Controlled L-Theory
(Preliminary announcement)

A. Ranicki and M. Yamasaki

Introduction.

This is a preliminary announcement of a controlled algebraic surgery theory, of the type first proposed by Quinn [1]. We define and study the ϵ-controlled L-groups $L_n(X, p_X, \epsilon)$, extending to L-theory the controlled K-theory of Ranicki and Yamasaki [4].

The most immediate application of the algebra to controlled geometric surgery is the controlled surgery obstruction: a normal map $(f, b) : K \to L$ from a closed n-dimensional manifold to a δ-controlled Poincaré complex determines an element

$$\sigma_*^\delta(f, b) \in L_n(X, 1_X, 100\delta) \ .$$

(The construction in Ranicki and Yamasaki [3] can be used to produce a 6ϵ n-dimensional quadratic Poincaré structure on an $(n+1)$-dimensional chain complex. There is a chain equivalence from this to an n-dimensional chain complex with a 100δ n-dimensional quadratic Poincaré structure, and $\sigma_*^\delta(f, b)$ is the cobordism class of this complex in $L_n(X, 1_X, 100\delta)$.) A relative construction shows that if (f, b) can be made into a δ-controlled homotopy equivalence by δ-controlled surgery then

$$\sigma_*^\delta(f, b) = 0 \in L_n(X, 1_X, 100\delta) \ .$$

Conversely, if $n \geq 5$ and (f, b) is such that

$$\sigma_*^\delta(f, b) = 0 \in L_n(X, 1_X, 100\delta)$$

then (f, b) can be made into an ϵ-controlled homotopy equivalence by ϵ-controlled surgery, where $\epsilon = C \times 100\delta$ for a certain constant $C > 1$ that depends on n. Proofs of difficult results and the applications of the algebra to topology are deferred to the final account.

The algebraic properties required to obtain these applications include the controlled L-theory analogues of the homology exact sequence of a pair (3.1, 3.2) and the Mayer-Vietoris sequence (3.3, 3.4).

The limit of the controlled L-groups

$$L_n^c(X; 1_X) \ = \ \varprojlim_{\epsilon} \varprojlim_{\delta} \mathrm{im}\{L_n(X, 1_X, \delta) \longrightarrow L_n(X, 1_X, \epsilon)\}$$

is the obstruction group for controlled surgery to ϵ-controlled homotopy equivalence for all $\epsilon > 0$.

Theorem. (5.4.) *Fix a compact polyhedron X and an integer $n(\geq 0)$. There exist numbers $\epsilon_0 > 0$ and $0 < \mu_0 \leq 1$ such that*

$$L^c_n(X; 1_X) = \mathrm{im}\{L_n(X, 1_X, \delta) \longrightarrow L_n(X, 1_X, \epsilon)\}$$

for every $\epsilon \leq \epsilon_0$ and every $\delta \leq \mu_0 \epsilon$.

Throughout this paper all the modules are assumed to be finitely generated unless otherwise stated explicitly. But note that all the definitions and the constructions are valid also for possibly-infinitely-generated modules and chain complexes. Actually we heavily use finite dimensional but infinitely generated chain complexes in the later part of the paper. (That is where the bounded-control over $\mathbb{R}$ comes into the game.) So we first pretend that everything is finitely generated, and later we introduce a possibly-infinitely-generated analogue without any details.

1. Epsilon-controlled L-groups.

In this section we introduce ϵ-controlled L-groups $L_n(X, p_X, \epsilon)$ and $L_n(X, Y, p_X, \epsilon)$ for $p_X : M \to X$, $Y \subset X$, $n \geq 0$, $\epsilon > 0$. These are defined using geometric module chain complexes with quadratic Poincaré structures, which were discussed in Yamasaki [5].

We use the convention in Ranicki and Yamasaki [4] for radii of geometric morphisms, etc. The dual of a geometric module is the geometric module itself, and the dual of a geometric morphism is defined by reversing the orientation of paths. Note that if f has radius ϵ then so does its dual f^* and that $f \sim_\epsilon g$ implies $f^* \sim_\epsilon g^*$, by our convention. For a geometric module chain complex C, its dual C^{n-*} is defined using the sign convention used in Ranicki [2].

For a subset S of a metric space X, S^ϵ will denote the closed ϵ neighborhood of S in X when $\epsilon \geq 0$. When $\epsilon < 0$, S^ϵ will denote the set $X - (X - S)^{-\epsilon}$.

Let C be a free chain complex on $p_X : M \to X$. An *n-dimensional ϵ quadratic structure ψ on C* is a collection $\{\psi_s | s \geq 0\}$ of geometric morphisms

$$\psi_s \ : \ C^{n-r-s} \ = \ (C_{n-r-s})^* \ \to \ C_r \quad (r \in \mathbb{Z})$$

of radius ϵ such that

$$(*) \quad d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) \sim_{3\epsilon} 0 : C^{n-r-s-1} \to C_r,$$

for $s \geq 0$. An n-dimensional free ϵ chain complex C on p_X equipped with an n-dimensional ϵ quadratic structure is called an *n-dimensional ϵ quadratic complex on p_X*. (Here, a complex C is n-dimensional if $C_i = 0$ for $i < 0$ and $i > n$.)

Next let $f: C \to D$ be a chain map between free chain complexes on p_X. An $(n+1)$-*dimensional* ϵ *quadratic structure* $(\delta\psi, \psi)$ *on* f is a collection $\{\delta\psi_s, \psi_s | s \geq 0\}$ of geometric morphisms

$$\delta\psi_s : D^{n+1-r-s} \to D_r \quad , \quad \psi_s : C^{n-r-s} \to C_r \quad (r \in \mathbb{Z})$$

of radius ϵ such that the following holds in addition to $(*)$:

$$d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}) + (-)^n f\psi_s f^* \sim_{3\epsilon} 0$$
$$: D^{n-r-s} \to D_r \quad (s \geq 0) \ .$$

An ϵ chain map $f: C \to D$ between an n-dimensional free ϵ chain complex C on p_X and an $(n+1)$-dimensional free ϵ chain complex D on p_X equipped with an $(n+1)$-dimensional ϵ quadratic structure is called an $(n+1)$-*dimensional* ϵ *quadratic pair on* p_X. Obviously its *boundary* (C, ψ) is an n-dimensional ϵ quadratic complex on p_X.

An ϵ *cobordism* of n-dimensional ϵ quadratic structures ψ on C and ψ' on C' is an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi \oplus -\psi')$ on some chain map $C \oplus C' \to D$. An ϵ *cobordism* of n-dimensional ϵ quadratic complexes (C, ψ), (C', ψ') on p_X is an $(n+1)$-dimensional ϵ quadratic pair on p_X

$$((f \quad f') : C \oplus C' \to D, \quad (\delta\psi, \psi \oplus -\psi'))$$

with boundary $(C \oplus C', \psi \oplus -\psi')$. The union of adjoining cobordisms are defined using the formula in Chapter 1.7 of Ranicki [2]. The union of adjoining ϵ cobordisms is a 2ϵ cobordism.

ΣC and ΩC will denote the suspension and the desuspension of C respectively, and $\mathcal{C}(f)$ will denote the algebraic mapping cone of a chain map f.

Definition. Let W be a subset of X. An n-dimensional ϵ quadratic structure ψ on C is ϵ *Poincaré (over W)* if the algebraic mapping cone of the duality 3ϵ chain map

$$\mathcal{D}_\psi = (1+T)\psi_0 : C^{n-*} \longrightarrow C$$

is 4ϵ contractible (over W). A quadratic complex (C, ψ) is ϵ *Poincaré (over W)* if ψ is ϵ Poincaré (over W). Similarly, an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f: C \to D$ is ϵ *Poincaré (over W)* if the algebraic mapping cone of the duality 4ϵ chain map

$$\mathcal{D}_{(\delta\psi,\psi)} = ((1+T)\delta\psi_0 \quad f(1+T)\psi_0) \ : \ \mathcal{C}(f)^{n+1-*} \longrightarrow D$$

is 4ϵ contractible (over W) (or equivalently the algebraic mapping cone of the 4ϵ chain map

$$\bar{\mathcal{D}}_{(\delta\psi,\psi)} = \begin{pmatrix} (1+T)\delta\psi_0 \\ (-)^{n+1-r}(1+T)\psi_0 f^* \end{pmatrix} : D^{n+1-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is 4ϵ contractible (over W)) and ψ is ϵ Poincaré (over W). A quadratic pair $(f, (\delta\psi, \psi))$ is ϵ *Poincaré (over W)* if $(\delta\psi, \psi)$ is ϵ Poincaré (over W). We will also use the notation $\mathcal{D}_{\delta\psi} = (1+T)\delta\psi_0$, although it does not define a chain map from D^{n+1-*} to D in general.

Definition. (1) A positive geometric chain complex C ($C_i = 0$ for $i < 0$) is ϵ *connected* if there exists a 4ϵ morphism $h : C_0 \to C_1$ such that $dh \sim_{8\epsilon} 1_{C_0}$.
(2) A chain map $f : C \to D$ of positive chain complexes is ϵ *connected* if $\mathcal{C}(f)$ is ϵ connected.
(3) A quadratic complex (C, ψ) is ϵ *connected* if $\mathcal{D}_\psi$ is ϵ connected.
(4) A quadratic pair $(f : C \to D, (\delta\psi, \psi))$ is ϵ *connected* if $\mathcal{D}_\psi$ and $\mathcal{D}_{(\delta\psi,\psi)}$ are ϵ connected.

Now we define the ϵ-controlled L-groups. Let Y be a subset of X.

Definition. For $n \geq 0$ and $\epsilon \geq 0$, $L_n(X, Y, p_X, \epsilon)$ is defined to be the equivalence classes of n-dimensional ϵ connected ϵ quadratic complexes on p_X that are ϵ Poincaré over $X - Y$. The equivalence relation is generated by ϵ connected ϵ cobordisms that are ϵ Poincaré over $X - Y$. For $Y = \emptyset$ write

$$L_n(X, p_X, \epsilon) \;=\; L_n(X, \emptyset, p_X, \epsilon) \;.$$

Remarks. (1) We use only n-dimensional complexes and not the complexes chain equivalent to n-dimensional ones in order to make sure we have size control on some constructions.
(2) The ϵ connectedness condition is automatic for complexes that are ϵ Poincaré over X. Connectedness condition is used to insure that the *boundary* $\partial C = \Omega\mathcal{C}(\mathcal{D}_\psi)$ is chain equivalent to a positive one. There is a quadratic structure $\partial\psi$ for ∂C so that $(\partial C, \partial\psi)$ is Poincaré (Ranicki [2]).
(3) Using locally-finitely generated chain complexes on M, one can similarly define ϵ-controlled locally-finite L-groups $L^{lf}_n(X, Y, p_X, \epsilon)$. All the results in sections 1 – 3 are valid for locally-finite L-groups.

Proposition 1.1. *The direct sum*

$$(C, \psi) \oplus (C', \psi') \;=\; (C \oplus C', \psi \oplus \psi')$$

induces an abelian group structure on $L_n(X, Y, p_X, \epsilon)$. *Furthermore, if*

$$[C, \psi] \;=\; [C', \psi'] \in L_n(X, Y, p_X, \epsilon) \;,$$

then there is a 100ϵ *connected* 2ϵ *cobordism between* (C, ψ) *and* (C', ψ') *that is* 100ϵ *Poincaré over* $X - Y^{100\epsilon}$.

Next we study the functoriality. A *map between control maps* $p_X : M \to X$ and $p_Y : N \to Y$ means a pair of continuous maps $(f : M \to N,\ \bar{f} : X \to Y)$ which makes the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p_X"'] & N \arrow[d, "p_Y"] \\ X \arrow[r, "\bar{f}"'] & Y. \end{tikzcd}$$

For example, given a control map $p_Y : N \to Y$ and a subset $X \subset Y$, let us denote the control map $p_Y|p_Y^{-1}(X) : p_Y^{-1}(X) \to X$ by $p_X : M \to X$. Then the inclusion maps $j : M \to N$, $\bar{j} : X \to Y$ form a map form p_X to p_Y.

Epsilon controlled L-groups are functorial with respect to maps and relaxation of control in the following sense.

Proposition 1.2. *Let $F = (f, \bar{f})$ be a map from $p_X : M \to X$ to $p_Y : N \to Y$, and suppose that $\bar{f}$ is Lipschitz continuous with Lipschitz constant λ, i.e., there exists a constant $\lambda > 0$ such that*

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

Then F induces a homomorphism

$$F_* \ : \ L_n(X, X', p_X, \delta) \longrightarrow L_n(Y, Y', p_Y, \epsilon)$$

if $\epsilon \geq \lambda\delta$ and $\bar{f}(X') \subset Y'$. If two maps $F = (f, \bar{f})$ and $G = (g, \bar{g})$ are homotopic through maps $H_t = (h_t, \bar{h}_t)$ such that each $\bar{h}_t$ is Lipschitz continuous with Lipschitz constant λ, $\epsilon \geq \lambda\delta$, $\epsilon' > \epsilon$, and $\bar{h}_t(X') \subset Y'$, then the following two compositions are the same:

$$L_n(X, X', p_X, \delta) \xrightarrow{F_*} L_n(Y, Y', p_Y, \epsilon) \longrightarrow L_n(Y, Y', p_Y, \epsilon')$$
$$L_n(X, X', p_X, \delta) \xrightarrow{G_*} L_n(Y, Y', p_Y, \epsilon) \longrightarrow L_n(Y, Y', p_Y, \epsilon')$$

Proof: The direct image construction for geometric modules and morphisms [4, p.7] can be used to define the direct images $f_\#(C, \psi)$ of quadratic complexes and the direct images of cobordism. And this induces the desired F_*. The first part is obvious. For the second part, split the homotopy in small pieces to construct small cobordisms. The size of the cobordism may be slightly bigger than the size of the object itself. ☐

Remark. The above is stated for Lipschitz continuous maps to simplify the statement. For a specific δ and a specific ϵ, the following condition, instead of the Lipschitz condition above, is sufficient for the existence of F_* :

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq k\epsilon \quad \text{whenever} \quad d(x_1, x_2) \leq k\delta,$$

for a certain finite set of integers k (more precisely, for $k = 1, 3, 4, 8$)

and similarly for the isomorphism in the second part. When X is compact and ϵ is given, the continuity of $\bar{f}$ implies that this condition is satisfied for sufficiently small δ's. [Use the continuity of the distance function $d : X \times X \to \mathbb{R}$ and the compactness of the diagonal set $\Delta \subset X \times X$.] And, in the second half of the proposition, there are cases when the equality $F_* = G_*$ holds without composing with the relax-control map; *e.g.,* see 4.1.

We are interested in the "limit" of ϵ-controlled L-groups.

Definition. Let $p_X : M \to X$ be a control map.

(1) Let ϵ, δ be positive numbers such that $\delta \leq \epsilon$. We define:

$$L_n^\epsilon(X, p_X, \delta) = \mathrm{im}\{L_n(X, p_X, \delta) \longrightarrow L_n(X, p_X, \epsilon)\}.$$

(2) For $\epsilon > 0$, we define the *stable ϵ-controlled L-group of X with coefficient p_X* by:

$$L_n^\epsilon(X; p_X) \ := \bigcap_{0<\delta<\epsilon} L_n^\epsilon(X, p_X, \delta).$$

(3) The *controlled L-group with coefficient p_X* is defined by:

$$L_n^c(X; p_X) \ := \ \varprojlim_{\epsilon} L_n^\epsilon(X; p_X),$$

where the limit is taken with respect to the obvious relax-control maps:

$$L_n^{\epsilon'}(X; p_X) \longrightarrow L_n^\epsilon(X; p_X), \qquad (\epsilon' < \epsilon).$$

In section 5, we study a certain stability result for the controlled L-groups in some special case.

2. Epsilon-controlled projective L-groups.

Fix a subset Y of X, and let $\mathcal{F}$ be a family of subsets of X such that $Z \supset Y$ for each $Z \in \mathcal{F}$. In this section we introduce intermediate ϵ-controlled L-groups $L_n^{\mathcal{F}}(Y, p_X, \epsilon)$, which will appear in the stable-exact sequence of a pair and also in the Mayer-Vietoris sequence. Roughly speaking, these are defined using "controlled projective quadratic chain complexes" $((C,p),\psi)$ with vanishing ϵ-controlled reduced projective class $[C,p] = 0 \in \widetilde{K}_0(Z, p_Z, n, \epsilon)$ (Ranicki and Yamasaki [4]) for each $Z \in \mathcal{F}$. Here p_Z denotes the restriction $p_X|p_X^{-1}(Z) : p_X^{-1}(Z) \to Z$ of p_X as in the previous section.

For a projective module (A,p) on p_X, its dual $(A,p)^*$ is the projective module (A^*, p^*) on p_X. If $f : (A,p) \to (B,q)$ is an ϵ morphism ([4]), then $f^* : (B,q)^* \to (A,p)^*$ is also an ϵ morphism. For an ϵ projective chain complex on p_X

$$(C,p) \ : \ \ldots \longrightarrow (C_r, p_r) \xrightarrow{d_r} (C_{r-1}, p_{r-1}) \xrightarrow{d_{r-1}} \ldots$$

in the sense of [4], $(C,p)^{n-*}$ will denote the ϵ projective chain complex on p_X defined by:

$$\cdots \longrightarrow (C^{n-r}, p^*_{n-r}) \xrightarrow{(-)^r d^*_{n-r+1}} (C^{n-r+1}, p^*_{n-r+1}) \longrightarrow \cdots .$$

An *n-dimensional ϵ quadratic structure* on a projective chain complex (C,p) on p_X is an n-dimensional ϵ quadratic structure ψ on C (in the sense of §1) such that $\psi_s : (C^{n-r-s}, p^*) \to (C_r, p)$ is an ϵ morphism for every $s \geq 0$ and $r \in \mathbb{Z}$. Similarly, an $(n+1)$*-dimensional ϵ quadratic structure* on a chain map $f : (C,p) \to (D,q)$ is an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : C \to D$ such that $\delta\psi_s : (D^{n+1-r-s}, q^*) \to (D_r, q)$ and $\psi_s : (C^{n-r-s}, p^*) \to (C_r, p)$ are ϵ morphisms for every $s \geq 0$ and $r \in \mathbb{Z}$. An n-dimensional ϵ projective chain complex (C,p) on p_X equipped with an n-dimensional ϵ quadratic structure is called an *n-dimensional ϵ projective quadratic complex on p_X*, and an ϵ chain map $f : (C,p) \to (D,q)$ between an n-dimensional ϵ projective chain complex (C,p) on p_X and an $(n+1)$-dimensional ϵ projective chain complex (D,q) on p_X equipped with an $(n+1)$-dimensional ϵ quadratic structure is called an *$(n+1)$-dimensional ϵ projective quadratic pair on p_X*.

An *ϵ cobordism* of n-dimensional ϵ projective quadratic complexes $((C,p),\psi)$, $((C',p'),\psi')$ on p_X is an $(n+1)$-dimensional ϵ projective quadratic pair on p_X

$$((\, f \quad f' \,) : (C,p) \oplus (C',p') \longrightarrow (D,q), (\delta\psi, \psi \oplus -\psi'))$$

with boundary $((C,p) \oplus (C',p'), \psi \oplus -\psi')$.

An n-dimensional ϵ quadratic structure ψ on (C,p) is *ϵ Poincaré* if

$$\partial(C,p) = \Omega\mathcal{C}((1+T)\psi_0 : (C^{n-*}, p^*) \longrightarrow (C,p))$$

is 4ϵ contractible. $((C,p),\psi)$ is *ϵ Poincaré* if ψ is ϵ Poincaré. Similarly, an $(n+1)$-dimensional ϵ quadratic structure $(\delta\psi,\psi)$ on $f : (C,p) \to (D,q)$ is *ϵ Poincaré* if $\partial(C,p)$ and

$$\partial(D,q) = \Omega\mathcal{C}(((1+T)\delta\psi_0 \quad f(1+T)\psi_0) : \mathcal{C}(f)^{n+1-*} \longrightarrow (D,q))$$

are both 4ϵ contractible. A pair $(f,(\delta\psi,\psi))$ is *ϵ Poincaré* if $(\delta\psi,\psi)$ is ϵ Poincaré.

Let Y and be a subset of X and $\mathcal{F}$ be a family of subsets of X such that $Z \supset Y$ for every $Z \in \mathcal{F}$.

Definition. Let $n \geq 0$ and $\epsilon \geq 0$. $L_n^{\mathcal{F}}(Y, p_X, \epsilon)$ is the equivalence classes of n-dimensional ϵ Poincaré ϵ projective quadratic complexes $((C,p),\psi)$ on p_Y such that $[C,p] = 0$ in $\widetilde{K}_0(Z, p_Z, n, \epsilon)$ for each $Z \in \mathcal{F}$. The equivalence relation is generated by ϵ Poincaré ϵ cobordisms $((\, f \quad f' \,) : (C,p) \oplus (C',p') \to (D,q), (\delta\psi, \psi \oplus -\psi'))$ on p_Y such that $[D,q] = 0$ in $\widetilde{K}_0(Z, p_Z, n+1, \epsilon)$ for each $Z \in \mathcal{F}$. When $\mathcal{F} = \{X\}$, we omit the braces and write $L_n^X(Y, p_X, \epsilon)$ instead of $L_n^{\{X\}}(Y, p_X, \epsilon)$. When $\mathcal{F} = \{\ \}$, then we use the notation $L_n^p(Y, p_Y, \epsilon)$, since it depends only on p_Y.

Proposition 2.1. *Direct sum induces an abelian group structure on* $L_n^{\mathcal{F}}(Y, p_X, \epsilon)$. *Furthermore, if*

$$[(C,p),\psi] = [(C',p'),\psi'] \in L_n^{\mathcal{F}}(Y, p_X, \epsilon) ,$$

then there is a 100ϵ *Poincaré* 2ϵ *cobordism on* p_Y

$$((f \quad f') : (C,p) \oplus (C',p') \to (D,q), (\delta\psi, \psi \oplus -\psi'))$$

such that $[D,q] = 0$ *in* $\widetilde{K}_0(Z, p_Z, n+1, 9\epsilon)$ *for each* $Z \in \mathcal{F}$.

A functoriality with respect to maps and relaxation of control similar to 1.2 holds for epsilon controlled projective L-groups.

Proposition 2.2. *Let* $F = (f, \bar{f})$ *be a map from* $p_X : M \to X$ *to* $p_Y : N \to Y$, *and suppose that* $\bar{f}$ *is Lipschitz continuous with Lipschitz constant* λ, *i.e., there exists a constant* $\lambda > 0$ *such that*

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq \lambda d(x_1, x_2) \qquad (x_1, x_2 \in X).$$

If $\epsilon \geq \lambda\delta$, $\bar{f}(A) \subset B$, *and there exists a* $Z \in \mathcal{F}$ *satisfying* $\bar{f}(Z) \subset Z'$ *for each* $Z' \in \mathcal{F}'$, *then* F *induces a homomorphism*

$$F_* \ : \ L_n^{\mathcal{F}}(A, p_X, \delta) \ \longrightarrow \ L_n^{\mathcal{F}'}(B, p_Y, \epsilon).$$

Remark. As in the remark to 1.2, for a specific δ and a ϵ, we do not need the full Lipschitz condition to guarantee the existence of F_*.

There is an obvious homomorphism

$$\iota_\epsilon \ : \ L_n(Y, p_Y, \epsilon) \longrightarrow L_n^{\mathcal{F}}(Y, p_X, \epsilon); \qquad [C, \psi] \mapsto [(C, 1), \psi].$$

On the other hand, the controlled K-theoretic condition posed in the definition can be used to construct a homomorphism from a projective L-group to a free L-group:

Proposition 2.3. *There exist a constant* $\alpha > 1$ *such that the following holds true: for any control map* $p_X : M \to X$, *any subset* $Y \subset X$, *any family of subsets* $\mathcal{F}$ *of* X *containing* Y , *any element* Z *of* $\mathcal{F}$, *any number* $n \geq 0$, *and any positive numbers* δ, ϵ *such that* $\epsilon \geq \alpha\delta$, *there is a well-defined homomorphism functorial with respect to relaxation of control:*

$$(i_Z)_* \ : \ L_n^{\mathcal{F}}(Y, p_X, \delta) \longrightarrow L_n(Z, p_Z, \epsilon)$$

such that the following compositions are equal to the maps induced from inclusion maps:

$$L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{(i_Z)_*} L_n(Z, p_Z, \epsilon) \xrightarrow{\iota_\epsilon} L_n^{\{Z\}}(Z, p_Z, \epsilon) ,$$

$$L_n(Y, p_Y, \delta) \xrightarrow{\iota_\delta} L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{(i_Z)_*} L_n(Z, p_Z, \epsilon) .$$

Remark. Actually $\alpha = 30000$ works.

3. Stably-exact sequences.

In this section we describe two 'stably-exact' sequences. The first is the stably-exact sequence of a pair:

$$\overset{\partial}{\cdots\triangleright} L_n^X(Y_n,p_X,\epsilon) \overset{i_*}{\cdots\triangleright} L_n(X,p_X,\epsilon) \xrightarrow{j_*} L_n(X,Y,p_X,\epsilon) \overset{\partial}{\cdots\triangleright} L_{n-1}^X(Y,p_X,\epsilon) \overset{i_*}{\cdots\triangleright}$$

where the dotted arrows are only 'stably' defined. The precise meaning will be explained below. The second is the Mayer-Vietoris-type stably-exact sequence:

$$\overset{\partial}{\cdots\triangleright} L_n^{\mathcal{F}}(C,p_X,\epsilon) \overset{i_*}{\cdots\triangleright} L_n(A,p_A,\epsilon)\oplus L_n(B,p_B,\epsilon) \xrightarrow{j_*} L_n(X,p_X,\epsilon)$$
$$\overset{\partial}{\cdots\triangleright} L_{n-1}^{\mathcal{F}}(C,p_X,\epsilon) \overset{i_*}{\cdots\triangleright}$$

where $X = A\cup B$, $C = A\cap B$, and $\mathcal{F} = \{A,B\}$.

Fix an integer $n \geq 0$, let Y_n, Z_n be subsets of X, and let γ_n, δ_n, ϵ_n be three positive numbers satisfying

$$\epsilon_n \geq \delta_n, \qquad \delta_n \geq \alpha\gamma_n$$

where α is the number (> 1) posited in 2.3. Then there is a sequence

$$L_n^X(Y_n,p_X,\gamma_n) \xrightarrow{i_*=(i_X)_*} L_n(X,p_X,\delta_n) \xrightarrow{j_*} L_n(X,Z_n,p_X,\epsilon_n),$$

where i_* is the homomorphism given in 2.3 and j_* is the homomorphism induced by the inclusion map and relaxation of control. (The subscripts are there just to remind the reader of the degrees of the relevant L-groups.)

Theorem 3.1. *There exist constants κ_0, κ_1, κ_2, $\ldots (> 1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$, $Z_n \supset Y_n^{\kappa_n\delta_n}$, and $\epsilon_n \geq \kappa_n\delta_n$, then the following composition j_*i_* is zero:*

$$j_*i_* = 0 \ : \ L_n^X(Y_n,p_X,\gamma_n) \xrightarrow{i_*} L_n(X,p_X,\delta_n) \xrightarrow{j_*} L_n(X,Z_n,p_X,\epsilon_n),$$

(2) *if $n \geq 1$, $Y_{n-1} \supset Z_n^{\kappa_n\epsilon_n}$ and $\gamma_{n-1} \geq \kappa_n\epsilon_n$, then there is a connecting homomorphism*

$$\partial \ : \ L_n(X,Z_n,p_X,\epsilon_n) \longrightarrow L_{n-1}^X(Y_{n-1},p_X,\gamma_{n-1}),$$

such that the following composition ∂j_* is zero:

$$\partial j_* = 0 \ : \ L_n(X,p_X,\delta_n) \xrightarrow{j_*} L_n(X,Z_n,p_X,\epsilon_n) \xrightarrow{\partial} L_{n-1}^X(Y_{n-1},p_X,\gamma_{n-1}),$$

and, if $\delta_{n-1} \geq \alpha\gamma_{n-1}$ (so that the homomorphism i_* is well-defined), the following composition $i_*\partial$ is zero:

$$i_*\partial = 0 \ : \ L_n(X,Z_n,p_X,\epsilon_n) \xrightarrow{\partial} L_{n-1}^X(Y_{n-1},p_X,\gamma_{n-1}) \xrightarrow{i_*} L_{n-1}(X,p_X,\delta_{n-1}).$$

Theorem 3.2. *There exist constants λ_0, λ_1, λ_2, $\ldots (> 1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$, $\delta_n \geq \alpha\gamma_n$ (so that i_* is well-defined), $\epsilon'_{n+1} \geq \lambda_n\delta_n$, $Z'_{n+1} \supset Y_n^{\lambda_n\delta_n}$, $Y'_n \supset {Z'_{n+1}}^{\kappa_{n+1}\epsilon'_{n+1}}$ and $\gamma'_n \geq \kappa_{n+1}\epsilon'_{n+1}$ (so that ∂' is well-defined), then the image of the kernel of i_* in $L_n^X(Y'_n, p_X, \gamma'_n)$ is in the image of ∂':*

$$\begin{tikzcd} & L_n^X(Y_n, p_X, \gamma_n) \arrow[r, "i_*"] \arrow[d] & L_n(X, p_X, \delta_n) \\ L_{n+1}(X, Z'_{n+1}, p_X, \epsilon'_{n+1}) \arrow[r, "\partial'"] & L_n^X(Y'_n, p_X, \gamma'_n) & \end{tikzcd}$$

(2) *if $n \geq 0$, $\epsilon_n \geq \delta_n$ (so that j_* is well-defined), $Y'_n \supset Z_n^{\lambda_n\epsilon_n}$, $\gamma'_n \geq \lambda_n\epsilon_n$, and $\delta'_n \geq \alpha\gamma'_n$ (so that i'_* is well-defined), then the image of the kernel of j_* in $L_n(X, p_X, \delta'_n)$ is in the image of i'_*:*

$$\begin{tikzcd} & L_n(X, p_X, \delta_n) \arrow[r, "j_*"] \arrow[d] & L_n(X, Z_n, p_X, \epsilon_n) \\ L_n^X(Y'_n, p_X, \gamma'_n) \arrow[r, "i'_*"] & L_n(X, p_X, \delta'_n) & \end{tikzcd}$$

(3) *if $n \geq 1$, $\gamma_{n-1} \geq \kappa_n\epsilon_n$ (so that ∂ is well-defined), $\epsilon'_n \geq \lambda_n\gamma_{n-1}$, and $Z'_n \supset Y_{n-1}^{\lambda_n\gamma_{n-1}}$, then the image of the kernel of ∂ in $L_n(X, Z'_n, p_X, \epsilon'_n)$ is in the image of j'_*:*

$$\begin{tikzcd} & L_n(X, Z_n, p_X, \epsilon_n) \arrow[r, "\partial"] \arrow[d] & L_{n-1}^X(Y_{n-1}, p_X, \gamma_{n-1}) \\ L_n(X, p_X, \epsilon'_n) \arrow[r, "j'_*"] & L_n(X, Z'_n, p_X, \epsilon'_n) & \end{tikzcd}$$

Here the vertical maps are the homomorphisms induced by inclusion maps and relaxation of control.

Next we investigate the Mayer-Vietoris-type stably-exact sequence. Fix an integer $n \geq 0$, and assume that X is the union of two closed subsets A_n and B_n with intersection $C_n = A_n \cap B_n$. Suppose three positive numbers γ_n, δ_n, ϵ_n satisfy

$$\delta_n \geq \alpha\gamma_n, \qquad \epsilon_n \geq \delta_n,$$

and define a family $\mathcal{F}_n$ to be $\{A_n, B_n\}$. Then we have a sequence

$$L_n^{\mathcal{F}_n}(C_n, p_X, \gamma_n) \xrightarrow{i_*} L_n(A_n, p_{A_n}, \delta_n) \oplus L_n(B_n, p_{B_n}, \delta_n) \xrightarrow{j_*} L_n(X, p_X, \epsilon_n).$$

Theorem 3.3. *There exist constants κ_0, κ_1, κ_2, $\ldots (> 1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$ and $\epsilon_n \geq \kappa_n \delta_n$, then the following composition $j_* i_*$ is zero:*

$$L_n^{\mathcal{F}_n}(C_n, p_X, \gamma_n) \xrightarrow{i_*} L_n(A_n, p_{A_n}, \delta_n) \oplus L_n(B_n, p_{B_n}, \delta_n) \xrightarrow{j_*} L_n(X, p_X, \epsilon_n).$$

(2) *if $n \geq 1$, $C_{n-1} \supset C_n^{\kappa_n \epsilon_n}$, $\gamma_{n-1} \geq \kappa_n \epsilon_n$, and if we set*

$$\mathcal{F}_{n-1} = \{A_{n-1} = A_n \cup C_{n-1}, B_{n-1} = B_n \cup C_{n-1}\},$$

then there is a connecting homomorphism

$$\partial \ : \ L_n(X, p_X, \epsilon_n) \longrightarrow L_{n-1}^{\mathcal{F}_{n-1}}(C_{n-1}, p_X, \gamma_{n-1}),$$

such that the following composition ∂j_* is zero:

$$L_n(A_n, p, \delta_n) \oplus L_n(B_n, p, \delta_n) \xrightarrow{j_*} L_n(X, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^{\mathcal{F}_{n-1}}(C_{n-1}, p_X, \gamma_{n-1}),$$

and, if $\delta_{n-1} \geq \alpha\gamma_{n-1}$ (so that the homomorphism i_* is well-defined), the following composition $i_* \partial$ is zero:

$$\begin{aligned} L_n(X, p_X, \epsilon_n) \xrightarrow{\partial} L_{n-1}^{\mathcal{F}_{n-1}}&(C_{n-1}, p_X, \gamma_{n-1}) \\ &\xrightarrow{i_*} L_{n-1}(A_{n-1}, p, \delta_{n-1}) \oplus L_{n-1}(B_{n-1}, p, \delta_{n-1}). \end{aligned}$$

Theorem 3.4. *There exist constants λ_0, λ_1, λ_2, $\ldots (> 1)$ which do not depend on p_X such that*

(1) *if $n \geq 0$, $\delta_n \geq \alpha\gamma_n$ (so that i_* is well-defined), $\epsilon'_{n+1} \geq \lambda_n \delta_n$, $C'_n \supset C_n^{\lambda_n \delta_n}$, $\gamma'_n \geq \kappa_{n+1}\epsilon'_{n+1}$ (so that ∂' is well-defined), then the image of the kernel of i_* in $L_{n-1}^{\mathcal{F}'_{n-1}}(C'_{n-1}, p_X, \gamma'_{n-1})$ is in the image of ∂':*

$$\begin{CD} @. L_n^{\mathcal{F}_n}(C_n, p_X, \gamma_n) @>{i_*}>> L_n(A_n, p, \delta_n) \oplus L_n(B_n, p, \delta_n) \\ @. @VVV \\ L_{n+1}(X, p_X, \epsilon'_{n+1}) @>{\partial'}>> L_n^{\mathcal{F}'_n}(C'_n, p_X, \gamma'_n) \end{CD}$$

(2) *if $n \geq 0$, $\epsilon_n \geq \delta_n$ (so that j_* is well-defined), $C'_n \supset C_n^{\lambda_n \epsilon_n}$, $\gamma'_n \geq \lambda_n \epsilon_n$, $\delta'_n \geq \alpha\gamma'_n$ (so that i'_* is well-defined), and $\mathcal{F}'_n = \{A'_n = A_n \cup C'_n, B'_n = B_n \cup C'_n\}$, then the image of the kernel of j_* in $L_n(A'_n, p, \delta'_n) \oplus L_n(B'_n, p, \delta'_n)$ is in the image of i'_*:*

$$\begin{CD} @. L_n(A_n, p, \delta_n) \oplus L_n(B_n, p, \delta_n) @>{j_*}>> L_n(X, p_X, \epsilon_n) \\ @. @VVV \\ L_n^{\mathcal{F}'_n}(C'_n, p_X, \gamma'_n) @>{i'_*}>> L_n(A'_n, p, \delta'_n) \oplus L_n(B'_n, p, \delta'_n) \end{CD}$$

(3) *if* $n \geq 1$, $C_{n-1} \supset C_n^{\kappa_n \epsilon_n}$, $\gamma_{n-1} \geq \kappa_n \epsilon_n$ *(so that* ∂ *is well-defined),* $\epsilon'_n \geq \lambda_n \gamma_{n-1}$, $C'_n \supset C_{n-1}^{\lambda_n \gamma_{n-1}}$, $A'_n = A_n \cup C'_n$, *and* $B'_n = B_n \cup C'_n$, *then the image of the kernel of* ∂ *in* $L_n(X, p_X, \epsilon'_n)$ *is in the image of* j'_**:*

$$\begin{tikzcd} & L_n(X, p_X, \epsilon_n) \arrow[r, "\partial"] \arrow[d] & L_{n-1}^{\mathcal{F}_{n-1}}(C_{n-1}, p_X, \gamma_{n-1}) \\ L_n(A'_n, p, \epsilon'_n) \oplus L_n(B'_n, p, \epsilon'_n) \arrow[r, "j'_*"] & L_n(X, p_X, \epsilon'_n) & \end{tikzcd}$$

Here the vertical maps are the homomorphisms induced by inclusion maps and relaxation of control.

Theorems 3.1 – 3.4 are all straightforward to prove.

4. Locally-finite analogues.

Up to this point, we considered only finitely generated modules and chain complexes. To study the behaviour of controlled L-groups, we need to use infinitely generated objects; such objects arise naturally when we take the pullback of a finitely generated object via an infinite-sheeted covering map.

Consider a control map $p_X : M \to X$, and take the product with another metric space N:

$$p_X \times 1_N \ : \ M \times N \longrightarrow X \times N.$$

Here we use the maximum metric on the product $X \times N$.

Definition. (Ranicki and Yamasaki [4, p.14]) A geometric module on the product space $M \times N$ is said to be *M-finite* if, for any $y \in N$, there is a neighbourhood U of y in N such that $M \times U$ contains only finitely many basis elements; a projective module (A, p) on $M \times N$ is said to be *M-finite* if A is M-finite; a projective chain complex (C, p) on $M \times N$ is *M-finite* if each (C_r, p_r) is M-finite. [In [4], we used the terminology "*M-locally finite*", but this does not sound right and we decided to use "*M-finite*" instead. "*N-locally M-finite*" may be describing the meaning better, but it is too long.] When M is compact, M-finiteness is equivalent to the ordinary locally-finiteness.

Definition. Using this notion, one can define *M-finite ϵ-controlled L-groups* $L_n^M(X \times N, Y \times N, p_X \times 1_N, \epsilon)$, and *$M$-finite ϵ-controlled projective L-groups* $L_n^{M,\mathcal{F}}(Y \times N, p_X \times 1_N, \epsilon)$ by requiring that every chain complexes concerned are M-finite.

Consider the case when $N = \mathbb{R}$. We would like to apply the M-finite version of the Mayer-Vietoris-type stable exact sequence with respect to the splitting $\mathbb{R} = (-\infty, 0] \cup [1, \infty)$. The following says that one of the three terms in the sequence vanishes.

Proposition 4.1. Let $p_X : M \to X$ be a control map. For any $\epsilon > 0$ and $r \in \mathbb{R}$,

$$L_n^M(X \times (\infty, r], p_X \times 1, \epsilon) = L_n^M(X \times [r, \infty), p_X \times 1, \epsilon) = 0.$$
$$\widetilde{K}_0^M(X \times (\infty, r], p_X \times 1, n, \epsilon) = \widetilde{K}_0^M(X \times [r, \infty), p_X \times 1, n, \epsilon) = 0.$$

Proof: This is done using repeated shifts towards infinity and the 'Eilenberg Swindle'. Let us consider the case of $L_n^M(X \times [r, \infty), p_X \times 1, \epsilon)$. Let $J = [r, \infty)$ and define $T : M \times J \to M \times J$ by $T(x, t) = (x, t+\epsilon)$. Take an element $[c] \in L_n^M(X \times J, p_X \times 1, \epsilon)$. It is zero, because there exist M-finite ϵ Poincaré cobordisms:

$$\begin{aligned} c \ &\sim \ c \oplus (T_\#(-c) \oplus T_\#^2(c)) \oplus (T_\#^3(-c) \oplus T_\#^4(c)) \oplus \ldots \\ &= \ (c \oplus T_\#(-c)) \oplus (T_\#^2(c) \oplus T_\#^3(-c)) \oplus \ldots \ \sim \ 0 \ . \end{aligned}$$

$\square$

Thus, the Mayer-Vietoris stably-exact sequence reduces to:

$$0 \longrightarrow L_n^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon) \xrightarrow{\partial} L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \longrightarrow 0,$$

where $\gamma = \kappa_n \epsilon$, $I = [-\delta, \delta]$, for some $\delta > 0$. A diagram chase shows that there exists a well-defined homomorphism:

$$\beta : L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \longrightarrow L_n^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon'),$$

where $\epsilon'' = \lambda_n \kappa_n \lambda_{n-1} \alpha \gamma$. The homomorphisms ∂ and β are stable inverses of each other; the compositions

$$\begin{aligned} \beta\partial \ &: \ L_n^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon) \longrightarrow L_n^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon') \\ \partial\beta \ &: \ L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \longrightarrow L_{n-1}^p(X \times I, p_X \times 1_I, \kappa_n \epsilon') \end{aligned}$$

are both relax-control maps.

Note that, for any γ, a projective L-group analogue of 1.2 gives an isomorphism:

$$L_{n-1}^p(X \times I, p_X \times 1_I, \gamma) \ \cong \ L_{n-1}^p(X \times \{0\}, p_X, \gamma).$$

In this case, no composition with relax-control map is necessary, because $X \times I$ is given the maximum metric. Thus, we have obtained:

Theorem 4.2. *There is a stable isomorphism:*

$$L_n^M(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon) \longrightarrow L_{n-1}^p(X, p_X, \gamma).$$

Similarly, we have:

Theorem 4.3. *There is a stable isomorphism:*

$$L_n^{lf}(X \times \mathbb{R}, p_X \times 1_{\mathbb{R}}, \epsilon) \longrightarrow L_{n-1}^{p,lf}(X, p_X, \gamma).$$

5. Stability in a special case.

In this section we treat the special case when the control map is the identity map. The following can be used to replace the controlled projective L-group terms in the previous section by controlled L-groups.

Proposition 5.1. *Suppose that $Y(\subset X)$ is a compact polyhedron or a compact metric ANR embedded in the Hilbert cube and that p_Y is the identity map 1_Y on Y. Then for any $\epsilon > 0$ and n, there exists a $\delta_0 > 0$ such that for any positive number δ satisfying $\delta \leq \delta_0$ there is a well-defined homomorphism functorial with respect to relaxation of control:*

$$\tau_{\epsilon,\delta} \ : \ L_n^{\mathcal{F}}(Y, p_X, \delta) \ \longrightarrow \ L_n(Y, 1_Y, \epsilon)$$

such that the compositions

$$L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{\tau_{\epsilon,\delta}} L_n(Y, 1_Y, \epsilon) \xrightarrow{\iota_\epsilon} L_n^{\mathcal{F}}(Y, p_X, \epsilon)$$
$$L_n(Y, 1_Y, \delta) \xrightarrow{\iota_\delta} L_n^{\mathcal{F}}(Y, p_X, \delta) \xrightarrow{\tau_{\epsilon,\delta}} L_n(Y, 1_Y, \epsilon)$$

are both relax-control maps. In particular $L_n^p(Y, 1_Y, \delta)$ and $L_n(Y, 1_Y, \epsilon)$ are stably isomorphic.

Proof: Let $\delta_1 = \epsilon/\alpha$, where α is the positive number posited in 2.3. By 8.2 and 8.3 of [4], there exists a $\delta_0 > 0$ such that the following map is a zero map:

$$\widetilde{K}_0(Y, 1_Y, n, \delta_0) \longrightarrow \widetilde{K}_0(Y, 1_Y, n, \delta_1); \qquad [C, p] \mapsto [C, p].$$

Therefore, if $\delta \leq \delta_0$, there is a homomorphism

$$L_n^{\mathcal{F}}(Y, p_X, \delta) \longrightarrow L_n^{\mathcal{F}\cup\{Y\}}(Y, p_X, \delta_1); \qquad [(C, p), \psi] \mapsto [(C, p), \psi].$$

The desired map $\tau_{\epsilon,\delta}$ is obtained by composing this with the map

$$(i_Y)_* \ : \ L_n^{\mathcal{F}\cup\{Y\}}(Y, p_X, \delta) \longrightarrow L_n(Y, 1_X, \epsilon)$$

corresponding to the subspace Y. $\square$

Remark. If Y is a compact polyhedron, then there is a constant $\kappa_n^Y > 1$ which depends on n and Y such that δ_0 above can be taken to be δ_1/κ_n^Y. For this we need to change the statement and the proof of 8.1 of [4] like those of 5.4 below.

Recall that in our Mayer-Vietoris-type stably-exact sequence, each piece of space tends to get bigger in the process. The following can be used to remedy this in certain cases. (It is stated here for the identity control map, but there is an obvious extension to general control maps.)

Proposition 5.2. Let $r : X \to A$ be a strong deformation retraction, with a Lipschitz continuous strong deformation of Lipschitz constant λ, and $i : A \to X$ be the inclusion map. Then r and i induce "stable" isomorphisms of controlled L-groups in the following sense: if $\epsilon > 0$, then for any δ ($0 < \delta \leq \epsilon/\lambda$) the compositions

$$L_n(X, 1_X, \delta) \xrightarrow{r_*} L_n(A, 1_A, \epsilon) \xrightarrow{i_*} L_n(X, 1_X, \epsilon)$$
$$L_n(A, 1_A, \delta) \xrightarrow{i_*} L_n(X, 1_X, \delta) \xrightarrow{r_*} L_n(A, 1_A, \epsilon)$$

are relax-control maps.

Proof: Obvious from 1.2. □

Theorem 5.3. *Fix a compact polyhedron X and an integer $n \geq 0$. Then there exist numbers $\epsilon_1 > 0$, $\kappa \geq 1$ and $\lambda \geq 1$ (which depend on n, X, and the triangulation) such that, for any subpolyhedrons A and B of X, any integer $k \geq 0$, and any number $0 < \epsilon \leq \epsilon_1$, there exists a ladder:*

$$\begin{array}{ccccc} L_n^{lf}(C \times \mathbb{R}^k, 1, \epsilon) & \xrightarrow{i_*} & L_n^{lf}(A \times \mathbb{R}^k, 1, \epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^k, 1, \epsilon) & \xrightarrow{j_*} & L_n^{lf}(K \times \mathbb{R}^k, 1, \epsilon) \\ \downarrow & & \downarrow & & \downarrow \\ L_n^{lf}(C \times \mathbb{R}^k, 1, \lambda\epsilon) & \xrightarrow[i_*]{} & L_n^{lf}(A \times \mathbb{R}^k, 1, \lambda\epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^k, 1, \lambda\epsilon) & \xrightarrow[j_*]{} & L_n^{lf}(K \times \mathbb{R}^k, 1, \lambda\epsilon) \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\partial} & L_n^{lf}(C \times \mathbb{R}^{k+1}, 1, \kappa\epsilon) & \xrightarrow{i_*} & L_n^{lf}(A \times \mathbb{R}^{k+1}, 1, \kappa\epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^{k+1}, 1, \kappa\epsilon) \\ & \downarrow & & \downarrow \\ \xrightarrow[\partial]{} & L_n^{lf}(C \times \mathbb{R}^{k+1}, 1, \kappa\lambda\epsilon) & \xrightarrow[i_*]{} & L_n^{lf}(A \times \mathbb{R}^{k+1}, 1, \kappa\lambda\epsilon) \oplus L_n^{lf}(B \times \mathbb{R}^{k+1}, 1, \kappa\lambda\epsilon) \end{array}$$

which is stably-exact in the sense that

(1) *the image of a horizontal map is contained in the kernel of the next map, and*
(2) *the relax-control image in the second row of the kernel of a map in the first row is contained in the image of a horizontal map from the left,*

where $C = A \cap B$ and $K = A \cup B$, and the vertical maps are relax-control maps.

Proof: This is obtained from the locally-finite versions of 3.3, 3.4 combined with 4.3, 5.1, and 5.2 (the strong deformations of the neighbourhoods of A and B in K can be chosen to be PL and hence Lipschitz). Since there are only finitely many subpolyhedrons of X (with a fixed triangulation), we may choose constants κ and λ independent of A and B. □

Theorem 5.4. *Suppose X is a compact polyhedron and $n \geq 0$ is an integer. Then there exist numbers $\epsilon_0 > 0$ and $0 < \mu_0 \leq 1$ which depend on X and n such that*

$$L_n^{\epsilon}(X, 1_X, \delta) = L_n^c(X; 1_X)$$

for every $\epsilon \leq \epsilon_0$ and every $\delta \leq \mu_0 \epsilon$.

Proof: We inductively construct sequences of positive numbers

$$\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \dots \ (> 0)$$
$$(1 \geq)\ \mu_1 \ \geq \ \mu_2 \ \geq \ \mu_3 \ \geq \ \dots \ (> 0)$$

such that for any subcomplex K of X with the number of simplices $\leq l$,

(1) if $0 < \epsilon \leq \epsilon_l$, $0 < \delta \leq \mu_l \epsilon$, and $k \geq 0$, then

$$L_n^{lf,\epsilon}(K \times \mathbb{R}^k, 1_K \times 1, \delta) \ = \ L_n^{lf,\epsilon}(K \times \mathbb{R}^k, 1_K \times 1, \mu_l \epsilon),$$

and

(2) if $0 < \epsilon \leq \epsilon_l$, then the homomorphism

$$L_n^{lf,\epsilon}(K \times \mathbb{R}^k; 1_K \times 1) \longrightarrow L_n^{lf,\epsilon_l}(K \times \mathbb{R}^k; 1_K \times 1)$$

is injective.

Here $\mathbb{R}^k$ is given the maximum metric.

First suppose $l = 1$ (*i.e.* K is a single point). Any object with bounded control on $\mathbb{R}^k$ can be squeezed to obtain an arbitrarily small control; therefore,

$$\epsilon_1 \ = \ \text{the number posited in 5.2}, \qquad \mu_1 \ = \ 1$$

works.

Next assume we have constructed ϵ_i and μ_i for $i \leq l$. We claim that

$$\epsilon_{l+1} \ = \ \min\{\frac{\mu_l}{\lambda}, \frac{1}{\kappa}\}\epsilon_l, \qquad \mu_{l+1} \ = \ \frac{\mu_l^2}{\lambda\kappa}$$

satisfy the required condition. Suppose the number of simplices of K is less than or equal to $l+1$. Choose a simplex of K of the highest dimension, and call the simplex (viewed as a subpolyhedron) A, and let $B = K - \mathrm{int}A$. Suppose $0 < \epsilon \leq \epsilon_{l+1}$ and $0 < \delta \leq \mu_l \epsilon$. A diagram chase starting from an element of

$$L_n^{lf}(K \times \mathbb{R}^k, 1, \mu_{l+1}\epsilon)$$

in the following diagram establishes the property (1). Here the entries in each of the columns are

$$L_n^{lf}(A \times \mathbb{R}^k, 1, \gamma) \oplus L_n^{lf}(B \times \mathbb{R}^k, 1, \gamma)\ , \quad L_n^{lf}(C \times \mathbb{R}^k, 1, \gamma)$$
$$L_n^{lf}(K \times \mathbb{R}^k, 1, \gamma)\ , \quad \text{and } L_n^{lf}(A \times \mathbb{R}^{k+1}, 1, \gamma) \oplus L_n^{lf}(B \times \mathbb{R}^{k+1}, 1, \gamma)\ ,$$

for various γ's specified in the diagram.

$$\begin{tikzcd}[row sep=small]
L_n^{lf}(A\ldots)\oplus L_n^{lf}(B\ldots) & L_n^{lf}(K\ldots) & L_n^{lf}(C\ldots) & L_n^{lf}(A\ldots)\oplus L_n^{lf}(B\ldots) \\
 & & \mu_l\frac{\delta}{\lambda} \arrow[r] \arrow[d] & \mu_l\frac{\delta}{\lambda} \arrow[d] \\
 & & \frac{\delta}{\lambda} \arrow[r] \arrow[d] & \frac{\delta}{\lambda} \arrow[ddd] \\
\delta \arrow[r] \arrow[ddd] & \delta \arrow[r] \arrow[d] & \kappa\delta \arrow[d] & \\
 & \mu_{l+1}\epsilon \arrow[r] \arrow[d] & \kappa\mu_{l+1}\epsilon \arrow[d] & \\
 & \frac{\mu_{l+1}}{\mu_l}\epsilon \arrow[r] \arrow[d] & \kappa\frac{\mu_{l+1}}{\mu_l}\epsilon \arrow[r] & \kappa\frac{\mu_{l+1}}{\mu_l}\epsilon \\
\lambda\kappa\frac{\mu_{l+1}}{\mu_l}\epsilon = \mu_l\epsilon \arrow[r] \arrow[d] & \mu_l\epsilon \arrow[d] & & \\
\epsilon \arrow[r] & \epsilon & &
\end{tikzcd}$$

Next suppose $0 < \epsilon \leq \epsilon_{l+1}$. A diagram chase starting from an element of

$$L_n^{lf}(K \times \mathbb{R}^k, 1, \mu_{l+1}\epsilon)$$

representing an element of

$$\ker\{L_n^{lf,\epsilon}(K \times \mathbb{R}^k; 1) \longrightarrow L_n^{lf,\epsilon_l}(K \times \mathbb{R}^k; 1)\}$$

establishes (2).

$$\begin{tikzcd}[row sep=small]
L_n^{lf}(C\ldots) & L_n^{lf}(A\ldots)\oplus L_n^{lf}(B\ldots) & L_n^{lf}(K\ldots) & L_n^{lf}(C\ldots) \\
 & & \mu_{l+1}\epsilon = \frac{\mu_l^2}{\lambda\kappa}\epsilon \arrow[r] \arrow[d] & \frac{\mu_l^2}{\lambda}\epsilon \arrow[d] \\
 & & \frac{\mu_l}{\lambda\kappa}\epsilon \arrow[r] \arrow[d] & \frac{\mu_l}{\lambda}\epsilon \arrow[ddd] \\
\mu_l\epsilon \arrow[r] \arrow[d] & \mu_l\epsilon \arrow[r] \arrow[d] & \mu_l\epsilon \arrow[d] & \\
\epsilon \arrow[r] \arrow[dd] & \epsilon \arrow[r] \arrow[d] & \epsilon \arrow[d] & \\
 & \epsilon_{l+1} \arrow[r] \arrow[d] & \epsilon_{l+1} \arrow[r] & \kappa\epsilon_{l+1} \\
\lambda\epsilon_{l+1} \arrow[r] \arrow[d] & \lambda\epsilon_{l+1} \arrow[d] & & \\
\frac{1}{\mu_l}\lambda\epsilon_{l+1} \arrow[r] & \frac{1}{\mu_l}\lambda\epsilon_{l+1} & &
\end{tikzcd}$$

$\square$

References.

[1] Quinn, F., Resolutions of homology manifolds, and the topological characterization of manifolds, Invent. Math. **72**, 267 – 284 (1983).

[2] Ranicki, A. A.,*Exact Sequences in the Algebraic Theory of Surgery*, Math. Notes, vol. **26**, Princeton Univ. Press, 1981.

[3] —— and Yamasaki, M., Symmetric and quadratic complexes with geometric control, Proc. of TGRC-KOSEF vol. **3**, 139 – 152 (1993), available electronically on WWW from `http://www.maths.ed.ac.uk/~aar/` and also from `http://math.josai.ac.jp/~yamasaki/`.

[4] —— and ——, Controlled K-theory, Topology Appl. **61**, 1 – 59 (1995).

[5] Yamasaki, M., L-groups of crystallographic groups, Invent. Math. **88**, 571 – 602 (1987).

Andrew Ranicki
Department of Mathematics and Statistics
University of Edinburgh
Edinburgh EH9 3JZ
Scotland
email : `aar@maths.ed.ac.uk`

Masayuki Yamasaki
Department of Mathematics
Josai University
1-1 Keyaki-dai, Sakado
Saitama 350-02
Japan
email : `yamasaki@math.josai.ac.jp`