CONTROLLED TOPOLOGICAL EQUIVALENCE OF MAPS IN THE THEORY OF STRATIFIED SPACES AND APPROXIMATE FIBRATIONS

Bruce Hughes

Abstract. Ideas from the theory of topological stability of smooth maps are transported into the controlled topological category. For example, the controlled topological equivalence of maps is discussed. These notions are related to the classification of manifold approximate fibrations and manifold stratified approximate fibrations. In turn, these maps form a bundle theory which can be used to describe neighborhoods of strata in topologically stratified spaces.

1. Introduction

We explore some connections among the theories of topological stability of maps, controlled topology, and stratified spaces. The notions of topological equivalence of maps and locally trivial families of maps play an important role in the theory of topological stability of smooth maps. We formulate the analogues of these notions in the controlled topological category for two reasons. First, the notion of controlled topological equivalence of maps is a starting point for formulating a topological version of Mather's theory of the topological stability of smooth maps. Recall that Mather proved that the topologically stable maps are generic for the space of all smooth maps (with the C^∞ topology) between closed smooth manifolds (see Mather [22], Gibson, Wirthmüller, du Plessis, and Looijenga [9]). The hope is to identify an analogous generic class for the space of all maps (with the compact-open topology) between closed topological manifolds. Controlled topology at least gives a place to begin speculations. Second, the controlled analogue of local triviality for families of maps is directly related to the classification of approximate fibrations between manifolds due to Hughes, Taylor and Williams [17], [18]. We elucidate that relation in §8.

Another important topic in the theory of topological stability of smooth maps is that of smoothly stratified spaces (cf. Mather [21]). Quinn [26] initiated the study of topologically stratified spaces and Hughes [12], [13] has shown that 'manifold

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stratified approximate fibrations' form the correct bundle theory for those spaces. The classification of manifold approximate fibrations via controlled topology mentioned above extends to manifold stratified approximate fibrations; hence, we have another connection between controlled topology and stratified spaces. This classification of manifold stratified approximate fibrations is the main new result of this paper.

Two essential tools in stability theory are Thom's two isotopy lemmas [21]. In §9 we formulate an analogue of the first of these lemmas for topologically stratified spaces. A non-proper version is also stated.

It should be noted that in his address to the International Congress in 1986, Quinn predicted that controlled topology would have applications to the topological stability of smooth maps [25]. In particular, controlled topology should be applicable to the problem of characterizing the topologically stable maps among all smooth maps. More recently, Cappell and Shaneson [1] suggested that topologically stratified spaces should play a role in the study of the local and global topological type of topologically smooth maps (the connection is via the mapping cylinder of the smooth map). While the speculations in this paper are related to these suggestions, they differ in that it is suggested here that controlled topology might be used to study a generic class of topological, rather than smooth, maps.

2. Topological equivalence and locally trivial families of maps

We recall some definitions from the theory of topological stability of smooth maps (see Damon [3], du Plessis and Wall [5], Gibson, Wirthmüller, du Plessis, and Looijenga [9], Mather [21], [22]).

Definition 2.1. Two maps $p_0 : X_0 \to Y_0$, $p_1 : X_1 \to Y_1$ are *topologically equivalent* if there exist homeomorphisms $h : X_0 \to X_1$ and $g : Y_0 \to Y_1$ such that $p_1 h = g p_0$, so that there is a commuting diagram:

$$\begin{tikzcd} X_0 \arrow[r, "h"] \arrow[d, "p_0"'] & X_1 \arrow[d, "p_1"] \\ Y_0 \arrow[r, "g"] & Y_1. \end{tikzcd}$$

Definition 2.2. A smooth map $p_0 : M \to N$ between smooth manifolds is *topologically stable* if there exists a neighborhood V of p_0 in the space of all smooth maps $C^\infty(M, N)$ such that for all $p_1 \in V$, p_0 is topologically equivalent to p_1.

The space $C^\infty(M, N)$ is given the Whitney C^∞ topology. Thom conjectured and Mather proved that the topologically stable maps are generic in $C^\infty(M, N)$; in fact, they form an open dense subset (see [9], [21], [22]). The proof yields a stronger result, namely that the *strongly* topologically stable maps are dense (see [9]).

Definition 2.3. A smooth map $p_0 : M \to N$ between smooth manifolds is *strongly topologically stable* if there exists a neighborhood V of p_0 in $C^\infty(M, N)$ such that for all $p_1 \in V$, there exists a (*topologically*) *trivial* smooth one-parameter family $p : M \times I \to N$ joining p_0 to p_1. This means there exist continuous families

$\{h_t : M \to M \mid 0 \le t \le 1\}$ and $\{g_t : N \to N \mid 0 \le t \le 1\}$ of homeomorphisms such that $p_0 = g_t^{-1} \circ p_t \circ h_t$ for all $t \in I$, so that there is a commuting diagram:

$$\begin{array}{ccc} M & \xrightarrow{h_t} & M \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_t}\downarrow \\ N & \xrightarrow{g_t} & N. \end{array}$$

The notion of triviality for the one-parameter family of maps in the definition above can be generalized to arbitrary families of maps. We now recall that definition and the related notion of local triviality (cf. [21]).

Definition 2.4. Consider a commuting diagram of spaces and maps:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

(1) f is *trivial* over B if there exist spaces F_1 and F_2, a map $q : F_1 \to F_2$ and homeomorphisms $h : E_1 \to F_1 \times B$, $g : E_2 \to F_2 \times B$ such that the following diagram commutes:

$$\begin{array}{ccccccc} B & \xleftarrow{p_1} & E_1 & \xrightarrow{f} & E_2 & \xrightarrow{p_2} & B \\ {\scriptstyle \mathrm{id}_B}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\ B & \xleftarrow{\mathrm{proj}} & F_1 \times B & \xrightarrow{q \times \mathrm{id}_B} & F_2 \times B & \xrightarrow{\mathrm{proj}} & B \end{array}$$

(2) f is *locally trivial* over B if for every $x \in B$ there exists an open neighborhood U of x in B such that $f| : p_1^{-1}(U) \to p_2^{-1}(U)$ is trivial over U.
(3) In either case, $q : F_1 \to F_2$ is the *model* of the family f.

Remarks 2.5.

(1) The model $q : F_1 \to F_2$ is well-defined up to topological equivalence.
(2) Both $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are fibre bundle projections with fibre F_1 and F_2, respectively.
(3) For every $x \in B$, $f_x = f| : p_1^{-1}(x) \to p_2^{-1}(x)$ is topologically equivalent to $q : F_1 \to F_2$.
(4) One step in Mather's proof that the topologically stable smooth maps form an open dense subset is to show that certain families of maps are locally trivial. Thom's second isotopy lemma is used for this.

A *fibre preserving map* is a map which preserves the fibres of maps to a given space, usually a k-simplex or an arbitrary space B. Specifically, if $\rho : X \to B$ and $\sigma : Y \to B$ are maps, then a map $f : X \to Y$ is fibre preserving over B if $\sigma f = \rho$.

There is a notion of equivalence for families of maps over B.

Definition 2.6.

(1) Two locally trivial families of maps over B

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_2'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

are *topologically equivalent* provided there exist homeomorphisms

$$h_1 : E_1 \to E_1' \quad \text{and} \quad h_2 : E_2 \to E_2'$$

which are fibre preserving over B and $f'h_1 = h_2 f$; that is, the following diagram commutes:

$$\begin{array}{ccccccc} B & \xleftarrow{p_1} & E_1 & \xrightarrow{f} & E_2 & \xrightarrow{p_2} & B \\ {\scriptstyle \mathrm{id}_B}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\ B & \xleftarrow{p_1'} & E_1' & \xrightarrow{f'} & E_2' & \xrightarrow{p_2'} & B \end{array}$$

(2) Let $\mathcal{A}_1(q, B)$ denote the set of topological equivalence classes of locally trivial families of maps over B with model $q : F_1 \to F_2$.

The set $\mathcal{A}_1(q, B)$ can be interpreted as a set of equivalence classes of certain fibre bundles over B as follows. Let $\mathrm{TOP}(q)$ be the topological group given by the pull-back diagram

$$\begin{array}{ccc} \mathrm{TOP}(q) & \longrightarrow & \mathrm{TOP}(F_2) \\ \downarrow & & \downarrow{\scriptstyle q^\sharp} \\ \mathrm{TOP}(F_1) & \xrightarrow{q_\sharp} & \mathrm{Map}(F_1, F_2) \end{array}$$

where $q_\sharp(h) = q \circ h$ and $q^\sharp(g) = g \circ q$. That is,

$$\mathrm{TOP}(q) = \{(h, g) \in \mathrm{TOP}(F_1) \times \mathrm{TOP}(F_2) \mid qh = gq\}.$$

Note that $\mathrm{TOP}(q)$ is naturally a subgroup of $\mathrm{TOP}(F_1 \amalg F_2)$ via $(h, g) \mapsto h \amalg g$. Let $\mathcal{A}_2(q, B)$ denote the set of bundle equivalence classes of fibre bundles over B with fibre $F_1 \amalg F_2$ and structure group $\mathrm{TOP}(q)$.

Proposition 2.7. *There is a bijection $\alpha : \mathcal{A}_1(q, B) \to \mathcal{A}_2(q, B)$. In particular, if B is a separable metric space, then there is a bijection $\mathcal{A}_1(q, B) \to [B, \mathrm{BTOP}(q)]$.*

The function α is defined by sending a locally trivial family

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

to the fibre bundle $p_1 \amalg p_2 : E_1 \amalg E_2 \to B$ whose total space is the disjoint union of E_1 and E_2. The fact that α is a bijection is fairly straightforward to prove. At any rate, it follows from a more general result in §5 (see Theorem 5.5 and the comments following it).

3. Controlled topological equivalence

We propose a definition of topological equivalence in the setting of controlled topology and use it to make some speculations about generic maps between topological manifolds.

The mapping cylinder of a map $p : X \to Y$ is the space

$$\mathrm{cyl}(p) = (X \times I \amalg Y)/\{(x, 1) \sim p(x) \mid x \in X\}.$$

There is a natural map $\pi : \mathrm{cyl}(p) \to I$ defined by

$$\begin{cases} \pi([x,t]) = t, & \text{if } (x,t) \in X \times I \\ \pi([y]) = 1, & \text{if } y \in Y. \end{cases}$$

For clarification the map π will sometimes be denoted $\pi_p : \mathrm{cyl}(p) \to I$. If $p : X \to Y$ and $p' : X' \to Y'$ are maps and $\pi_p : \mathrm{cyl}(p) \to I$ and $\pi_{p'} : \mathrm{cyl}(p') \to I$ are the natural maps, then a homeomorphism $h : \mathrm{cyl}(p) \to \mathrm{cyl}(p')$ is *level* if $\pi_p = \pi_{p'} h$. Let $\mathrm{TOP}^{\mathrm{level}}(p)$ denote the simplicial group of level homeomorphisms from $\mathrm{cyl}(p)$ onto itself. That is, a k–simplex of $\mathrm{TOP}^{\mathrm{level}}(p)$ consists of a Δ^k–parameter family of level homeomorphisms $h : \mathrm{cyl}(p) \times \Delta^k \to \mathrm{cyl}(p) \times \Delta^k$. The group $\mathrm{TOP}(p)$ as defined in the previous section has a simplicial version (the singular complex of the topological group) and, as such, is a simplicial subgroup of $\mathrm{TOP}^{\mathrm{level}}(p)$. For example, a pair of homeomorphisms $(h : X \to X, g : Y \to Y)$ such that $ph = gp$ induces a level homeomorphism

$$\mathrm{cyl}(p) \to \mathrm{cyl}(p), \quad \begin{cases} [x,t] \mapsto [h(x), t], & \text{if } x \in X \\ [y] \mapsto [g(y)], & \text{if } y \in Y \end{cases}$$

Definition 3.1. Two maps $p_0 : X_0 \to Y_0$, $p_1 : X_1 \to Y_1$ are *controlled topologically equivalent* if there exists a level homeomorphism $h : cyl(p_0) \to \mathrm{cyl}(p_1)$.

Note that a level homeomorphism $h : \mathrm{cyl}(p_0) \to \mathrm{cyl}(p_1)$ induces (by restriction) a one-parameter family $h_t : X_0 \to X_1, 0 \leq t < 1$, of homeomorphisms and a homeomorphism $g : Y_0 \to Y_1$. If all the spaces involved are compact metric, then

$$gp_0 = \lim_{t \to 1} p_1 h_t$$

and such data is equivalent to having a level homeomorphism (cf. [16], [17], [19], [20]). This formulation should be compared with the formulation of topological equivalence in Definition 2.1.

Definition 3.2. Two maps $p_0 : X_0 \to Y_0$, $p_1 : X_1 \to Y_1$ between compact metric spaces are *weakly controlled topologically equivalent* if there exist continuous families $\{h_t : X_0 \to X_1 \mid 0 \leq t < 1\}$ and $\{g_t : Y_0 \to Y_1 \mid 0 \leq t < 1\}$ of homeomorphisms such that $p_0 = \lim_{t \to 1} g_t^{-1} \circ p_1 \circ h_t$.

The limit above is taken in the sup metric which is the metric for the compact-open topology. The space $C(X, Y)$ of maps from X to Y is given the compact-open topology.

Definition 3.3. A map $p_0 : X \to Y$ between compact metric spaces is *weakly controlled topologically stable* if there exists a neighborhood V of p_0 in $C(X,Y)$ such that for all $p_1 \in V$ and $\epsilon > 0$, there exists a map $p_1' : X \to Y$ such that p_0 is weakly controlled topologically equivalent to p_1' and p_1' is ϵ–close to p_1.

Many of the results in the theory of singularities have a mixture of smooth and topological hypotheses and conclusions. This is the case in Mather's theorem on the genericness of topologically stable maps among smooth maps. One direction that controlled topology is likely to take is in finding the topological underpinnings in this area. The following speculation is meant to be a step towards formulating what might be true.

Speculation 3.4. *If M and N are closed topological manifolds, then the weakly controlled topologically stable maps from M to N are generic in $C(M,N)$.*

This might be established by showing that the stratified systems of approximate fibrations are dense and also weakly controlled topologically stable (see Hughes [14] and Quinn [27] for stratified systems of approximate fibrations). As evidence for this line of reasoning, note that Chapman's work [2] shows that manifold approximate fibrations are weakly controlled topologically stable.

Another line of speculation concerns polynomial maps between euclidean spaces. It is known that the classification of polynomial maps up to smooth equivalence differs from their classification up to topological equivalence (cf. Thom [32], Fakuda [8], Nakai [23]). What can be said about the classification of polynomial maps up to controlled topological equivalence?

4. Controlled locally trivial families of maps

Analogues in controlled topology of locally trivial families of maps are defined. In fact, we define a moduli space of all such families.

Definition 4.1. Consider a commuting diagram of spaces and maps:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

(1) f is *controlled trivial* over B if there exist spaces F_1 and F_2, a map $q : F_1 \to F_2$ and a homeomorphism $H : \mathrm{cyl}(f) \to \mathrm{cyl}(q) \times B$ such that the following diagram commutes:

$$\begin{array}{ccccc} B & \xleftarrow{c} & \mathrm{cyl}(f) & \xrightarrow{\pi_f} & I \\ {\scriptstyle \mathrm{id}_B}\downarrow & & {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_I} \\ B & \xleftarrow{\mathrm{proj}} & \mathrm{cyl}(q) \times B & \xrightarrow{\pi_q'} & I \end{array}$$

where $c : \mathrm{cyl}(f) \to B$ is given by

$$\begin{cases} c([x,t]) = p_1(x) = p_2 f(x), & \text{if } (x,t) \in E_1 \times I \\ c([y]) = p_2(y), & \text{if } y \in E_2 \end{cases}$$

and π'_q is the composition $\mathrm{cyl}(q) \times B \xrightarrow{\mathrm{proj}} \mathrm{cyl}(q) \xrightarrow{\pi_q} I$.

(2) f is *controlled locally trivial* over B if for every $x \in B$ there exists an open neighborhood U of x in B such that $f| : p_1^{-1}(U) \to p_2^{-1}(U)$ is controlled trivial over U.

(3) In either case, $q : F_1 \to F_2$ is the *model* of the family f.

Remarks 4.2.

(1) The model $q : F_1 \to F_2$ is well-defined up to controlled topological equivalence.

(2) Both $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are fibre bundle projections with fibre F_1 and F_2, respectively.

(3) For every $x \in B$, $f_x = f| : p_1^{-1}(x) \to p_2^{-1}(x)$ is controlled topologically equivalent to $q : F_1 \to F_2$.

There is a notion of controlled equivalence for families of maps over B.

Definition 4.3.

(1) Two controlled locally trivial families of maps over B

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} E'_1 & \xrightarrow{f'} & E'_2 \\ {\scriptstyle p'_1}\downarrow & & \downarrow{\scriptstyle p'_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

are *controlled topologically equivalent* provided there exists a level homeomorphism

$$H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$$

which is fibre preserving over B in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{cyl}(f) & \xrightarrow{H} & \mathrm{cyl}(f') \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

where c is given by

$$\begin{cases} c([x,t]) = p_2 f(x) = p_1(x), & \text{if } (x,t) \in E_1 \times I \\ c([y]) = p_2(y), & \text{if } y \in E_2 \end{cases}$$

and c' is given by

$$\begin{cases} c'([x,t]) = p'_2 f'(x) = p'_1(x), & \text{if } (x,t) \in E'_1 \times I \\ c'([y]) = p'_2(y), & \text{if } y \in E'_2. \end{cases}$$

(2) Let $\mathcal{B}_1(q, B)$ denote the set of controlled topological equivalence classes of locally trivial families of maps over B with model $q : F_1 \to F_2$.

In the next section we will show that the set $\mathcal{B}_1(q, B)$ can be interpreted as a set of equivalence classes of certain fibre bundles over B in analogy with Proposition 2.7 (see Theorem 5.5). But first we will define the moduli space of all controlled locally trivial families of maps over B with model $q : F_1 \to F_2$. This is done in the setting of simplicial sets as follows.

Define a simplicial set $\mathbf{B}_1(q, B)$ so that a typical k–simplex of $\mathbf{B}_1(q, B)$ consists of a commuting diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B \times \Delta^k & \xrightarrow{\mathrm{id}_{B\times\Delta^k}} & B \times \Delta^k \end{array}$$

which is a controlled locally trivial family of maps over $B \times \Delta^k$ with model $q : F_1 \to F_2$. Thus, a vertex of $\mathbf{B}_1(q, B)$ is a controlled locally trivial family of maps over B with model $q : F_1 \to F_2$. (As in [17], [18] we also need to require that these spaces are reasonably embedded in an ambient universe, but we will ignore that technicality in this paper.) Face and degeneracy operations are induced from those on the standard simplexes. As in [18], this simplicial set satisfies the Kan condition.

Definition 4.4. The *mapping cylinder construction* μ takes a controlled locally trivial family of maps

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

to the mapping cylinder $\mathrm{cyl}(f)$ together with the natural map $\mu(f) : \mathrm{cyl}(f) \to B$.

Note that the controlled locally trivial condition on f means that $\mu(f) : \mathrm{cyl}(f) \to B$ is a fibre bundle with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$ where q is the model of f. If

$$\begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_2'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

is another controlled locally trivial family of maps over B with model q, then to have a controlled topological equivalence $H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$ as in Definition 4.3 means precisely to have a bundle isomorphism from $\mu(f)$ to $\mu(f')$.

Proposition 4.5. *There is a bijection* $\pi_0\mathbf{B}_1(q, B) \approx \mathcal{B}_1(q, B)$.

Proof. In order to see that the natural function $\pi_0\mathbf{B}_1(q, B) \to \mathcal{B}_1(q, B)$ is well-defined, suppose

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B \times \Delta^1 & \xrightarrow{\mathrm{id}_{B\times\Delta^1}} & B \times \Delta^1 \end{array}$$

is a locally trivial family of maps with model $q : F_1 \to F_2$. Then by the remarks above $\mu(f) : \mathrm{cyl}(f) \to B \times \Delta^1$ is a fibre bundle with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$. Thus, there is a bundle isomorphism from the restriction of $\mu(f)$ over $B \times \{0\}$ to the restriction of $\mu(f)$ over $B \times \{1\}$, and the remarks above further show that this isomorphism gives a controlled topological equivalence from

$$\begin{array}{ccc} p_1^{-1}(B \times \{0\}) & \xrightarrow{f|} & p_2^{-1}(B \times \{0\}) \\ {\scriptstyle p_1|}\downarrow & & \downarrow{\scriptstyle p_2|} \\ B \times \{0\} & \xrightarrow{\mathrm{id}_B} & B \times \{0\} \end{array} \quad \text{to} \quad \begin{array}{ccc} p_1^{-1}(B \times \{1\}) & \xrightarrow{f|} & p_2^{-1}(B \times \{1\}) \\ {\scriptstyle p_1|}\downarrow & & \downarrow{\scriptstyle p_2|} \\ B \times \{1\} & \xrightarrow{\mathrm{id}_B} & B \times \{1\} \end{array}$$

showing that the function is well-defined. The function is obviously surjective, so it remains to see that it is injective. To this end suppose that

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_2'} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

are controlled topologically equivalent with a level homeomorphism $H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$ as in Definition 4.3. Let $h_0 : E_1 \to E_1'$ and $h_1 : E_2 \to E_2'$ be the restrictions of H to the top and bottom of the mapping cylinders, respectively. Then there is an induced commutative diagram

$$\begin{array}{ccc} \mathrm{cyl}(h_0) & \longrightarrow & \mathrm{cyl}(h_1) \\ \downarrow & & \downarrow \\ B \times \Delta^1 & \longrightarrow & B \times \Delta^1 \end{array}$$

which is a 1–simplex in $\mathbf{B}_1(q, B)$ from f to f'. $\square$

5. Bundles with mapping cylinder fibres

In this section we show that controlled locally trivial families of maps over B can be interpreted as fibre bundles over B with fibre the mapping cylinder of the model. Reduced structure groups are discussed as well as a relative situation in which the target bundle over B is fixed.

Let B be a fixed separable metric space. Let $\mathcal{B}_2(q, B)$ denote the set of bundle equivalence classes of fibre bundles over B with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$. Define $\mathbf{B}_2(q, B)$ to be the simplicial set whose k–simplices are fibre bundles over $B \times \Delta^k$ with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$. The following result is well-known (cf. [17]).

Proposition 5.1. *There are bijections*

$$\pi_0 \mathbf{B}_2(q, B) \approx \mathcal{B}_2(q, B) \approx [B, \mathrm{BTOP}^{\mathrm{level}}(q)].$$

The mapping cylinder construction of Definition 4.4 has the following simplicial version.

Definition 5.2. *The mapping cylinder construction* is the simplicial map

$$\mu : \mathbf{B}_1(q,B) \to \mathbf{B}_2(q,B)$$

defined by sending a diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B \times \Delta^k & \xrightarrow{\mathrm{id}_{B\times\Delta^k}} & B \times \Delta^k \end{array}$$

to $\mathrm{cyl}(f) \to B \times \Delta^k$. Note that the local triviality condition on f implies that $\mathrm{cyl}(f) \to B \times \Delta^k$ is a fibre bundle projection with fibre $\mathrm{cyl}(q)$ and structure group $\mathrm{TOP}^{\mathrm{level}}(q)$.

The first part of the following result is proved in [18]. The second part follows from the first part together with Propositions 4.5 and 5.1.

Theorem 5.3. *The mapping cylinder construction defines a homotopy equivalence $\mu : \mathbf{B}_1(q,B) \to \mathbf{B}_2(q,B)$. In particular, $\mathcal{B}_1(q,B) \approx \mathcal{B}_2(q,B) \approx [B, \mathrm{BTOP}^{\mathrm{level}}(q)]$.*

Reduced structure groups. Let G be a simplicial subgroup of $\mathrm{TOP}^{\mathrm{level}}(q)$. We will now generalize the discussion above to the situation where the structure group is reduced to G.

Definition 5.4. Consider a controlled locally trivial family

$$\begin{array}{ccc} E_1 & \xrightarrow{f_1} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

with model $q : F_1 \to F_2$. Then f is *G–locally trivial* over B provided there exists an open cover $\mathcal{U}$ of B such that f is controlled trivial over U for each $U \in \mathcal{U}$ via a trivializing homeomorphism

$$H_U : \mathrm{cyl}(f| : p_1^{-1}(U) \to p_2^{-1}(U)) \to \mathrm{cyl}(q) \times B.$$

These trivializing homeomorphisms are required to have the property that if $U, V \in \mathcal{U}$ and $x \in U \cap V$, then

$$H_V \circ H_U^{-1}| : \mathrm{cyl}(q) \times \{x\} \to \mathrm{cyl}(q) \times \{x\}$$

is an element of G.

Let $\mathbf{B}_1(q,B,G)$ be the simplicial set whose k–simplices are the G–locally trivial families of maps over $B \times \Delta^k$ with model $q : F_1 \to F_2$. For example,

$$\mathbf{B}_1(q,B,\mathrm{TOP}^{\mathrm{level}}(q)) = \mathbf{B}_1(q,B).$$

Definition 4.3 can be extended in the obvious way to define what it means for two G–locally trivial families to be *G–controlled topologically equivalent* (the homeomorphism H is required to be a family of homeomorphisms in the group G) and $\mathcal{B}_1(q, B, G)$ denotes the set of equivalence classes. In analogy with Proposition 4.5 there is a bijection

$$\pi_0 \mathbf{B}_1(q, B, G) \approx \mathcal{B}_1(q, B, G).$$

Likewise $\mathcal{B}_2(q, B, G)$ denotes the set of bundle equivalence classes of fibre bundles over B with fibre $\mathrm{cyl}(q)$ and structure group G, and $\mathbf{B}_2(q, B, G)$ is the simplicial set whose k–simplices are fibre bundles over $B \times \Delta^k$ with fibre $\mathrm{cyl}(q)$ and structure group G. In analogy with Proposition 5.1 there are bijections

$$\pi_0 \mathbf{B}_2(q, B, G) \approx \mathcal{B}_2(q, B, G) \approx [B, \mathrm{B}G].$$

Moreover, the proof of Theorem 5.3 can be seen to give a proof of the following result (cf. [18, §2]).

Theorem 5.5. *The mapping cylinder construction defines a homotopy equivalence $\mu : \mathbf{B}_1(q, B, G) \to \mathbf{B}_2(q, B, G)$. In particular, $\mathcal{B}_1(q, B, G) \approx \mathcal{B}_2(q, B, G) \approx [B, \mathrm{B}G]$.*

As an example, consider the group $\mathrm{TOP}(q)$ of §2. It was pointed out at the beginning of §3 that $\mathrm{TOP}(q)$ is naturally a subgroup of $\mathrm{TOP}^{\mathrm{level}}(q)$. Note that $\mathcal{B}_1(q, B, \mathrm{TOP}(q)) = \mathcal{A}_1(q, B)$ and $\mathcal{B}_2(q, B, \mathrm{TOP}(q)) = \mathcal{A}_2(q, B)$, so that Proposition 2.7 follows directly from Theorem 5.5.

Fixed target bundle. There are also relative versions of the preceding results in which the bundle $p_2 : E_2 \to B$ is fixed. For example, $\mathcal{B}_1(q \text{ rel } p_2 : E_2 \to B)$ is the set of controlled locally trivial families of maps of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}_B} & B. \end{array}$$

Two such families $f : E_1 \to E_2$ and $f' : E_1' \to E_2$ are *controlled topologically equivalent* rel p_2 if the homeomorphism $H : \mathrm{cyl}(f) \to \mathrm{cyl}(f')$ of Definition 4.3 is required to be the identity on E_2. There are analogous definitions of the following:

(1) $\mathbf{B}_1(q \text{ rel } p_2 : E_2 \to B)$,
(2) $\mathcal{B}_2(q \text{ rel } p_2 : E_2 \to B)$,
(3) $\mathbf{B}_2(q \text{ rel } p_2 : E_2 \to B)$.

Definition 5.6. The group of *controlled homeomorphisms of* q is the subgroup $\mathrm{TOP}^c(q)$ of $\mathrm{TOP}^{\mathrm{level}}(q)$ consisting of all level homeomorphisms $h : \mathrm{cyl}(q) \times \Delta^k \to \mathrm{cyl}(q) \times \Delta^k$ such that $h | F_2 \times \Delta^k = \mathrm{id}_{F_2 \times \Delta^k}$.

Note that $\mathrm{TOP}^c(q)$ is the kernel of the restriction homomorphism

$$\mathrm{TOP}^{\mathrm{level}}(q) \to \mathrm{TOP}(F_2).$$

Let $\widehat{p_2} : B \to \mathrm{BTOP}(F_2)$ be the classifying map for the bundle p_2. Thus, $\mathcal{B}_2(q \text{ rel } p_2 : E_2 \to B)$ is in one-to-one correspondence with the set of vertical homotopy classes of lifts of $\widehat{p_2} : B \to \mathrm{BTOP}(F_2)$ to $\mathrm{BTOP}^{\mathrm{level}}(q) \to \mathrm{BTOP}(F_2)$:

$$\begin{array}{ccc} & & \mathrm{BTOP}^{\mathrm{level}}(q) \\ & & \downarrow \\ B & \xrightarrow{\widehat{p_2}} & \mathrm{BTOP}(F_2). \end{array}$$

The following result follows from the proofs of the preceding results.

Proposition 5.7.

(1) $\pi_0\mathbf{B}_1(q \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_1(q \text{ rel } p_2 : E_2 \to B)$,
(2) $\pi_0\mathbf{B}_2(q \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_2(q \text{ rel } p_2 : E_2 \to B)$,
(3) *the mapping cylinder construction defines a homotopy equivalence*

$$\mu : \mathbf{B}_1(q \text{ rel } p_2 : E_2 \to B) \to \mathbf{B}_2(q \text{ rel } p_2 : E_2 \to B).$$

Reduced structure group and fixed target bundle. There are versions of these relative results when the structure groups are reduced to G as before. The sets and simplicial sets involved are denoted as follows:

(1) $\mathcal{B}_1(q, G \text{ rel } p_2 : E_2 \to B)$,
(2) $\mathbf{B}_1(q, G \text{ rel } p_2 : E_2 \to B)$,
(3) $\mathcal{B}_2(q, G \text{ rel } p_2 : E_2 \to B)$,
(4) $\mathbf{B}_2(q, G \text{ rel } p_2 : E_2 \to B)$.

The following result records the analogous bijections and homotopy equivalences.

Proposition 5.8.

(1) $\pi_0\mathbf{B}_1(q, G \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_1(q, G \text{ rel } p_2 : E_2 \to B)$,
(2) $\pi_0\mathbf{B}_2(q, G \text{ rel } p_2 : E_2 \to B) \approx \mathcal{B}_2(q, G \text{ rel } p_2 : E_2 \to B)$,
(3) *the mapping cylinder construction defines a homotopy equivalence*

$$\mu : \mathbf{B}_1(q, G \text{ rel } p_2 : E_2 \to B) \to \mathbf{B}_2(q, G \text{ rel } p_2 : E_2 \to B).$$

6. Manifold stratified spaces

There are many naturally occurring spaces which are not manifolds but which are composed of manifold pieces, those pieces being called the *strata* of the space. Examples include polyhedra, algebraic varieties, orbit spaces of many group actions on manifolds, and mapping cylinders of maps between manifolds. Quinn [26] has introduced a class of stratified spaces called by him 'manifold homotopically stratified sets' with the objective 'to give a setting for the study of purely topological stratified phenomena' as opposed to the smooth and piecewise linear phenomena previously studied.

Roughly, the stratified spaces of Quinn are spaces X together with a finite filtration by closed subsets

$$X = X^m \supseteq X^{m-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset,$$

such that the strata $X_i = X^i \setminus X^{i-1}$ are manifolds with neighborhoods in $X_i \cup X_k$ (for $k > i$) which have the local homotopy properties of mapping cylinders of fibrations. These spaces include the smoothly stratified spaces of Whitney [35], Thom [31] and Mather [21] (for historical remarks on smoothly stratified spaces see Goresky and MacPherson [10]) as well as the locally conelike stratified spaces of Siebenmann [29] and, hence, orbit spaces of finite groups acting locally linearly on manifolds.

Cappell and Shaneson [1] have shown that mapping cylinders of 'smoothly stratified maps' between smoothly stratified spaces are in this class of topologically stratified spaces even though it is known that such mapping cylinders need not be smoothly stratified (see [1] and [32]). Hence, the stratified spaces of Quinn arise naturally in the category of smoothly stratified spaces. For a comprehensive survey of the classification and applications of stratified spaces, see Weinberger [34].

Smoothly stratified spaces have the property that strata have neighborhoods which are mapping cylinders of fibre bundles, a fact which is often used in arguments involving induction on the number of strata. Such neighborhoods fail to exist in general for Siebenmann's locally conelike stratified spaces. For example, it is known that a (topologically) locally flat submanifold of a topological manifold (which is an example of a locally conelike stratified space with two strata) may fail to have a tubular neighborhood (see Rourke and Sanderson [28]). However, Edwards [6] proved that such submanifolds do have neighborhoods which are mapping cylinders of manifold approximate fibrations (see also [18]). On the other hand, examples of Quinn [24] and Steinberger and West [30] show that strata in orbit spaces of finite groups acting locally linearly on manifolds may fail to have mapping cylinder neighborhoods. In Quinn's general setting, mapping cylinder neighborhoods may fail to exist even locally.

The main result announced in [12] (and restated here in §8) gives an effective substitute for neighborhoods which are mapping cylinders of bundles. Instead of fibre bundles, we use 'manifold stratified approximate fibrations,' and instead of mapping cylinders, we use 'teardrops'. This result should be thought of as a tubular neighborhood theorem for strata in manifold stratified spaces.

We now recall the concepts needed to precisely define the manifold stratified spaces of interest (see [26], [12], [15], [16]). A subset $Y \subseteq X$ is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I, h_1(U) = Y$, and $h((U \setminus Y) \times [0,1)) \subseteq X \setminus Y$.

Define the *homotopy link* of Y in X by

$$\text{holink}(X,Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{iff } t = 0\}.$$

Evaluation at 0 defines a map $q : \text{holink}(X,Y) \to Y$ called *holink evaluation*.

Let $X = X^m \supseteq X^{m-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset$ be a space with a finite filtration by closed subsets. Then X^i is the *i-skeleton* and the difference $X_i = X^i \setminus X^{i-1}$ is called the *i-stratum*.

A subset A of a filtered space X is called a *pure* subset if A is closed and a union of components of strata of X. For example, the skeleta are pure subsets.

The *stratified homotopy link* of Y in X, denoted $\mathrm{holink}_{\mathrm{s}}(X,Y)$ consists of all ω in $\mathrm{holink}(X,Y)$ such that $\omega((0,1])$ lies in a single stratum of X. The stratified homotopy link has a natural filtration with i–skeleton

$$\mathrm{holink}_{\mathrm{s}}(X,Y)^i = \{\omega \in \mathrm{holink}_{\mathrm{s}}(X,Y) \mid \omega(1) \in X^i\}.$$

The holink evaluation (at 0) restricts to a map $q : \mathrm{holink}_{\mathrm{s}}(X,Y) \to Y$.

If X is a filtered space, then a map $f : Z \times A \to X$ is *stratum preserving along* A if for each $z \in Z$, $f(\{z\} \times A)$ lies in a single stratum of X. In particular, a map $f : Z \times I \to X$ is a *stratum preserving homotopy* if f is stratum preserving along I.

Definition 6.1. A filtered space X is a *manifold stratified space* if the following four conditions are satisfied:

(1) **Manifold strata.** X is a locally compact, separable metric space and each stratum X_i is a topological manifold (without boundary).
(2) **Forward tameness.** For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.
(3) **Normal fibrations.** For each $k > i$, the holink evaluation $q : \mathrm{holink}(X_i \cup X_k, X_i) \to X_i$ is a fibration.
(4) **Finite domination.** For each i there exists a closed subset K of the stratified homotopy link $\mathrm{holink}_{\mathrm{s}}(X, X^i)$ such that the holink evaluation map $K \to X^i$ is proper, together with a stratum preserving homotopy
$$h : \mathrm{holink}_{\mathrm{s}}(X, X^i) \times I \to \mathrm{holink}_{\mathrm{s}}(X, X^i)$$
which is also fibre preserving over X^i (i.e., $qh_t = q$ for each $t \in I$) such that $h_0 = \mathrm{id}$ and $h_1(\mathrm{holink}_{\mathrm{s}}(X, X^i)) \subseteq K$.

7. Manifold stratified approximate fibrations

The definition of an approximate fibration (as given in [17]) was generalized in [12] to the stratified setting. Let $X = X^m \supseteq \cdots \supseteq X^0$ and $Y = Y^n \supseteq \cdots \supseteq Y^0$ be filtered spaces and let $p : X \to Y$ be a map (p is not assumed to be stratum preserving). Then p is said to be a *stratified approximate fibration* provided given any space Z and any commuting diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & Y \end{array}$$

where F is a stratum preserving homotopy, there exists a *stratified controlled solution*; i.e., a map $\tilde{F} : Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\tilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\bar{F} : Z \times I \times [0,1] \to Y$ defined by $\bar{F}|Z \times I \times [0,1) = p\tilde{F}$ and $\bar{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous.

A stratified approximate fibration between manifold stratified spaces is a *manifold stratified approximate fibration* if, in addition, it is a proper map (i.e., inverse images of compact sets are compact).

8. Teardrop neighborhoods

Given spaces X, Y and a map $p : X \to Y \times \mathbb{R}$, the *teardrop* of p (see [16]) is the space denoted by $X \cup_p Y$ whose underlying set is the disjoint union $X \amalg Y$ with the minimal topology such that

(1) $X \subseteq X \cup_p Y$ is an open embedding, and
(2) the function $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ defined by

$$c(x) = \begin{cases} p(x), & \text{if } x \in X \\ (x, +\infty), & \text{if } x \in Y. \end{cases}$$

is continuous.

The map c is called *the tubular map of the teardrop* or *the teardrop collapse*. The tubular map terminology comes from the smoothly stratified case (see [4], [21], [33]). This is a generalization of the construction of the open mapping cylinder of a map $g : X \to Y$. Namely, $\overset{\circ}{\mathrm{cyl}}(g)$ is the teardrop $(X \times \mathbb{R}) \cup_{g \times \mathrm{id}} Y$.

Theorem 8.1. *If X and Y are manifold stratified spaces and $p : X \to Y \times \mathbb{R}$ is a manifold stratified approximate fibration, then $X \cup_p Y$ is a manifold stratified space with Y a pure subset.*

In this statement, $Y \times \mathbb{R}$ and $X \cup_p Y$ are given the natural stratifications.

The next result from [12] is a kind of converse to this proposition. First, some more definitions. A subset Y of a space X has a *teardrop neighborhood* if there exist a neighborhood U of Y in X and a map $p : U \setminus Y \to Y \times \mathbb{R}$ such that the natural function $(U \setminus Y) \cup_p Y \to U$ is a homeomorphism. In this case, U is the *teardrop neighborhood* and p is the restriction of the tubular map.

Theorem 8.2 (Teardrop Neighborhood Existence). *Let X be a manifold stratified space such that all components of strata have dimension greater than 4, and let Y be a pure subset. Then Y has a teardrop neighborhood whose tubular map*

$$c : U \to Y \times (-\infty, +\infty]$$

is a manifold stratified approximate fibration.

A complete proof of this result will be given in [13], but special cases are in [15] and [16].

The next result from [12] concerns the classification of neighborhoods of pure subsets of a manifold stratified space. Given a manifold stratified space Y, a *stratified neighborhood* of Y consists of a manifold stratified space containing Y as a pure subset. Two stratified neighborhoods X, X' of Y are *equivalent* if there exist neighborhoods U, U' of Y in X, X', respectively, and a stratum preserving homeomorphism $h : U \to U'$ such that $h|Y = \mathrm{id}$. A *neighborhood germ* of Y is an equivalence class of stratified neighborhoods of Y.

Theorem 8.3 (Neighborhood Germ Classification). *Let Y be a manifold stratified space such that all components of strata have dimension greater than 4. Then the teardrop construction induces a one-to-one correspondence from controlled, stratum preserving homeomorphism classes of manifold stratified approximate fibrations over $Y \times \mathbb{R}$ to neighborhood germs of Y.*

9. Applications of Teardrop Neighborhoods

Teardrop neighborhoods can also be used in conjunction with the geometric theory of manifold approximate fibrations [11] to study the geometric topology of manifold stratified pairs. Examples of results proved using teardrop technology are stated in this section. Details will appear in [13].

Theorem 9.1 (Parametrized Isotopy Extension). *Let X be a manifold stratified space such that all components of strata have dimension greater than 4, let Y be a pure subset of X, let U be a neighborhood of Y in X, and let $h : Y \times \Delta^k \to Y \times \Delta^k$ be a k-parameter stratum preserving isotopy. Then there exists a k-parameter isotopy $\tilde{h} : X \times \Delta^k \to X \times \Delta^k$ extending h and supported on $U \times \Delta^k$.*

This generalizes results of Edwards and Kirby [7], Siebenmann [29] and Quinn [26].

The next result is a topological analogue of Thom's First Isotopy Theorem [31] and can be viewed as a first step towards a topological theory of topological stability.

Theorem 9.2 (First Topological Isotopy). *Let X be a manifold stratified space and let $p : X \to \mathbb{R}^n$ be a map such that*

(i) *p is proper,*
(ii) *for each stratum X_i of X, $p| : X_i \to \mathbb{R}^n$ is a topological submersion,*
(iii) *for each $t \in \mathbb{R}^n$, the filtration of X restricts to a filtration of $p^{-1}(t)$ giving $p^{-1}(t)$ the structure of a manifold stratified space such that all components of strata have dimension greater than 4.*

Then p is a bundle and can be trivialized by a stratum preserving homeomorphism; that is, there exists a stratum preserving homeomorphism $h : p^{-1}(0) \times \mathbb{R}^n \to X$ such that ph is projection.

Here is a non-proper version of the preceding result.

Theorem 9.3 (Non-proper First Topological Isotopy). *Let X be a manifold stratified space and let $p : X \to \mathbb{R}^n$ be a map such that*

(i) *if $\rho : X \to [0, \infty)$ is a proper map and $p' = \rho \times p : X \to \mathbb{R}^n \times [0, \infty)$, then the teardrop $X \cup_{p'} \mathbb{R}^n$ is a manifold stratified space,*
(ii) *for each stratum X_i of X, $p| : X_i \to \mathbb{R}^n$ is a topological submersion,*
(iii) *for each $t \in \mathbb{R}^n$, the filtration of X restricts to a filtration of $p^{-1}(t)$ giving $p^{-1}(t)$ the structure of a manifold stratified space such that all components of strata have dimension greater than 4.*

Then p is a bundle and can be trivialized by a stratum preserving homeomorphism; that is, there exists a stratum preserving homeomorphism $h : p^{-1}(0) \times \mathbb{R}^n \to X$ such that ph is projection.

10. Classifying manifold stratified approximate fibrations

Some applications of teardrop neighborhoods are combined with the material in §5 on bundles with mapping cylinder fibres in order to present a classification of manifold stratified approximate fibrations, at least when the range is a manifold, generalizing the classification of manifold approximate fibrations in [17] and [18].

For notation, let B be a connected i-manifold without boundary and let $q : V \to \mathbb{R}^i$ be a manifold stratified approximate fibration where all components of strata of V have dimension greater than 4. A stratified manifold approximate fibration $p : X \to B$ has *fibre germ* q if there exists an embedding $\mathbb{R}^i \subseteq B$ such that $p| : p^{-1}(\mathbb{R}^i) \to \mathbb{R}^i$ is controlled, stratum preserving homeomorphic to q; that is, there exists a stratum preserving, level homeomorphism $\mathrm{cyl}(q) \to \mathrm{cyl}(p| : p^{-1}(\mathbb{R}^i) \to \mathbb{R}^i)$ where the mapping cylinders have the natural stratifications.

The following result shows that fibre germs are essentially unique. For notation, let $r : \mathbb{R}^i \to \mathbb{R}^i$ be the orientation reversing homeomorphism defined by $r(x_1, x_2, \dots, x_i) = (-x_1, x_2, \dots, x_i)$.

Theorem 10.1. *Let $p : X \to B$ be a manifold stratified approximate fibration such that all components of strata have dimension greater than 4. Let $g_k : \mathbb{R}^i \to B$, $k = 1, 2$, be two open embeddings. Then $p| : p^{-1}(g_0(\mathbb{R}^i)) \to g_0(\mathbb{R}^i)$ is controlled, stratum preserving homeomorphic to either $p| : p^{-1}(g_1(\mathbb{R}^i)) \to g_1(\mathbb{R}^i)$ or $p| : p^{-1}(g_1(\mathbb{R}^i)) \to rg_1(\mathbb{R}^i)$.*

Proof. The proof follows that of the corresponding result for manifold approximate fibrations in [17, Cor. 14.6]. The stratified analogues of the straightening phenomena are consequences of the teardrop neighborhood results [12], [13]. The use of Siebenmann's Technical Bundle Theorem is replaced with the non-proper topological version of Thom's First Isotopy Lemma in §9. □

There is a moduli space $\mathrm{MSAF}(B)_q$ of all manifold stratified approximate fibrations over B with fibre germ q. It is defined as a simplicial set with a typical k–simplex given by a map $p : X \to B \times \Delta^k$ such that for each $t \in \Delta^k$, $p| : p^{-1}(t) \to B \times \{t\}$ is a manifold stratified approximate fibration with fibre germ q and there exists a stratum preserving homeomorphism $p^{-1}(0) \times \Delta^k \to X$ which is fibre preserving over Δ^k. (There is also a technical condition giving an embedding in an ambient universe; cf. [17]).

The proof of the next proposition follows that of the corresponding result for manifold approximate fibrations in [17]. The necessary stratified versions of the manifold approximate fibration tools are in [12] and [13] and follow from teardrop technology.

Proposition 10.2. *$\pi_0 \mathrm{MSAF}(B)_q$ is in one-to-one correspondence with the set of controlled, stratum preserving homeomorphism classes of stratified manifold approximate fibrations over B with fibre germ q.*

Let $\mathrm{TOP}_s^{\mathrm{level}}(q)$ denote the simplicial group of self homeomorphisms of the mapping cylinder $\mathrm{cyl}(p)$ which preserve the mapping cylinder levels and are stratum preserving with respect to the induced stratification of $\mathrm{cyl}(q)$. Note that there is a restriction homomorphism $\mathrm{TOP}_s^{\mathrm{level}}(q) \to \mathrm{TOP}_i$.

Let $\tau B \to B$ denote the topological tangent bundle of B. Consider τB as an open neighborhood of the diagonal in $B \times B$ so that $\tau B \to B$ is first coordinate projection. As in §5 we can form the simplicial set $\mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B \to B)$ which we denote simply by $\mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B)$.

The *differential*

$$d : \mathrm{MSAF}(B)_q \to \mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B)$$

is a simplicial map whose definition is illustrated on vertices as follows (for higher dimensional simplices, the construction is analogous; cf. [17]). If $p : X \to B$ is a vertex of $\mathrm{MSAF}(B)_q$, then form

$$\mathrm{id}_B \times p : B \times X \to B \times B$$

and let

$$\hat{p} = p| : E = p^{-1}(\tau B) \to \tau B.$$

Thus, there is a commuting diagram

$$\begin{array}{ccc} E & \xrightarrow{\hat{p}} & \tau B \\ \downarrow & & \downarrow \\ B & \xrightarrow{\mathrm{id}_B} & B. \end{array}$$

It follows from the stratified straightening phenomena [13] that the local triviality condition is satisfied, so that the diagram is a vertex of

$$\mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B).$$

Once again the proof of the following result follows that of the corresponding manifold approximate fibration result in [17] using the stratified results of [12] and [13].

Theorem 10.3 (MSAF Classification). *The differential*

$$d : \mathrm{MSAF}(B)_q \to \mathbf{B}_1(q, \mathrm{TOP}_s^{\mathrm{level}}(q) \text{ rel } \tau B)$$

is a homotopy equivalence.

Corollary 10.4. *Controlled, stratum preserving homeomorphism classes of stratified manifold approximate fibrations over B with fibre germ q are in one-to-one correspondence with homotopy classes of lifts of the map $\tau : B \to \mathrm{BTOP}_i$ which classifies the tangent bundle of B, to $\mathrm{BTOP}_s^{\mathrm{level}}(q)$:*

$$\begin{array}{ccc} & & \mathrm{BTOP}_s^{\mathrm{level}}(q) \\ & & \downarrow \\ B & \xrightarrow{\tau} & \mathrm{BTOP}_i\,. \end{array}$$

Proof. Combine Theorem 10.3, Proposition 10.2 and Proposition 5.8. □

Finally, observe that Corollary 10.4 can be combined with Theorem 8.3 to give a classification of neighborhood germs of B with fixed local type.

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Department of Mathematics, Vanderbilt University, Nashville, TN 37240
E-mail address: hughes@math.vanderbilt.edu