45 SLIDES ON CHAIN DUALITY

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Abstract The texts of 45 slides[1] on the applications of chain duality to the homological analysis of the singularities of Poincaré complexes, the double points of maps of manifolds, and to surgery theory.

1. Introduction

- Poincaré duality
$$H^{n-*}(M) \;\cong\; H_*(M)$$
is the basic algebraic property of an n-dimensional manifold M.
- A chain complex C with n-dimensional Poincaré duality
$$H^{n-*}(C) \;\cong\; H_*(C)$$
is an algebraic model for an n-dimensional manifold, generalizing the intersection form.
- Spaces with Poincaré duality (such as manifolds) determine Poincaré duality chain complexes in additive categories with chain duality, giving rise to interesting invariants, old and new.

2. What is chain duality?

- $\mathbb{A}$ = additive category.
- $\mathbb{B}(\mathbb{A})$ = additive category of finite chain complexes in $\mathbb{A}$.
- A contravariant additive functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ extends to
$$T : \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A}) \ ; \ C \to T(C)$$
by the total double complex
$$T(C)_n = \sum_{p+q=n} T(C_{-p})_q \ .$$
- **Definition**: A chain duality (T, e) on $\mathbb{A}$ is a contravariant additive functor $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$, together with a natural transformation $e : T^2 \to 1 : \mathbb{A} \to \mathbb{B}(\mathbb{A})$ such that for each object A in $\mathbb{A}$:
  - $e(T(A))\,.\,T(e(A)) = 1 : T(A) \to T(A)$,
  - $e(A)\colon T^2(A) \to A$ is a chain equivalence.

[1]The lecture at the conference on Surgery and Geometric Topology, Josai University, Japan on 17 September, 1996 used slides 1.–36.

3. Properties of chain duality

- The dual of an object A is a chain complex $T(A)$.
- The dual of a chain complex C is a chain complex $T(C)$.
- Motivated by Verdier duality in sheaf theory.
- A.Ranicki, Algebraic L-theory and topological manifolds, Tracts in Mathematics 102, Cambridge (1992)

4. Involutions

- An involution (T, e) on an additive category $\mathbb{A}$ is a chain duality such that $T(A)$ is a 0-dimensional chain complex (= object) for each object A in $\mathbb{A}$, with $e(A) : T^2(A) \to A$ an isomorphism.
- **Example**: An involution $R \to R; r \to \overline{r}$ on a ring R determines the involution (T, e) on the additive category $\mathbb{A}(R)$ of f.g. free left R-modules:
  - $T(A) = \mathrm{Hom}_R(A, R)$
  - $R \times T(A) \to T(A)$; $(r, f) \to (x \to f(x)\overline{r})$
  - $e(A)^{-1} : A \to T^2(A)$; $x \to (f \to \overline{f(x)})$.

5. Manifolds and homeomorphisms up to homotopy

- Traditional questions of surgery theory:
  - Is a space with Poincaré duality homotopy equivalent to a manifold?
  - Is a homotopy equivalence of manifolds homotopic to a homeomorphism?
- Answered for dimensions ≥ 5 by surgery exact sequence in terms of the assembly map

$$A : H_*(X; \mathbb{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi_1(X)]) .$$

- L-theory of additive categories with involution suffices for surgery groups $L_*(\mathbb{Z}[\pi_1(X)])$.
- Need chain duality for the generalized homology groups $H_*(X; \mathbb{L}_\bullet(\mathbb{Z}))$ and A.

6. Manifolds and homeomorphisms

- Will use chain duality to answer questions of the type:
  - Is a space with Poincaré duality a manifold?
  - Is a homotopy equivalence of manifolds a homeomorphism?

7. Controlled topology

- Controlled topology (Chapman-Ferry-Quinn) considers:
  - the approximation of manifolds by Poincaré complexes,
  - the approximation of homeomorphisms of manifolds by homotopy equivalences.
- Philosophy of controlled topology, with control map $1 : X \to X$:
  - A Poincaré complex X is a homology manifold if and only if it is an ϵ-controlled Poincaré complex for all $\epsilon > 0$.
  - A map of homology manifolds $f : M \to X$ has contractible point inverses if and only if it is an ϵ-controlled homotopy equivalence for all $\epsilon > 0$.

8. Simplicial complexes

- In dealing with applications of chain duality to topology will only work with (connected, finite) simplicial complexes and (oriented) polyhedral homology manifolds and Poincaré complexes.
- Can also work with Δ-sets and topological spaces, using the methods of:
  - M.Weiss, Visible L-theory, Forum Math. 4, 465–498 (1992)
  - S.Hutt, Poincaré sheaves on topological spaces, Trans. A.M.S. (1996)

9. Simplicial control

- Additive category $\mathbb{A}(\mathbb{Z}, X)$ of X-controlled $\mathbb{Z}$-modules for a simplicial complex X.
  - A.Ranicki and M.Weiss, Chain complexes and assembly, Math. Z. 204, 157–186 (1990)
- Will use chain duality on $\mathbb{A}(\mathbb{Z}, X)$ to obtain homological obstructions for deciding:
  - Is a simplicial Poincaré complex X a homology manifold? (Singularities)
  - Does a degree 1 map $f : M \to X$ of polyhedral homology manifolds have acyclic point inverses? (Double points)
- Acyclic point inverses $\widetilde{H}_*(f^{-1}(x)) = 0$ is analogue of homeomorphism in the world of homology.

10. The X-controlled $\mathbb{Z}$-module category $\mathbb{A}(\mathbb{Z}, X)$

- X = simplicial complex.
- A $(\mathbb{Z}, X)$-module is a finitely generated free $\mathbb{Z}$-module A with direct sum decomposition
$$A \;=\; \sum_{\sigma \in X} A(\sigma)\ .$$
- A $(\mathbb{Z}, X)$-module morphism $f : A \to B$ is a $\mathbb{Z}$-module morphism such that
$$f(A(\sigma)) \subseteq \sum_{\tau \geq \sigma} B(\tau)\ .$$
- **Proposition**: A $(\mathbb{Z}, X)$-module chain map $f : C \to D$ is a chain equivalence if and only if the $\mathbb{Z}$-module chain maps
$$f(\sigma,\sigma)\ :\ C(\sigma)\ \to\ D(\sigma)\ (\sigma \in X)$$
are chain equivalences.

11. Functorial formulation

- Regard simplicial complex X as the category with:
  - objects: simplexes $\sigma \in X$
  - morphisms: face inclusions $\sigma \leq \tau$.
- A $(\mathbb{Z}, X)$-module $A = \sum\limits_{\sigma \in X} A(\sigma)$ determines a contravariant functor
$$[A] : X \to \mathbb{A}(\mathbb{Z}) = \{\text{f.g. free abelian groups}\}\ ;\ \sigma \to [A][\sigma] = \sum_{\tau \geq \sigma} A(\tau)\ .$$

- The $(\mathbb{Z},X)$-module category $\mathbb{A}(\mathbb{Z},X)$ is a full subcategory of the category of contravariant functors $X \to \mathbb{A}(\mathbb{Z})$.

12. Dual cells

- The barycentric subdivision X' of X is the simplicial complex with one n-simplex $\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n$ for each sequence of simplexes in X
$$\sigma_0 < \sigma_1 < \dots < \sigma_n \ .$$
- The dual cell of a simplex $\sigma \in X$ is the contractible subcomplex
$$D(\sigma, X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n \,|\, \sigma \leq \sigma_0\} \subseteq X' \ ,$$
with boundary
$$\partial D(\sigma, X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n \,|\, \sigma < \sigma_0\} \subseteq D(\sigma, X) \ .$$
- Introduced by Poincaré to prove duality.
- A simplicial map $f : M \to X'$ has acyclic point inverses if and only if
$$(f|)_* : H_*(f^{-1}D(\sigma,X)) \cong H_*(D(\sigma,X)) \ (\sigma \in X) \ .$$

13. Where do $(\mathbb{Z},X)$-module chain complexes come from?

- For any simplicial map $f : M \to X'$ the simplicial chain complex $\Delta(M)$ is a $(\mathbb{Z},X)$-module chain complex:
$$\Delta(M)(\sigma) = \Delta(f^{-1}D(\sigma,X), f^{-1}\partial D(\sigma,X))$$
with a degreewise direct sum decomposition
$$[\Delta(M)][\sigma] = \sum_{\tau \geq \sigma} \Delta(M)(\tau) = \Delta(f^{-1}D(\sigma,X)) \ .$$
- The simplicial cochain complex $\Delta(X)^{-*}$ is a $(\mathbb{Z},X)$-module chain complex with:
$$\Delta(X)^{-*}(\sigma)_r = \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

14. The $(\mathbb{Z},X)$-module chain duality

- **Proposition**: The additive category $\mathbb{A}(\mathbb{Z},X)$ of $(\mathbb{Z},X)$-modules has a chain duality (T,e) with
$$\boxed{T(A) \ = \ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X)^{-*}, A), \mathbb{Z})}$$
- $TA(\sigma) = [A][\sigma]^{|\sigma|-*}$
- $T(A)_r(\sigma) \ = \ \begin{cases} \sum\limits_{\tau \geq \sigma} \mathrm{Hom}_{\mathbb{Z}}(A(\tau),\mathbb{Z}) & \text{if } r = -|\sigma| \\ 0 & \text{if } r \neq -|\sigma| \end{cases}$
- $T(C) \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C, \Delta(X'))^{-*} \simeq_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z})^{-*}$
- $T(\Delta(X')) \simeq_{(\mathbb{Z},X)} \Delta(X)^{-*}$
- Terminology $T(C)^{n-*} \ = \ T(C_{*+n}) \ (n \geq 0)$

15. Products

- The product of $(\mathbb{Z}, X)$-modules A, B is the $(\mathbb{Z}, X)$-module

$$A \otimes_{(\mathbb{Z},X)} B = \sum_{\lambda,\mu \in X, \lambda \cap \mu \neq \emptyset} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \subseteq A \otimes_{\mathbb{Z}} B \ ,$$

$$(A \otimes_{(\mathbb{Z},X)} B)(\sigma) = \sum_{\lambda,\mu \in X, \lambda \cap \mu = \sigma} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \ .$$

- $C \otimes_{(\mathbb{Z},X)} \Delta(X') \simeq_{(\mathbb{Z},X)} C$.
- $T(C) \otimes_{(\mathbb{Z},X)} D \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C, D)$.
- For simplicial maps $f : M \to X'$, $g : N \to X'$
  - $\Delta(M) \otimes_{(\mathbb{Z},X)} \Delta(N) \simeq_{(\mathbb{Z},X)} \Delta((f \times g)^{-1}\Delta_X)$
  - $T\Delta(M) \otimes_{(\mathbb{Z},X)} T\Delta(N) \simeq_{\mathbb{Z}} \Delta(M \times N, M \times N \backslash (f \times g)^{-1}\Delta_X)^{-*}$.

16. Cap product

- The Alexander-Whitney diagonal chain approximation

$$\begin{aligned} \Delta : \Delta(X') &\to \Delta(X') \otimes_{\mathbb{Z}} \Delta(X') \ ; \\ (\widehat{x}_0 \ldots \widehat{x}_n) &\to \sum_{i=0}^{n} (\widehat{x}_0 \ldots \widehat{x}_i) \otimes (\widehat{x}_i \ldots \widehat{x}_n) \end{aligned}$$

  is the composite of a chain equivalence

$$\Delta(X') \simeq_{(\mathbb{Z},X)} \Delta(X') \otimes_{(\mathbb{Z},X)} \Delta(X')$$

  and the inclusion

$$\Delta(X') \otimes_{(\mathbb{Z},X)} \Delta(X') \subseteq \Delta(X') \otimes_{\mathbb{Z}} \Delta(X') \ .$$

- Homology classes $[X] \in H_n(X)$ are in one-one correspondence with the chain homotopy classes of $(\mathbb{Z}, X)$-module chain maps

$$[X] \cap - \ : \ \Delta(X)^{n-*} \ \to \ \Delta(X') \ .$$

17. Homology manifolds

- **Definition**: A simplicial complex X is an n-dimensional homology manifold if

$$H_*(X, X \backslash \widehat{\sigma}) \ = \ \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases} \ (\sigma \in X) \ .$$

- **Proposition**: A simplicial complex X is an n-dimensional homology manifold if and only if there exists a homology class $[X] \in H_n(X)$ such that the cap product

$$[X] \cap - \ : \ \Delta(X)^{n-*} \ \to \ \Delta(X')$$

  is a $(\mathbb{Z}, X)$-module chain equivalence.
- **Proof**: For any simplicial complex X

$$H_*(X, X \backslash \widehat{\sigma}) \ = \ H_{*-|\sigma|}(D(\sigma, X), \partial D(\sigma, X)) \ ,$$

$$H^{n-*}(D(\sigma, X)) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases} \ (\sigma \in X) \ .$$

18. Poincaré complexes

- **Definition**: An <u>n-dimensional Poincaré complex</u> X is a simplicial complex with a homology class $[X] \in H_n(X)$ such that

$$[X] \cap - : H^{n-*}(X) \cong H_*(X) \ .$$

- **Poincaré duality theorem**: An n-dimensional homology manifold X is an n-dimensional Poincaré complex.
- **Proof**: A $(\mathbb{Z}, X)$-module chain equivalence

$$[X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$$

is a $\mathbb{Z}$-module chain equivalence.
- There is also a $\mathbb{Z}[\pi_1(X)]$-version.

19. McCrory's Theorem

- X = n-dimensional Poincaré complex
  - $X \times X$ is a $2n$-dimensional Poincaré complex.
  - Let $V \in H^n(X \times X)$ be the Poincaré dual of $\Delta_*[X] \in H_n(X \times X)$.
  - Exact sequence

$$H^n(X \times X, X \times X \backslash \Delta_X) \to H^n(X \times X) \to H^n(X \times X \backslash \Delta_X) \ .$$

Theorem (McCrory) X is an n-dimensional homology manifold if and only if V has image $0 \in H^n(X \times X \backslash \Delta_X)$.

- <u>A characterization of homology manifolds</u>, J. Lond. Math. Soc. 16 (2), 149–159 (1977)

20. Chain duality proof of McCrory's Theorem

- V has image $0 \in H^n(X \times X \backslash \Delta_X)$ if and only if there exists $U \in H^n(X \times X, X \times X \backslash \Delta_X)$ with image V.
- U is a chain homotopy class of $(\mathbb{Z}, X)$-module chain maps $\Delta(X') \to \Delta(X)^{n-*}$, since

$$\begin{aligned} H^n(X \times X, X \times X \backslash \Delta_X) &= H_n(T\Delta(X) \otimes_{(\mathbb{Z},X)} T\Delta(X)) \\ &= H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X'), \Delta(X)^{n-*})) \ . \end{aligned}$$

- U is a chain homotopy inverse of

$$\phi = [X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$$

with

$$\begin{aligned} &\phi U = 1 \in H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X'), \Delta(X'))) = H^0(X) \ , \\ &\phi = T\phi \ , \ (TU)\phi = (TU)(T\phi) = T(\phi U) = 1 \ . \end{aligned}$$

21. The homology tangent bundle

- The tangent bundle τ_X of a manifold X is the normal bundle of the diagonal embedding
$$\Delta : X \to X \times X \; ; \; x \to (x,x) \; .$$
- The homology tangent bundle τ_X of an n-dimensional homology manifold X is the fibration
$$(X, X\backslash\{*\}) \longrightarrow (X \times X, X \times X\backslash\Delta_X) \longrightarrow X$$
with $X \times X \to X; (x,y) \to x$.
- Thom space of τ_X
$$T(\tau_X) \;=\; (X \times X)/(X \times X\backslash\Delta_X) \; .$$
- Thom class of τ_X
$$U \in \widetilde{H}^n(T(\tau_X)) = H^n(X \times X, X \times X\backslash\Delta_X)$$
has image $V \in H^n(X \times X)$.

22. Euler

- The Euler characteristic of a simplicial complex X is
$$\chi(X) = \sum_{r=0}^{\infty}(-)^r \dim_{\mathbb{R}} H_r(X;\mathbb{R}) \in \mathbb{Z} \; .$$
- For an n-dimensional Poincaré complex X
$$\chi(X) = \Delta^*(V) \in H^n(X) = \mathbb{Z} \; .$$
- The Euler class of n-plane bundle η over X
$$e(\eta) = [U] \in \mathrm{im}(\widetilde{H}^n(T(\eta)) \to H^n(X)) \; .$$
- Reformulation of McCrory's Theorem:
an n-dimensional Poincaré complex X is a homology manifold if and only if $V \in H^n(X \times X)$ is the image of Thom class $U \in \widetilde{H}^n(T(\tau_X))$, in which case
$$\chi(X) = e(\tau_X) \in H^n(X) = \mathbb{Z} \; .$$

23. Degree 1 maps

- A map $f : M \to X$ of n-dimensional Poincaré complexes has degree 1 if
$$f_*[M] = [X] \in H_n(X) \; .$$
- A homology equivalence has degree 1.
- The Umkehr $\mathbb{Z}$-module chain map of a degree 1 map $f : M \to X$
$$f^! \;:\; \Delta(X) \simeq \Delta(X)^{n-*} \xrightarrow{f^*} \Delta(M)^{n-*} \simeq \Delta(M)$$
is such that $ff^! \simeq 1 : \Delta(X) \to \Delta(X)$.
- A degree 1 map f is a homology equivalence if and only if
$$f^! f \simeq 1 : \Delta(M) \to \Delta(M) \; ,$$
if and only if
$$(f^! \otimes f^!)\Delta_*[X] = \Delta_*[M] \in H_n(M \times M) \; .$$

24. The double point set

- Does a degree 1 map of n-dimensional homology manifolds $f : M \to X$ have acyclic point inverses?
- Obstruction in homology of double point set

$$(f \times f)^{-1}\Delta_X = \{(x, y) \in M \times M \mid f(x) = f(y) \in X\}.$$

- Define maps

$$i : M \to (f \times f)^{-1}\Delta_X \ ; \ a \to (a, a) \ ,$$
$$j : (f \times f)^{-1}\Delta_X \to X \ ; \ (x, y) \to f(x) = f(y)$$

such that $f = ji : M \to X$.
- The Umkehr map

$$\begin{aligned} j^{\,!} : H_n(X) &\cong H^n(X \times X, X \times X \backslash \Delta_X) \\ &\to H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) \\ &\cong H_n((f \times f)^{-1}\Delta_X) \text{ (Lefschetz duality)} \end{aligned}$$

is such that $j_* j^{\,!} = 1$.

25. Lefschetz

- Lefschetz duality: If W is an m-dimensional homology manifold and $A \subseteq W$ is a subcomplex then

$$H^*(W, W\backslash A) \cong H_{m-*}(A) \ .$$

- Proof: For any regular neighbourhood $(V, \partial V)$ of A in W there are defined isomorphisms

$$\begin{aligned} H^*(W, W\backslash A) &\cong H^*(W, W\backslash V) \text{ (homotopy invariance)} \\ &\cong H^*(W, \overline{W\backslash V}) \text{ (collaring)} \\ &\cong H^*(V, \partial V) \text{ (excision)} \\ &\cong H_{m-*}(V) \text{ (Poincaré-Lefschetz duality)} \\ &\cong H_{m-*}(A) \text{ (homotopy invariance).} \end{aligned}$$

- Alexander duality is the special case $W = S^m$.

26. Acyclic Point Inverse Theorem

Theorem A degree 1 map $f : M \to X$ of n-dimensional homology manifolds has acyclic point inverses if and only if

$$i_*[M] = j^{\,!}[X] \in H_n((f \times f)^{-1}\Delta_X) \ .$$

- Equivalent conditions:
  - $i_* : H_n(M) \cong H_n((f \times f)^{-1}\Delta_X)$,
  - $i_* : H_*(M) \cong H_*((f \times f)^{-1}\Delta_X)$,
  - $H_*^{lf}((f \times f)^{-1}\Delta_X \backslash \Delta_M) = 0$.

- Conditions satisfied if $f : M \to X$ is injective, with
$$(f \times f)^{-1}\Delta_X = \Delta_M \ .$$
- In general, $i_* \neq j^! f_*$ and $i_*[M] \neq j^![X]$.

27. Proof of Theorem - Part I

- A simplicial map $f : M \to X'$ has acyclic point inverses if and only if $f : \Delta(M) \to \Delta(X')$ is a $(\mathbb{Z}, X)$-module chain equivalence.
- For degree 1 map $f : M \to X'$ of n-dimensional homology manifolds define the Umkehr $(\mathbb{Z}, X)$-module chain map
$$f^! \ : \ \Delta(X') \simeq \Delta(X')^{n-*} \xrightarrow{f^*} \Delta(M)^{n-*} \simeq \Delta(M) \ .$$
- $f^!$ is a chain homotopy right inverse for f
$$ff^! \simeq 1 : \Delta(X') \to \Delta(X') \ .$$
- $f^!$ is also a chain homotopy left inverse for f if and only if
$$f^!f = 1 \in H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(M), \Delta(M))) \ .$$

28. Proof of Theorem - Part II

- Use the $(\mathbb{Z}, X)$-Poincaré duality
$$\Delta(M)^{n-*} \simeq \Delta(M)$$
and the properties of chain duality in $\mathbb{A}(\mathbb{Z}, X)$ to identify
$$\begin{aligned} 1 \ = \ i_*[M] \ , \ f^!f \ = \ j^![X] &\in H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(M), \Delta(M))) \\ &= H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(M)^{n-*}, \Delta(M))) \\ &= H_n(\Delta(M) \otimes_{(\mathbb{Z},X)} \Delta(M)) \\ &= H_n((f \times f)^{-1}\Delta_X) \ . \end{aligned}$$

29. Cohomology version of Theorem

Theorem* A degree 1 map $f : M \to X$ of n-dimensional homology manifolds has acyclic point inverses if and only if the Thom classes $U_M \in H^n(M \times M, M \times M \backslash \Delta_M)$, $U_X \in H^n(X \times X, X \times X \backslash \Delta_X)$ have the same image in $H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X)$.

- Same proof as homology version, after Lefschetz duality identifications
$$\begin{aligned} &U_M = [M] \in H^n(M \times M, M \times M \backslash \Delta_M) = H_n(M) \ , \\ &U_X = [X] \in H^n(X \times X, X \times X \backslash \Delta_X) = H_n(X) \ , \\ &H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) = H_n((f \times f)^{-1}\Delta_X) \ . \end{aligned}$$

30. The double point obstruction

- The double point obstruction of a degree 1 map $f : M \to X$ of homology manifolds
$$i_*[M] - j^![X] \in H_n((f \times f)^{-1}\Delta_X)$$
is 0 if and only if f has acyclic point inverses.
- The obstruction has image
$$\chi(M) - \chi(X) \in H^n(M) = \mathbb{Z} .$$
- If f is covered by a map of homology tangent bundles
$$b : (M \times M, M \times M \backslash \Delta_M) \to (X \times X, X \times X \backslash \Delta_X)$$
then
  - $U_M = b^* U_X \in H^n(M \times M, M \times M \backslash \Delta_M)$,
  - the double point obstruction is 0, and f has acyclic point inverses.

31. Normal maps

- A degree 1 map $f : M \to X$ of n-dimensional homology manifolds is normal if it is covered by a map $b : \tau_M \oplus \epsilon^\infty \to \tau_X \oplus \epsilon^\infty$ of the stable tangent bundles.
- The stable map of Thom spaces
$$T(b) : \Sigma^\infty T(\tau_M) \to \Sigma^\infty T(\tau_X)$$
induces a map in cohomology
$$\begin{aligned} T(b)^* : \widetilde{H}^n(T(\tau_X)) &= H^n(X \times X, X \times X \backslash \Delta_X) \\ &\to \widetilde{H}^n(T(\tau_M)) = H^n(M \times M, M \times M \backslash \Delta_M) \end{aligned}$$
which sends the Thom class U_X to U_M.
- However, Theorem* may not apply to a normal map $(f, b) : M \to X$, since in general
$$\begin{aligned} (f \times f)^* \neq (\text{inclusion})^* T(b)^* : \widetilde{H}^n(T(\tau_X)) \\ \to H^n(M \times M, M \times M \backslash (f \times f)^{-1}\Delta_X) \end{aligned}$$
(dual of $i_* \neq j^! f_*$).

32. The surgery obstruction

- The Wall surgery obstruction of a degree 1 normal map $(f, b) : M \to X$ of n-dimensional homology manifolds
$$\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$$
is 0 if (and for $n \geq 5$ only if) (f, b) is normal bordant to a homotopy equivalence.
- A degree 1 map $f : M \to X$ with acyclic point inverses is a normal map with zero surgery obstruction.
- What is the relationship between the double point obstruction of a degree 1 normal map $(f, b) : M \to X$ and the surgery obstruction?
- Use chain level surgery obstruction theory:
A.Ranicki, The algebraic theory of surgery, Proc. Lond. Math. Soc. (3) 40, 87–283 (1980)

33. Quadratic Poincaré complexes

- The simply-connected surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z})$ is the cobordism class of the n-dimensional quadratic Poincaré complex

$$(C,\psi) = (C(f^!),(e\otimes e)\psi_b)$$

where
  - $e : \Delta(M) \to C(f^!)$ is the inclusion in the algebraic mapping cone of the $\mathbb{Z}$-module chain map $f^! : \Delta(X) \to \Delta(M)$,
  - the quadratic structure ψ is the image of

$$\psi_b \in H_n(E\Sigma_2 \times_{\Sigma_2} (M\times M)) = H_n(W \otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M)\otimes_{\mathbb{Z}} \Delta(M))) ~,$$

  - $E\Sigma_2 = S^\infty$, a contractible space with a free Σ_2-action,
  - $W = \Delta(E\Sigma_2)$.
- There is also a $\mathbb{Z}[\pi_1(X)]$-version.

34. The double point and surgery obstructions - Part I

- For any degree 1 map $f : M \to X$ of n-dimensional homology manifolds the composite of

$$i_*f^! - j^! : H_*(X) \to H_*((f\times f)^{-1}\Delta_X)$$

and $H_*((f\times f)^{-1}\Delta_X) \to H_*(M\times M)$ is

$$\Delta_*f^! - (f^!\otimes f^!)\Delta_* : H_*(X) \to H_*(M\times M) ~.$$

- For a degree 1 normal map $(f,b) : M \to X$

$$H_n((f\times f)^{-1}\Delta_X) \to H_n(M\times M)$$

sends the double point obstruction $i_*[M] - j^![X]$ to

$$(1+T)\psi_b = \Delta_*[M] - (f^!\otimes f^!)\Delta_*[X] \in H_n(M\times M) ~.$$

- $(1+T)\psi_b = 0$ if and only if f is a homology equivalence.

35. The double point and surgery obstructions - Part II

- A degree 1 normal map $(f,b) : M \to X$ of n-dimensional homology manifolds determines the X-controlled quadratic structure

$$\begin{aligned}\psi_{b,X} &\in H_n(E\Sigma_2 \times_{\Sigma_2} (f\times f)^{-1}\Delta_X)\\ &= H_n(W\otimes_{\mathbb{Z}[\Sigma_2]} (\Delta(M)\otimes_{(\mathbb{Z},X)} \Delta(M))) ~.\end{aligned}$$

- $\psi_{b,X}$ has images
  - the quadratic structure

$$[\psi_{b,X}] = \psi_b \in H_n(E\Sigma_2 \times_{\Sigma_2} (M\times M)) ~,$$

  - the double point obstruction

$$(1+T)\psi_{b,X} = i_*[M] - j^![X] \in H_n((f\times f)^{-1}\Delta_X) ~.$$

36. The normal invariant

- The X-controlled quadratic Poincaré cobordism class
$$\sigma_*^X(f,b) = (C(f^!), (e\otimes e)\psi_{b,X}) \in L_n(\mathbb{A}(\mathbb{Z},X)) = H_n(X;\mathbb{L}_\bullet(\mathbb{Z}))$$
is the normal invariant of an n-dimensional degree 1 normal map $(f,b): M \to X$.
- $\sigma_*^X(f,b) = 0$ if (and for $n \geq 5$ only if) (f,b) is normal bordant to a map with acyclic point inverses.
- The non-simply-connected surgery obstruction of (f,b) is the assembly of the normal invariant
$$\sigma_*(f,b) = A\sigma_*^X(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]) \ .$$

37. Hom and Derived Hom

- For $(\mathbb{Z},X)$-modules A,B the additive group $\mathrm{Hom}_{(\mathbb{Z},X)}(A,B)$ does not have a natural $(\mathbb{Z},X)$-module structure, but the chain duality determines a natural $(\mathbb{Z},X)$-module resolution.
- Derived Hom of $(\mathbb{Z},X)$-module chain complexes C,D
$$\mathrm{RHom}_{(\mathbb{Z},X)}(C,D) \ = \ T(C)\otimes_{(\mathbb{Z},X)} D \ .$$
- Adjoint properties:
$$\begin{aligned}&\mathrm{RHom}_{(\mathbb{Z},X)}(C,D) \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C,D)\\ &\mathrm{RHom}_{(\mathbb{Z},X)}(T(C),D) \simeq_{(\mathbb{Z},X)} C\otimes_{(\mathbb{Z},X)} D \ .\end{aligned}$$
- $D = \Delta(X')$ is the dualizing complex for chain duality
$$T(C) \simeq_{(\mathbb{Z},X)} \mathrm{RHom}_{(\mathbb{Z},X)}(C,\Delta(X'))$$
as for Verdier duality in sheaf theory.

38. When is a Poincaré complex homotopy equivalent to a manifold?

- Every n-dimensional topological manifold is homotopy equivalent to an n-dimensional Poincaré complex
- Is every n-dimensional Poincaré complex homotopy equivalent to an n-dimensional topological manifold?
- From now on $n \geq 5$
- Browder-Novikov-Sullivan-Wall obstruction theory has been reformulated in terms of chain duality
  - the total surgery obstruction.

39. Browder-Novikov-Sullivan-Wall theory

- An n-dimensional Poincaré complex X is homotopy equivalent to an n-dimensional topological manifold if and only if
  1. the Spivak normal fibration of X admits a topological reduction,
  2. there exists a reduction such that the corresponding normal map $(f,b): M \to X$ has surgery obstruction
$$\sigma_*(f,b) = 0 \in L_n(\mathbb{Z}[\pi_1(X)]) \ .$$

40. Algebraic Poincaré cobordism

- Λ = ring with involution.
- $L_n(\Lambda)$ = Wall surgery obstruction group
  = the cobordism group of n-dimensional quadratic Poincaré complexes over Λ
  - n-dimensional f.g. free Λ-module chain complexes C with
  $$H^{n-*}(C) \cong H_*(C) \ ,$$
  - uses ordinary duality
  $$C^{n-*} \ = \ \mathrm{Hom}_\Lambda(C,\Lambda)_{*-n} \ .$$

41. Assembly

- X = connected simplicial complex
  - $\widetilde{X}$ = universal cover
  - $p : \widetilde{X} \to X$ covering projection.
- Assembly functor

$$A \ : \ \mathbb{A}(\mathbb{Z},X) = \{(\mathbb{Z},X)\text{-modules}\} \to \mathbb{A}(\mathbb{Z}[\pi_1(X)]) = \{\mathbb{Z}[\pi_1(X)]\text{-modules}\} \ ;$$

$$M = \sum_{\sigma\in X} M(\sigma) \to M(\widetilde{X}) = \sum_{\widetilde{\sigma}\in\widetilde{X}} M(p\widetilde{\sigma}) \ .$$

- The assembly $A(T(M))$ of dual $(\mathbb{Z},X)$-module chain complex
$$T(M) \ \simeq_{\mathbb{Z}} \ \mathrm{Hom}_{(\mathbb{Z},X)}(M,\Delta(X'))$$
is chain equivalent to dual $\mathbb{Z}[\pi_1(X)]$-module
$$M(\widetilde{X})^* \ = \ \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(M(\widetilde{X}),\mathbb{Z}[\pi_1(X)]) \ .$$

42. The algebraic surgery exact sequence

- For any simplicial complex X exact sequence

$$\dots \to H_n(X;\mathbb{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet(\mathbb{Z})) \to \dots$$

with
- A = assembly,
- $\mathbb{L}_\bullet(\mathbb{Z})$ = the 1-connective simply-connected surgery spectrum
  - $\pi_*(\mathbb{L}_\bullet(\mathbb{Z})) = L_*(\mathbb{Z})$,
- $H_n(X;\mathbb{L}_\bullet(\mathbb{Z}))$ = generalized homology group
  - cobordism group of n-dimensional quadratic Poincaré $(\mathbb{Z},X)$-module complexes $C \simeq T(C)^{n-*}$
  - uses chain duality
  $$T(C)^{n-*} \ \simeq_{(\mathbb{Z},X)} \ \mathrm{RHom}_{(\mathbb{Z},X)}(C,\Delta(X'))_{*-n} \ .$$

43. The structure group

- X = simplicial complex.
- $\mathbb{S}_n(X)$ = structure group.
- $\mathbb{S}_n(X)$ = cobordism group of
  - $(n-1)$-dimensional quadratic Poincaré $(\mathbb{Z},X)$-module complexes
  - with contractible $\mathbb{Z}[\pi_1(X)]$-module assembly.

44. Local and global Poincaré duality

- X = n-dimensional Poincaré complex.
- The cap product $[X] \cap - : \Delta(X)^{n-*} \to \Delta(X')$:
  - is a $(\mathbb{Z}, X)$-module chain map,
  - assembles to $\mathbb{Z}[\pi_1(X)]$-module chain equivalence
$$[X] \cap - : \Delta(\widetilde{X})^{n-*} \to \Delta(\widetilde{X}') \ .$$
- The algebraic mapping cone
$$C \ = \ \mathcal{C}([X] \cap - : \Delta(X)^{n-*} \to \Delta(X'))_{*-1}$$
  - is an $(n-1)$-dimensional quadratic Poincaré $(\mathbb{Z}, X)$-module complex,
  - with contractible $\mathbb{Z}[\pi_1(X)]$-assembly.
- X is a homology manifold if and only if C is $(\mathbb{Z}, X)$-contractible.

45. The total surgery obstruction

- X = n-dimensional Poincaré complex.
- The <u>total surgery obstruction</u> of X is the cobordism class
$$s(X) \ = \ \mathcal{C}([X] \cap -)_{*-1} \in \mathbb{S}_n(X) \ .$$
- **Theorem 1**: X is homotopy equivalent to an n-dimensional topological manifold if and only if $s(X) = 0 \in \mathbb{S}_n(X)$.
- **Theorem 2**: A homotopy equivalence $f : M \to N$ of n-dimensional topological manifolds has a total surgery obstruction $s(f) \in \mathbb{S}_{n+1}(N)$ such that f is homotopic to a homeomorphism if and only if $s(f) = 0$.
  - Should also consider Whitehead torsion.

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