ is locally contractible.*

[**2**] $\mathcal{H}^{\mathrm{PL}}(M)$

Suppose M^n is a compact n-dimensional PL-manifold.

Basic Facts.
(1) (J. Keesling-D. Wilson [21]) $(\mathcal{H}(M), \mathcal{H}^{\mathrm{PL}}(M))$ is (s, σ)-stable.
(2) (D. B. Gauld [13]) $\mathcal{H}^{\mathrm{PL}}(M)$ is locally contractible.
(3) (R. Geoghegan [15]) $\mathcal{H}^{\mathrm{PL}}(M)$ is σ-fd-compact.
(4) (W.E. Haver [17]) A countable dimensional metric space is an ANR iff it is locally contractible.

From (2),(3),(4) it follows that $\mathcal{H}^{\mathrm{PL}}(M)$ is always an ANR. Hence by the characterization of σ-manifold (Theorem 3.5), we have:

Main Theorem. (J. Keesling-D. Wilson [21]) *$\mathcal{H}^{\mathrm{PL}}(M)$ is an σ-manifold.*

Let $\mathcal{H}(M)^* = cl\, \mathcal{H}^{\mathrm{PL}}(M)$. Consider the condition:

$$(*) \quad \underline{n \neq 4 \text{ and } \partial M = \emptyset \text{ for } n = 5.}$$

Under this condition $\mathcal{H}(M)^*$ is the union of some components of $\mathcal{H}(M)$.

Theorem 7.2. (R. Geoghegan, W. E. Haver [16])
If $\mathcal{H}(X)$ is an s-manifold and M satisfies $(*)$, then $(\mathcal{H}(X)^*, \mathcal{H}^{\mathrm{PL}}(X))$ is an (s,σ)-manifold.

[**3**] $\mathcal{H}^{\mathrm{LIP}}(M)$ (K. Sakai-R.Y. Wong [26])

Suppose M^n is a compact n-dimensional Euclidean PL-manifold.

Basic Facts. ([26])
(1) $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M))$ is (s,Σ)-stable.
(2) $\mathcal{H}^{\mathrm{LIP}}(M)$ is σ-compact.

Theorem 7.3. ([26])
If $\mathcal{H}(X)$ is an s-manifold and M satisfies $(*)$, then $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X))$ is an (s,Σ)-manifold.

[**4**] The triple $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{LIP}}(M))$ (T. Yagasaki [32])

Suppose M^n is a compact n-dimensional Euclidean PL-manifold.

Basic Facts.
(1) (K. Sakai-R. Y. Wong [26]) $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M), \mathcal{H}^{\mathrm{PL}}(M))$ is (s,Σ,σ)-stable.

Let $\mathcal{H}^{\mathrm{LIP}}(X)^* = \mathcal{H}^{\mathrm{LIP}}(X) \cap cl\mathcal{H}^{\mathrm{PL}}(M)$. From Theorem 7.2, Basic Facts and the characterization of (s,Σ,σ)-manifolds (Theorem 4.6) it follows that:

Theorem 7.4. ([32])
(1) *If $\mathcal{H}(X)$ is an s-manifold and M satisfies $(*)$, then $(\mathcal{H}(X)^*, \mathcal{H}^{\mathrm{LIP}}(X)^*, \mathcal{H}^{\mathrm{PL}}(X))$ is an (s,Σ,σ)-manifold.*
(2) *If X is a 1 or 2-dimensional compact Euclidean polyhedron with the standard metric, then $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{PL}}(X))$ is (s,Σ,σ)-manifold.*

8. Other triples

[**1**] The triple $(\mathcal{H}(X,K), \mathcal{H}(X;\mathrm{PL}(K)), \mathcal{H}^{\mathrm{PL}}(X,K))$ (T. Yagasaki [32])

Theorem 8.1.
(i) *Suppose M^n is a compact PL n-manifold with $\partial M \neq \emptyset$. If $n \geq 2$, $n \neq 4,5$ and $\mathcal{H}(M)$ is an s-manifold, then $(\mathcal{H}(M), \mathcal{H}(M;\mathrm{PL}(\partial M)), \mathcal{H}^{\mathrm{PL}}(M))$ is an $(s^2, s\times\sigma, \sigma^2)$-manifold.*
(ii) *Suppose (X,K) is a compact polyhedral pair such that $\dim X = 1,2$, $\dim K \geq 1$ and $\dim(X \setminus K) \geq 1$. Then $(\mathcal{H}(X,K), \mathcal{H}(X;\mathrm{PL}(K)), \mathcal{H}^{\mathrm{PL}}(X,K))$ is an $(s^2, s\times\sigma, \sigma^2)$-manifold.*

[**2**] The triples $(\mathcal{H}(X), \mathcal{H}^{\mathrm{PL}}(X), \mathcal{H}^{\mathrm{PL,c}}(X))$ and $(\mathcal{H}(X), \mathcal{H}^{\mathrm{LIP}}(X), \mathcal{H}^{\mathrm{LIP,c}}(X))$ (T. Yagasaki [33])

(1) 1-dim. case: $(\mathcal{H}_+(\mathbb{R}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}), \mathcal{H}^{\mathrm{PL,c}}(\mathbb{R})) \cong (s^\infty, \sigma^\infty, \sigma_f^\infty)$.

(2) 2-dim. case:

Theorem 8.2. *If M is a noncompact connected PL 2-manifold, then* $(\mathcal{H}(M)_0$, $\mathcal{H}^{\mathrm{PL}}(M)_0$, $\mathcal{H}^{\mathrm{PL,c}}(M)_0)$ *is an* $(s^\infty, \sigma^\infty, \sigma_f^\infty)$*-manifold.*

Corollary 8.1.
(i) *If* $M \cong \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}, \mathbb{S}^1 \times [0,1), \mathbb{P}^2\mathbb{R} \setminus 1pt$*, then* $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0) \cong \mathbb{S}^1 \times (s^\infty, \sigma^\infty, \sigma_f^\infty)$
(ii) *In the remaining cases,* $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}^{\mathrm{PL,c}}(M)_0) \cong (s^\infty, \sigma^\infty, \sigma_f^\infty)$

(3) There exist a (LIP, Σ)-version of the (PL, σ)-case.

[**3**] The group of quasiconformal (QC-)homeomorphisms of a Riemann surface (T. Yagasaki [34])

Suppose M is a connected Riemann surface. Let $\mathcal{H}^{\mathrm{QC}}(M)$ denote the subgroup of QC-homeomorphisms of M.

Theorem 8.3.
(i) *If M is compact, then* $(\mathcal{H}_+(M), \mathcal{H}^{\mathrm{QC}}(M))$ *is an* (s, Σ)*-manifold.*
(ii) *If M is noncompact, then* $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0)$ *is an* (s, Σ)*-manifold*

[**4**] The space of embeddings (T. Yagasaki [33])

Suppose M is a Euclidean PL 2-manifold.

Theorem 8.4. *If X is a compact subpolyhedron of M, then* $(\mathcal{E}(X,M), \mathcal{E}^{\mathrm{LIP}}(X,M), \mathcal{E}^{\mathrm{PL}}(X,M))$ *is an* (s, Σ, σ)*-manifold.*

Example. The case $X = I \equiv [0,1]$:

$$(\mathcal{E}(I,M), \mathcal{E}^{\mathrm{LIP}}(I,M), \mathcal{E}^{\mathrm{PL}}(I,M)) \cong S(TM) \times (s, \Sigma, \sigma)$$

where $S(TM)$ is the sphere bundle of the tangent bundle of M with respect to some Riemannian metric.

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