CONTROLLED ALGEBRA AND TOPOLOGY

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Let R be a ring and $(\overline{X}, \partial X)$ a pair of compact Hausdorff spaces. We assume $X = \overline{X} - \partial X$ is dense in $\overline{X}$.

Definition 1. The continuously controlled category $\mathcal{B}(\overline{X}, \partial X; R)$ has objects $A = \{A_x\}_{x \in X}$, A_x a finitely generated free R-module, satisfying that $\{x | A_x \neq 0\}$ is locally finite in X.

Given a subset U in $\overline{X}$ we define $A|U$ by

$$(A|U)_x = \begin{cases} A_x & \text{if } x \in U \cap X \\ 0 & \text{if } x \notin U \cap X \end{cases}$$

A morphism $\phi \in \mathcal{B}(\overline{X}, \partial X; R)$, is an R-module morphism $\phi : \oplus A_x \to \oplus B_y$ satisfying a continuously controlled condition:

$$\forall z \in \partial X, \forall U \text{ open in } \overline{X}, z \in U, \exists V \text{ open in } \overline{X}, z \in V$$
$$\text{such that } \phi(A|V) \subset A|U \text{ and } \phi(A|X - U) \subset A|X - V$$

Clearly $\mathcal{B}(\overline{X}, \partial X; R)$ is an additive category with $(A \oplus B)_x = A_x \oplus B_x$ as direct sum.

If A is an object of $\mathcal{B}(\overline{X}, \partial X; R)$, then $\{x | A_x \neq 0\}$ has no limit point in X, all limit points must be in ∂X. We denote the set of limit points by $\operatorname{supp}_\infty(A)$. The full subcategory of $\mathcal{B}(\overline{X}, \partial X; R)$ on objects A with

$$\operatorname{supp}_\infty(A) \subset Z \subset \partial X$$

is denoted by $\mathcal{B}(\overline{X}, \partial X; R)_Z$. Putting $\mathcal{U} = \mathcal{B}(\overline{X}, \partial X; R)$ and $\mathcal{A} = \mathcal{B}(\overline{X}, \partial X; R)_Z$, this is a typical example of an $\mathcal{A}$-filtered additive category $\mathcal{U}$ in the sense of Karoubi [6]. The quotient category $\mathcal{U}/\mathcal{A}$ has the same objects as $\mathcal{U}$, but two morphisms are identified if the difference factors through an object of $\mathcal{A}$. In the present example this means two morphisms are identified if they agree on the object restricted to a neighborhood of $\partial X - Z$. We denote $\mathcal{U}/\mathcal{A}$ in this case by $\mathcal{B}(\overline{X}, \partial X; R)^{\partial X - Z}$. Given an object A and a neighborhood W of $\partial X - Z$ we have $A \cong A|W$ in this category.

If R is a ring with involution these categories become additive categories with involution in the sense of Ranicki [7]. It was proved in [2] that

Theorem 2. *There is a fibration of spectra*

$$\mathbb{L}^k(\mathcal{A}) \to \mathbb{L}^h(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A})$$

where k consists of projectives, i.e. objects in the idempotent completion of $\mathcal{A}$, that become free in $\mathcal{U}$, i.e. stably, by adding objects in $\mathcal{U}$ become isomorphic to an object of $\mathcal{U}$.

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Indication of proof. Using the bordism definition of L-spectra of Quinn and Ranicki, it is immediate that we have a fibration of spectra

$$\mathbb{L}^h(\mathcal{A}) \to \mathbb{L}^h(\mathcal{U}) \to \mathbb{L}^h(\mathcal{U}, \mathcal{A})$$

An element in $L^h_n(\mathcal{U}, \mathcal{A})$, the n-th homotopy group of $\mathbb{L}^h(\mathcal{U}, \mathcal{A})$ is a pair of chain complexes with boundary in $\mathcal{A}$ and a quadratic Poincaré structure. The boundary is isomorphic to 0 in $\mathcal{U}/\mathcal{A}$ since all $\mathcal{A}$-objects are isomorphic to 0 in $\mathcal{U}/\mathcal{A}$. This produces a map

$$\mathbb{L}^h(\mathcal{U}, \mathcal{A}) \to \mathbb{L}^h(\mathcal{U}/\mathcal{A})$$

which we ideally would like to be a homotopy equivalence. Given a quadratic Poincaré complex in $\mathcal{U}/\mathcal{A}$, it is easy to lift the chain complex to a chain complex in $\mathcal{U}$, and to lift the quadratic structure, but it is no longer a Poincaré quadratic structure. We may use [8, Prop. 13.1] to add a boundary so that we lift to a Poincaré pair. It follows that the boundary is contractible in $\mathcal{U}/\mathcal{A}$. It turns out that a chain complex in $\mathcal{U}$ is contractible in $\mathcal{U}/\mathcal{A}$ if and only if the chain complex is dominated by a chain complex in $\mathcal{A}$, and such a chain complex is homotopy equivalent to a chain complex in the idempotent completion of $\mathcal{A}$. This is the reason for the variation in the decorations in this theorem. See [2] for more details. □

Lemma 3. *If $(\overline{X}, \partial X)$ is a compact pair then*

$$\mathcal{B}(\overline{X}, \partial X; R) \cong \mathcal{B}(C\partial X, \partial X; R)$$

Proof. The isomorphism is given by moving the modules A_x, $x \in X$ to point in $C\partial X$, the same module, and if two are put the same place we take the direct sum. On morphisms the isomorphism is induced by the identity, so we have to ensure the continuously controlled condition is not violated. Choose a metric on $\overline{X}$ so that all distances are ≤ 1. Given $z \in X$, let y be a point in ∂X closest to z, and send z to $(1 - d(z, y))y$. Clearly, as z approaches the boundary it is moved very little. In the other direction send $t \cdot y$ to a point in $B(y; 1-t)$, the ball with center y and radius $1 - t$, which is furthest away from ∂X. Again moves become small as t approaches 1 or equivalently as the point approaches ∂X. □

Lemma 4. $\mathbb{L}^h(\mathcal{B}(\overline{X}, \partial X; R)_*) \simeq *$

Proof. The first $*$ denotes a point in ∂X and the second that the spectrum is contractible. The proof is an Eilenberg swindle towards the point. □

Theorem 5. [2] *The functor*

$$Y \to L^h_*(\mathcal{B}(CY, Y; \mathbb{Z}))$$

is a generalized homology theory on compact metric spaces

Proof. We have a fibration

$$\mathbb{L}^h(\mathcal{B}(CY, Y; \mathbb{Z}))_Z \to \mathbb{L}^h(\mathcal{B}(CY, Y; \mathbb{Z})) \to \mathbb{L}^h(\mathcal{B}(CY, Y; \mathbb{Z}))^{Y-Z}$$

But an argument similar to the one used in Lemma 3 shows

$$\mathcal{B}(CY, Y; \mathbb{Z})_Z \cong \mathcal{B}(CZ, Z; \mathbb{Z}).$$

When everything is away from Z it does not matter if we collapse Z so we have

$$\mathcal{B}(CY, Y; \mathbb{Z})^{Y-Z} \cong \mathcal{B}((CY)/Z, Y/Z; \mathbb{Z})^{Y/Z - Z/Z},$$

but $Y/Z - Z/Z$ is only one point from Y/Z so by Lemma 4 the $\mathbb{L}$-spectrum is homotopy equivalent to $\mathbb{L}((CY)/Z, Y/Z; \mathbb{Z})$. Finally Lemma 3 shows that

$$\mathcal{B}(CY/Z, Y/Z; \mathbb{Z}) \cong \mathcal{B}(C(Y/Z), Y/Z); \mathbb{Z}).$$

and we are done. □

Consider a compact pair (X, Y) so that $X - Y$ is a CW-complex. If we subdivide so that cells in $X - Y$ become small near Y, the cellular chain complex $C_\sharp(X - Y; \mathbb{Z})$ may be thought of as a chain complex in $\mathcal{B}(X, Y; \mathbb{Z})$ simply by choosing a point in each cell (a choice which is no worse than the choice of the cellular structure.) If we have a strict map

$$(f, 1_Y) : (W, Y) \to (X, Y)$$

(meaning $f^{-1}(X - Y) \subset W - Y$) it is easy to see that given appropriate local simple connectedness conditions, this map is a strict homotopy equivalence (homotopies through strict maps) if and only if the induced map is a homotopy equivalence of chain complexes in $\mathcal{B}(X, Y; \mathbb{Z})$. If the fundamental group of $X - Y$ is π and the universal cover satisfies the appropriate simply connectedness conditions, strict homotopy equivalence is measured by chain homotopy equivalences in $\mathcal{B}(X, Y; \mathbb{Z}\pi)$. We have the ingredients of a surgery theory which may be developed along the lines of [4] with a surgery exact sequence

$$\to L^h_{n+1}(\mathcal{B}(X, Y; \mathbb{Z}\pi)) \to \mathcal{S}^h_{cc}\begin{pmatrix} X-Y \\ \downarrow \\ X \end{pmatrix} \to [X - Y; F/\operatorname{Top}] \to$$

We will use this sequence to discuss a question originally considered in [1].

Suppose a finite group π acts freely on S^{n+k} fixing S^{k-1}, a standard $k-1$-dimensional subsphere. We may suspend this action to an action on S^{n+k+1} fixing S^k and the question arises whether a given action can be desuspended. Notice this question is only interesting in the topological category. In the PL or differentiable category it is clear that all such actions can be maximally desuspended, by taking a link or by an equivariant smooth normal bundle consideration.

Denoting $(S^{n+k} - S^{k-1})/\pi$ by X, X is the homotopy type of a Swan complex (a finitely dominated space with universal cover homotopy equivalent to a sphere). The strict homotopy type of $(S^{n+k}/\pi, S^{k-1})$ can be seen to be $(X * S^{k-1}, S^{k-1})$, [1], and if we have a strict homotopy equivalence from a manifold to $X * S^{k-1} - S^{k-1}$ it is easy to see that we may complete to get a semifree action on a sphere fixing a standard subsphere. This means that this kind of semifree action is classified by the surgery exact sequence

$$\to L_{n+1}(\mathcal{B}(D^k, S^{k-1}; \mathbb{Z}\pi)) \to \mathcal{S}^h_{cc}\begin{pmatrix} X*S^{k-1}-S^{k-1} \\ \downarrow \\ X*S^{k-1} \end{pmatrix} \to [X, F/\operatorname{Top}] \to$$

Now let $\mathcal{C}(\mathbb{R}^n; R)$ denote the subcategory of $\mathcal{B}(\mathbb{R}^n, \emptyset; R)$ where the morphisms are required to be bounded i. e. $\phi : A \to B$ has to satisfy that there exists $k = k(\phi)$ so that $\phi^y_x = 0$ if $|x - y| > k$. Radial shrinking defines a functor $\mathcal{C}(\mathbb{R}^n, R) \to \mathcal{B}(D^n, S^{n-1}; R)$, and it is easy to see by the kind of arguments developed above that this functor induces isomorphism in L-theory. We get a map from the bounded surgery exact sequence to the continuously controlled surgery exact sequence

$$\begin{tikzcd}
{} \arrow[r] & L^h_{n+1}(\mathcal{C}(\mathbb{R}^k;\mathbb{Z}\pi)) \arrow[r] \arrow[d] & \mathcal{S}^h_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \arrow[r] \arrow[d] & {[X, F/\operatorname{Top}]} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & L^h_{n+1}(\mathcal{B}(D^k, S^{k-1};\mathbb{Z}\pi)) \arrow[r] & \mathcal{S}^h_{cc}\begin{pmatrix} X*S^{k-1}-S^{k-1} \\ \downarrow \\ D^k \end{pmatrix} \arrow[r] & {[X, F/\operatorname{Top}]} \arrow[r] & {}
\end{tikzcd}$$

which is an isomorphism on two out of three terms, hence also on the structure set. This is useful because we can not define an operation corresponding to suspension of the action on the continuously controlled structure set. An attempt would be to cross with an open interval, but an open interval would have to have a specific cell structure to get a controlled algebraic Poincaré structure on the interval, but then we would lose control along the suspension lines. In the bounded context suspension corresponds precisely to crossing with the reals, and giving the reals a bounded triangulation we evidently have no trouble getting a map corresponding to crossing with $\mathbb{R}$. Since crossing with $\mathbb{R}$ kills torsion (think of crossing with $\mathbb{R}$ as crossing with S^1 and pass to the universal cover), we get a map from the h-structure set to the s-structure set. The desuspension problem is now determined by the diagram

$$\begin{tikzcd}
{} \arrow[r] & L^h_{n+1}(\mathcal{C}(\mathbb{R}^k;\mathbb{Z}\pi)) \arrow[r] \arrow[d] & \mathcal{S}^h_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \arrow[r] \arrow[d] & {[X, F/\operatorname{Top}]} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & L^s_{n+2}(\mathcal{C}(\mathbb{R}^{k+1};\mathbb{Z}\pi)) \arrow[r] & \mathcal{S}^s_b\begin{pmatrix} X\times\mathbb{R}^{k+1} \\ \downarrow \\ \mathbb{R}^{k+1} \end{pmatrix} \arrow[r] & {[X, F/\operatorname{Top}]} \arrow[r] & {}
\end{tikzcd}$$

with two out of three maps isomorphisms once again. This shows we may desuspend if and only if the element in the structure set can be thought of as a simple structure, i.e. if and only if an obstruction in

$$Wh(\mathcal{C}(\mathbb{R}^{k+1};\mathbb{Z}\pi)) = K_1(\mathcal{C}(\mathbb{R}^{k+1};\mathbb{Z}\pi))/\{\pm\pi\} = K_{-k}(\mathbb{Z}\pi)$$

vanishes. Since $K_{-k}(\mathbb{Z}\pi) = 0$ for $k \geq 2$ [3], this means we can always desuspend untill we have a fixed circle, but then we encounter a possible obstruction. The computations in [5] show these obstructions are realized.

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