NOTES ON SURGERY AND C^*-ALGEBRAS

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1. Introduction

A C^**-algebra* is a complex Banach algebra A with an involution $*$, which satisfies the identity

$$\|x^*x\| = \|x\|^2 \quad \forall x \in A.$$

The study of C^*-algebras seems to belong entirely within the realm of functional analysis, but in the past twenty years they have played an increasing rôle in geometric topology. The reason for this is that C^*-algebra K-theory is a natural receptacle for 'higher indices' of elliptic operators, including the 'higher signatures' which feature as surgery obstructions. The 'big picture' was originated by Atiyah [1, 2] and Connes [5, 6]; in these notes, based on my talk at the Josai conference, I want to explain part of the connection with particular reference to surgery theory. For more details one could consult [24].

2. About C^*-algebras

The following are key examples of C^*-algebras

- The algebra $C(X)$ of continuous complex-valued functions on a compact Hausdorff space X.
- The algebra $\mathfrak{B}(H)$ of bounded linear operators on a Hilbert space H.

Gelfand and Naimark (about 1950) proved: Any commutative C^*-algebra with unit is of the form $C(X)$; any C^*-algebra is a subalgebra of some $\mathfrak{B}(H)$.

Let A be a unital C^*-algebra. Let $x \in A$ be *normal*, that is $xx^* = x^*x$. Then x generates a commutative C^*-subalgebra of A which must be of the form $C(X)$. In fact we can identify X as the *spectrum*

$$X = \sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ has no inverse}\}$$

with x itself corresponding to the canonical $X \to \mathbb{C}$.

Hence we get the *Spectral Theorem*: for any $\varphi \in C(\sigma(x))$ we can define $\varphi(x) \in A$ so that the assignment $\varphi \mapsto \varphi(x)$ is a ring homomorphism.

If x is *self adjoint* ($x = x^*$), then $\sigma(x) \subseteq \mathbb{R}$.

One can define K-theory groups for C^*-algebras. For A unital

- $K_0(A)$ = Grothendieck group of f.g. projective A-modules
- $K_1(A) = \pi_0 GL_\infty(A)$

with a simple modification for non-unital A. These groups agree with the ordinary topological K-theory groups of the space X in case A is the commutative C^*-algebra $C(X)$.

For any integer i define $K_i = K_{i\pm 2}$. Then to any short exact sequence of C^*-algebras

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$$0 \to J \to A \to A/J \to 0$$

there is a long exact K-theory sequence

$$\dots K_i(J) \to K_i(A) \to K_i(A/J) \to K_{i-1}(J) \dots$$

The 2-periodicity is a version of the Bott periodicity theorem. Notice that *algebraic* K-theory does not satisfy Bott periodicity; analysis is essential here.

A good reference for this material is [26].

Classical Fredholm theory provides a useful example of C^*-algebra K-theory at work. Recall that an operator T on a Hilbert space H is called *Fredholm* if it has finite-dimensional kernel and cokernel. Then the *index* of T is the difference of the dimensions of the kernel and cokernel.

(2.1) DEFINITION: *The algebra of* compact operators, *$\mathfrak{K}(H)$, is the C^*-algebra generated by the operators with finite-dimensional range.*

Compact and Fredholm operators are related by *Atkinson's Theorem*, which states that $T \in \mathfrak{B}(H)$ is Fredholm if and only if its image in $\mathfrak{B}(H)/\mathfrak{K}(H)$ is invertible.

Thus a Fredholm operator T defines a class $[T]$ in $K_1(\mathfrak{B}/\mathfrak{K})$. Under the connecting map this passes to $\partial[T] \in K_0(\mathfrak{K}) = \mathbb{Z}$; this is the index.

3. ABSTRACT SIGNATURES

Recall that in symmetric L-theory we have isomorphisms $L^0(\mathbb{Z}) \to L^0(\mathbb{R}) \to \mathbb{Z}$. The second map associates to a nonsingular real symmetric matrix its *signature* = (Number of positive eigenvalues) − (Number of negative eigenvalues).

Can we generalize this to other rings?

If M is a nonsingular symmetric matrix over a C^*-algebra A we can use the spectral theorem to define projections p_+ and p_- corresponding to the positive and negative parts of the spectrum. Their difference is a class in $K_0(A)$.

This procedure defines a map $L_0^p(A) \to K_0(A)$ for every C^*-algebra A, and it can be shown that this map is an isomorphism [25]. There is a similar isomorphism on the level of L_1.

Now let Γ be a discrete group. The group ring $\mathbb{Z}\Gamma$ acts faithfully by convolution on the Hilbert space $\ell^2\Gamma$. The C^*-subalgebra of $\mathfrak{B}(\ell^2\Gamma)$ generated by $\mathbb{Z}\Gamma$ acting in this way is called the *group C^*-algebra*, $C_r^*\Gamma$.

We have a map $L_0(\mathbb{Z}\Gamma) \to K_0(C_r^*\Gamma)$.

Gelfand and Mishchenko [10] observed that this map is a *rational isomorphism* for Γ free abelian. (Then $C_r^*\Gamma = C(T^k)$ by Fourier analysis.)

REMARK: Our map from L_0 to K_0 is special to C^*-algebras; if it extended naturally to a map on all rings, we would have for a free abelian group Γ a diagram

$$\begin{array}{ccc} L_0(\mathbb{Z}\Gamma) & \longrightarrow & L_0(C_r^*\Gamma) \\ \downarrow & & \downarrow \\ K_0(\mathbb{Z}\Gamma) & \longrightarrow & K_0(C_r^*\Gamma) \end{array}$$

Going round the diagram via the top right we get Gelfand and Mishchenko's map, a rational isomorphism. But the bottom left-hand group is of rank one, by the Bass-Heller-Swan theorem [4, Chapter XII]. This contradiction shows that the left-hand vertical map cannot exist.

4. The signature operator

Let M be a complete oriented Riemannian manifold of even dimension (for simplicity). Define the operator $F = D(1+D^2)^{-\frac{1}{2}}$ on L^2 differential forms, where $D = d + d^*$, d =exterior derivative, d^* = its adjoint.

F is *graded* by an involution $\varepsilon = i^? *$ (here $i = \sqrt{-1}$ and the power ? depends on the dimension and the degree of forms, see [3] for the correct formula). Thus graded it is called the *signature operator*.

If M is compact, then F is Fredholm. Moreover the index of F is the signature of M. This is a simple consequence of Hodge theory [3].

Remark: The choice of normalizing function $\varphi(x) = x(1+x^2)^{-\frac{1}{2}}$ in $F = \varphi(D)$ does not matter as long as it has the right asymptotic behaviour.

Consider now the signature operator on the universal cover $\widetilde{M}$ of a compact manifold M. F belongs to the algebra A of $\Gamma = \pi_1 M$ equivariant operators. Moreover it is invertible modulo the ideal J of Γ equivariant *locally compact* operators. This follows from the theory of elliptic operators.

Thus via the connecting map $\partial\colon K_1(A/J) \to K_0(J)$ we get an 'index' in $K_0(J)$.

(4.1) Lemma: *$J \equiv C_r^*\Gamma \otimes \mathfrak{K}$. Consequently $K_0(J) = K_0(C_r^*\Gamma)$.*

We have defined the *analytic signature* of M as an element of $K_0(C_r^*\Gamma)$. In general it can be defined in $K_i(M)$ where i is the dimension of M mod 2.

(4.2) Proposition: *The analytic signature is the image of the Mishchenko-Ranicki symmetric signature under the map $L^0 \to K_0$.*

(4.3) Corollary: *The analytic signature is invariant under orientation preserving homotopy equivalence.*

Direct proofs of this can be given [13].

We can now define an 'analytic surgery obstruction' (= difference of analytic signatures) for a degree one normal map.

Can we mimic the rest of the surgery exact sequence?

5. K-homology

Let A be a C^*-algebra. A *Fredholm module* for A is made up of the following things.

- A representation $\rho\colon A \to \mathfrak{B}(H)$ of A on a Hilbert space
- An operator $F \in \mathfrak{B}(H)$ such that for all $a \in A$ the operators

$$F\rho(a) - \rho(a)F, \quad (F^2 - 1)\rho(a), \quad (F - F^*)\rho(a)$$

belong to $\mathfrak{K}(H)$.

The signature operator is an example with $A = C_0(M)$.

One can define both 'graded' and 'ungraded' Fredholm modules. These objects can be organized into Grothendieck groups to obtain Kasparov's *K-homology groups* $K^i(A)$ [15]. ($i = 0$ for graded and $i = 1$ for ungraded modules). They are contravariant functors of A.

REMARK: The critical condition in the definition is that $[F, \rho(a)] \in \mathfrak{K}$ for all a. One should regard this as a continuous control condition. In fact, if A is commutative it was shown by Kasparov that the condition is equivalent to $\rho(f)F\rho(g) \in \mathfrak{K}$ whenever f and g have disjoint supports — which is to say that F has 'only finite rank propagation' between open sets with disjoint closures.

Kasparov proved that the name 'K-homology' is justified.

(5.1) THEOREM: [15, 16] *Let $A = C(X)$ be a commutative C^*-algebra. Then $K^i(A)$ is naturally isomorphic to $H_i(X; \mathbb{K}(\mathbb{C}))$, the topological K-homology of X.*

We assume that X is metrizable here. If X is a 'bad' space (not a finite complex) then H refers to the Steenrod extension of K-homology [14, 9]; if X is only *locally* compact and we take $A = C_0(X)$ (the continuous functions vanishing at infinity), then we get locally finite K-homology.

Kasparov's definition was reformulated in the language of 'duality' by Paschke [20] and Higson. For a C^*-algebra A and ideal J define the algebra $\Psi(A//J)$ to consist of those $T \in \mathfrak{B}(H)$ such that

- $[T, \rho(a)] \in \mathfrak{K}\ \forall a \in A$, and
- $T\rho(j) \in \mathfrak{K}\ \forall j \in J$

where ρ is a good (i.e. sufficiently large) representation of A on H.

(5.2) PROPOSITION: (PASCHKE DUALITY THEOREM) *There is an isomorphism*

$$K^i(A) = K_{i+1}(\Psi(A//0)/\Psi(A//A))$$

for all separable C^*-algebras A.

Let us introduce some notation. For a locally compact space X, write $\Psi^0(X)$ for $\Psi(C_0(X)//0)$ (we call this the algebra of *pseudolocal* operators), and $\Psi^{-1}(X)$ for $\Psi(C_0(X)//C_0(X))$ (the algebra of locally compact operators).

Now let $X = \widetilde{M}$, the universal cover of a compact manifold M as above, and consider the exact sequence

$$0 \to \Psi^{-1}(\widetilde{M})^\Gamma \to \Psi^0(\widetilde{M})^\Gamma \to \Psi^0(M)/\Psi^{-1}(M) \to 0.$$

The superscript Γ denotes the Γ-equivariant part of the algebra. We have incorporated into the sequence the fundamental isomorphism

$$\Psi^0(\widetilde{M})^\Gamma/\Psi^{-1}(\widetilde{M})^\Gamma = \Psi^0(M)/\Psi^{-1}(M)$$

which exists because both sides consist of *local* objects — 'formal symbols' in some sense — and there is no difficulty in lifting a local object from a manifold to its universal cover.

Note that $\Psi^{-1}(\widetilde{M})$ = locally compact operators. Thus, applying the K-theory functor, we get a boundary map

$$A\colon K^i(M) = K_{i+1}(\Psi^0(M)/\Psi^{-1}(M)) \to K_i(C_r^*\Gamma).$$

This *analytic assembly map* takes the homology class of the signature operator F to the analytic signature.

6. On the Novikov conjecture

We use the above machinery to make a standard reduction of the Novikov conjecture. Assume $B\Gamma$ is compact and let $f\colon M \to B\Gamma$. Consider the diagram

$$\begin{tikzcd} H_*(M;\mathbb{Q}) \arrow[d, "f_*"'] & K_*(M) \arrow[l, "ch"'] \arrow[r] \arrow[d, "f_*"'] & K_*(C_r^*\Gamma) \\ H_*(B\Gamma;\mathbb{Q}) & K_*(B\Gamma) \arrow[l, "ch"'] \arrow[ur] & \end{tikzcd}$$

By the Atiyah-Singer index theorem $f_*(ch[F])$ is Novikov's higher signature (the push forward of the Poincaré dual of the L-class). So, if $A\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is injective, the Novikov conjecture is true for Γ.

This has led to a number of partial solutions to the Novikov conjecture using analysis. Methods used have included

- *Cyclic cohomology* [8, 7, 19] — pair $K_*(C_r^*\Gamma)$ with $H^*(B\Gamma;\mathbb{R})$. Need suitable dense subalgebras — very delicate.
- *Kasparov KK-theory* [16, 17] — sometimes allows one to construct an inverse of the assembly map as an 'analytic generalized transfer'.
- *Controlled C^*-algebra theory* [11] — parallel development to controlled topology, see later.

7. The analytic structure set

Recall the exact sequence

$$K_{i+1}(C_r^*\Gamma) \to K_i(\Psi^0(\widetilde{M})^\Gamma) \to K^i(M) \to K_i(C_r^*\Gamma)$$

The analogy with the surgery exact sequence suggests that we should think of $K_*(\Psi^0(\widetilde{M})^\Gamma)$ as the 'analytic structure set' of M.

Example: Suppose M is spin. Then one has the Dirac operator D and one can normalize as before to get a homology class
$[F]$, $F = \varphi(D)$.

If M has a metric of positive scalar curvature, then by Lichnerowicz there is a gap in the spectrum of D near zero. Thus we can choose the normalizing function φ so that $F^2 = 1$ *exactly*. Then $[F] \in K_*(\Psi^0(\widetilde{M})^\Gamma)$ gives the *structure invariant* of the positive scalar curvature metric.

Notice that Lichnerowicz' vanishing theorem [18] now follows from exactness in the analytic surgery sequence.

It is harder to give a map from the usual structure set to the analytic one! In the same way that the positive scalar curvature invariant gives a 'reason' for the Lichnerowicz vanishing theorem, we want an invariant which gives a 'reason' for the homotopy invariance of the symmetric signature.

Here is one possibility. Recall Pedersen's description (in these proceedings) of the structure set $\mathcal{S}^{TOP}(M)$, as the L-theory of the category

$$\mathcal{B}(\widetilde{M} \times I, \widetilde{M} \times 1; \mathbb{Z})^\Gamma.$$

Replacing $\mathbb{Z}$ by $\mathbb{C}$ we have a category

- whose objects can be completed to Hilbert spaces with $C_0(\widetilde{M})$-action
- whose morphisms are pseudolocal

Using Voiculescu's theorem (which says that all the objects can be embedded more or less canonically in a single 'sufficiently large' representation of $C_0(\widetilde{M})$) we should get a map from the structure set to $K^*(\Psi^0(\widetilde{M})^\Gamma)$. However, there is a significant problem: Are the morphisms bounded operators? Similar questions seem to come up elsewhere if one tries to use analysis to study homeomorphisms, and one needs some kind of torus trick to resolve them (compare [21]).

8. Controlled C^*-algebras

A more direct approach can be given [12] to obtaining a map from $\mathcal{S}^{DIFF}(M)$.

Let W be a metric space (noncompact) and suppose $\rho\colon C_0(W) \to \mathfrak{B}(H)$ as usual.

An operator T on H is *boundedly controlled* if there is $R = R(T)$ such that $\rho(\varphi)T\rho(\psi) = 0$ whenever distance from Support φ to Support ψ is greater than R.

Example: If D is a Dirac-type operator on complete Riemannian M, and φ has compactly supported Fourier transform, then $\varphi(D)$ is boundedly controlled [23].

Define $\Psi^j_{bc}(W)$, $j = 0, -1$, to be the C^* subalgebras generated by boundedly controlled elements. Then from the above one has that a Dirac type operator on a complete Riemannian manifold W has a 'boundedly controlled index' in $K_*(\Psi^{-1}_{bc}(W))$.

In fact *all* elliptic operators have boundedly controlled indices: in full generality one has a bounded assembly map $A\colon K^{lf}_*(W) \to K_*(\Psi^{-1}_{bc}(W))$, and

the assembly of the signature operator is the *bounded analytic signature*.

This bounded analytic signature can also be defined for suitable ('bounded, bounded') Poincaré complexes (bounded in both the analytic and geometric senses).

If W has a compactification $X = W \cup Y$ which is 'small at infinity', then there is a close relation between bounded and continuously controlled C^*-algebra theory [11].

In fact, consider a metrizable pair (X, Y), let $W = X \setminus Y$. We can define continuously controlled C^*-algebras, $\Psi^j_{cc}(W)$. Then one has

(8.1) Proposition: [11] *We have*

- $\Psi^0_{cc}(W) = \Psi^0(X) = \Psi(C(X)//0)$
- $\Psi^{-1}_{cc}(W) = \Psi(C(X)//C_0(W))$

The result for $\Psi^{-1}_{cc}(W)$ is an analytic counterpart to the theorem 'control means homology at infinity' (compare [22]).

Now we can define our map from the structure set; for simplicity we work in the simply connected case. Given a homotopy equivalence $f\colon M' \to M$, form the 'double trumpet space' W, consisting of open cones on M and M' joined by the mapping cylinder of f (there is a picture in [24]). This is a 'bounded, bounded' Poincaré space with a map to $M \times \mathbb{R}$, continuously controlled by $M \times S^0$.

Thus we have the analytic signature in $K_*(\Psi^{-1}_{cc}(X \times \mathbb{R}))$. Map this by the composite

$$\Psi^{-1}_{cc}(X \times \mathbb{R}) \to \Psi^0_{cc}(X \times \mathbb{R}) = \Psi^0(X \times I) \to \Psi^0(X)$$

using the preceding proposition. The image is the desired structure invariant.

The various maps we have defined fit into a diagram relating the geometric and C^* surgery exact sequences [12]. The diagram commutes up to some factors of 2, arising from the difference between the Dirac and signature operators.

9. Final remarks

- C^*-surgery can produce some information in a wide range of problems.
- Surjectivity of C^*-assembly maps is related to representation theory.
- Some techniques for Novikov are only available in the C^*-world.
- *But* We don't understand well how to do analysis on topological manifolds.
- Topologists construct; analysts only obstruct.

References

[1] M.F. Atiyah. Global theory of elliptic operators. In *Proceedings of the International Symposium on Functional Analysis*, pages 21–30. University of Tokyo Press, 1969.

[2] M.F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque*, 32:43–72, 1976.

[3] M.F. Atiyah and I.M. Singer. The index of elliptic operators III. *Annals of Mathematics*, 87:546–604, 1968.

[4] H. Bass. *Algebraic K-Theory*. Benjamin, 1968.

[5] A. Connes. A survey of foliations and operator algebras. In *Operator Algebras and Applications*, pages 521–628. American Mathematical Society, 1982. Proceedings of Symposia in Pure Mathematics 38.

[6] A. Connes. *Non-Commutative Geometry*. Academic Press, 1995.

[7] A. Connes, M. Gromov, and H. Moscovici. Group cohomology with Lipschitz control and higher signatures. *Geometric and Functional Analysis*, 3:1–78, 1993.

[8] A. Connes and H. Moscovici. Cyclic cohomology, the Novikov conjecture, and hyperbolic groups. *Topology*, 29:345–388, 1990.

[9] S.C. Ferry. Remarks on Steenrod homology. In S. Ferry, A. Ranicki, and J. Rosenberg, editors, *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, volume 227 of *LMS Lecture Notes*, pages 148–166, 1995.

[10] I.M. Gelfand and A.S. Mishchenko. Quadratic forms over commutative group rings and K-theory. *Functional Analysis and its Applications*, 3:277–281, 1969.

[11] N. Higson, E.K. Pedersen, and J. Roe. C^*-algebras and controlled topology. *K-Theory*, 1996. To appear.

[12] N. Higson and J. Roe. Mapping surgery to analysis. In preparation.

[13] J. Kaminker and J.G. Miller. Homotopy invariance of the index of signature operators over C^*-algebras. *Journal of Operator Theory*, 14:113–127, 1985.

[14] J. Kaminker and C. Schochet. K-theory and Steenrod homology: applications to the Brown-Douglas-Fillmore theory of operator algebras. *Transactions of the American Mathematical Society*, 227:63–107, 1977.

[15] G.G. Kasparov. Topological invariants of elliptic operators I: K-homology. *Mathematics of the USSR — Izvestija*, 9:751–792, 1975.

[16] G.G. Kasparov. The operator K-functor and extensions of C^*-algebras. *Mathematics of the USSR — Izvestija*, 16:513–572, 1981.

[17] G.G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Inventiones Mathematicae*, 91:147–201, 1988.

[18] A. Lichnerowicz. Spineurs harmoniques. *Comptes Rendus de l'Académie des Sciences de Paris*, 257:7–9, 1963.

[19] C. Ogle. Cyclic homology and the strong Novikov conjecture. Preprint, 1996.

[20] W. Paschke. K-theory for commutants in the Calkin algebra. *Pacific Journal of Mathematics*, 95:427–437, 1981.

[21] E.K. Pedersen, J. Roe, and S. Weinberger. On the homotopy invariance of the boundedly controlled analytic signature of a manifold over an open cone. In S. Ferry, A. Ranicki, and J. Rosenberg, editors, *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, volume 227 of *LMS Lecture Notes*, pages 285–300, 1995.

[22] E.K. Pedersen and C.A. Weibel. A nonconnective delooping of algebraic K-theory. In A.A. Ranicki et al., editor, *Algebraic and Geometric Topology, Rutgers 1983*, volume 1126 of *Springer Lecture Notes in Mathematics*, pages 306–320, 1985.

[23] J. Roe. Coarse cohomology and index theory on complete Riemannian manifolds. *Memoirs of the American Mathematical Society*, 497, 1993.

[24] J. Roe. *Index theory, coarse geometry, and the topology of manifolds*, volume 90 of *CBMS Conference Proceedings*. American Mathematical Society, 1996.

[25] J. Rosenberg. Analytic Novikov for topologists. In S. Ferry, A. Ranicki, and J. Rosenberg, editors, *Proceedings of the 1993 Oberwolfach Conference on the Novikov Conjecture*, volume 226 of *LMS Lecture Notes*, pages 338–372, 1995.

[26] N.E. Wegge-Olsen. *K-theory and C^*-algebras*. Oxford University Press, 1993.

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