COHOMOLOGY AND EULER CHARACTERISTICS OF COXETER GROUPS

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1. Introduction

Coxeter groups are familiar objects in many branches of mathematics. The connections with semisimple Lie theory have been a major motivation for the study of Coxeter groups. (Crystallographic) Coxeter groups are involved in Kac-Moody Lie algebras, which generalize the entire theory of semisimple Lie algebras. Coxeter groups of finite order are known to be finite reflection groups, which appear in invariant theory. Coxeter groups also arise as the transformation groups generated by reflections on manifolds (in a suitable sense). Finally, Coxeter groups are classical objects in combinatorial group theory.

In this paper, we discuss the cohomology and the Euler characteristics of (finitely generated) Coxeter groups. Our emphasis is on the rôle of the parabolic subgroups of finite order in both the Euler characteristics and the cohomology of Coxeter groups.

The Euler characteristic is defined for groups satisfying a suitable cohomological finiteness condition. The definition is motivated by topology, but it has applications to group theory as well. The study of Euler characteristics of Coxeter groups was initiated by J.-P. Serre [22], who obtained the formulae for the Euler characteristics of Coxeter groups, as well as the relation between the Euler characteristics and the Poincaré series of Coxeter groups. The formulae for the Euler characteristics of Coxeter groups were simplified by I. M. Chiswell [7]. From his result, one knows that the Euler characteristics of Coxeter groups can be computed in terms of the orders of parabolic subgroups of finite order.

On the other hand, for a Coxeter group W, the family of parabolic subgroups of finite order forms a finite simplicial complex $\mathcal{F}(W)$. In general, given a simplicial complex K, the Euler characteristics of Coxeter groups W with $\mathcal{F}(W) = K$ are bounded, but are not unique. However, it follows from the result of M. W. Davis that $e(W) = 0$ if $\mathcal{F}(W)$ is a generalized homology $2n$-sphere (Theorem 4). Inspired by this result, the author investigated the relation between the Euler characteristics of Coxeter groups W and the simplicial complexes $\mathcal{F}(W)$, and obtained the following results:

1. If $\mathcal{F}(W)$ is a PL-triangulation of some closed $2n$-manifold M, then
$$e(W) = 1 - \frac{\chi(M)}{2}.$$
2. If $\mathcal{F}(W)$ is a connected graph, then $e(W) \geq \gamma(\mathcal{F}(W))$, where $\gamma(-)$ denotes the genus of the graph.

See Theorem 5 and 7. Conversely, given a PL-triangulation K of a closed $2n$-manifold M, we obtain an equation for the number of i-simplices of K $(0 \leq i \leq 2n)$ by considering a Coxeter group W with $K = \mathcal{F}(W)$ (Theorem 6 and its corollary).

The family of parabolic subgroups of finite order is also important in understanding the cohomology of a Coxeter group W. For instance, let k be a commutative ring with unity, ρ a ring homomorphism

$$\rho : H^*(W, k) \to \prod_{W_F} H^*(W_F, k).$$

induced by restriction maps, where W_F ranges all the parabolic subgroups of finite order. Then $u \in \ker \rho$ is nilpotent and cannot be detected by any finite subgroup of W. And we can say more about the homomorphism ρ.

We remark that, according to the results of D. Quillen [19] and K. S. Brown [5], the family of elementary abelian p-subgroups also plays an important rôle. However, it is p-local. The rôle of the parabolic subgroups of finite order is *not* p-local, a phenomenon in which I am very interested.

Notation 1. For a finite set X, the cardinality of X is denoted by $|X|$. In particular, for a finite group G, the order of G is denoted by $|G|$.

2. Definitions and Examples

In this section, we give the definition and elementary examples of Coxeter groups.

Definition 2.1. Let S be a finite set. Let $m : S \times S \to \mathbb{N} \cup \{\infty\}$ be a map satisfying the following three conditions:

1. $m(s,t) = m(t,s)$ for all $s, t \in S$,
2. $m(s,s) = 1$ for all $s \in S$,
3. $2 \leq m(s,t) \leq \infty$ for all distinct $s, t \in S$.

The group W defined by the set of generators S and the fundamental relation $(s \cdot t)^{m(s,t)} = 1$ $(m(s,t) \neq \infty)$ is called a *Coxeter group*. Some authors permit S to be an infinite set.

Remark 1. We frequently write (W, S) or (W, S, m) instead of W to emphasize S and m. The pair (W, S) is sometimes called a *Coxeter system* in the literature.

Remark 2. Each generator $s \in S$ is an element of order 2 in W. Hence W is generated by involutions.

Example 2.1. Let (W, S) be a Coxeter group with $S = \{s, t\}$. If $m(s,t) < \infty$, then W is isomorphic to $D_{2m(s,t)}$, the dihedral group of order $2m(s,t)$. If $m(s,t) = \infty$, then W is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, the free product of two copies of the cyclic group of order 2.

Example 2.2. A *finite reflection group* is a finite subgroup of the orthogonal group $O(n)$ (for some n) generated by orthogonal reflections in the Euclidean space. A finite reflection group is known to be a Coxeter group, i.e., it admits a presentation of Coxeter groups. Conversely, any Coxeter group of finite order can be realized as a finite reflection group. Hence one can identify Coxeter groups of finite order with finite reflection groups in this way.

For example, an elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^n$ and a symmetric group Σ_n can be regarded as Coxeter groups. Finite reflection groups are completely classified and their list is short. By using the list, it is easy to determine if a given Coxeter group is of finite order. See [4], [15] for details.

Example 2.3. Coxeter groups are closed under free products and direct products.

Example 2.4 (Full triangular group)**.** Let p, q, r be integers greater than 1. The group $T^*(p,q,r)$ defined by the presentation

$$T^*(p,q,r) =< s_1, s_2, s_3 | s_i^2 = (s_1s_2)^p = (s_2s_3)^q = (s_3s_1)^r = 1 >$$

is called the *full triangular group*. It is obvious from the presentation that $T^*(p,q,r)$ is a Coxeter group. The group $T^*(p,q,r)$ is known to be of finite order if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

The *triangular group*

$$T(p,q,r) =< u, v | u^p = v^q = (uv)^r = 1 >$$

is a subgroup of $T^*(p,q,r)$ of index 2 (via $u = s_1s_2$ and $v = s_2s_3$).

The full triangular group $T^*(p,q,r)$ can be realized as a planar discontinuous group acting on a sphere S^2 (if $1/p + 1/q + 1/r > 1$), on the Euclidean plane E^2 (if $1/p + 1/q + 1/r = 1$), or on the hyperbolic plane H^2 (if $1/p + 1/q + 1/r < 1$). The orbit space of the action of $T^*(p,q,r)$ on S^2, E^2, or H^2 is homeomorphic to a disk D^2.

Example 2.5. Given integers p, q, r greater than 1, let $O(p,q,r)$ be the orbifold defined as follows. (See [21] for the notion of orbifolds.) The underlying space of O is a standard 2-simplex Δ^2. Vertices v_0, v_1, and v_2 of Δ^2 are corner reflection points of order $2p$, $2q$, and $2r$. The points in the interior of edges are reflection points, while the points in the interior of the whole Δ^2 are manifold points.

The orbifold $O(p,q,r)$ is uniformable (i.e., it has a manifold cover). Indeed, the orbifold $O(p,q,r)$ comes from the orbit space of the action of the full triangular group $T^*(p,q,r)$ on one of S^2, E^2, or H^2 mentioned in the bottom of Example 2.4. The orbifold fundamental group $\pi_1^{orb}(O(p,q,r))$ is isomorphic to $T^*(p,q,r)$.

Let $O'(p,q,r)$ be the orbifold, whose underlying space is a 2-sphere S^2, with three cone points of order p, q, and r. Then there is a double orbifold covering

$$O'(p,q,r) \to O(p,q,r).$$

The orbifold fundamental group $\pi_1^{orb}(O'(p,q,r))$ is isomorphic to the triangular group $T(p,q,r)$. See [17] and [21] for the details.

Example 2.6. Example 2.4 and 2.5 are special cases of reflection orbifolds and groups generated by reflections on a manifold, both of which are closely related to Coxeter groups. See [8] and [16] for the general theory.

3. Parabolic Subgroups

Let (W, S, m) be a Coxeter group. For a subset $T \subset S$, define W_T to be the subgroup of W generated by the elements of T (i.e. $W_T =< T > \subseteq W$). In particular, $W_\emptyset = \{1\}$ and $W_S = W$. W_T is called a *parabolic subgroup* (or special subgroup) of W. The subgroup W_T is known to be a Coxeter group. Indeed, $(W_T, T, m|T \times T)$ is a Coxeter group. It is obvious from the definition that the number of parabolic subgroups of a Coxeter group is finite.

Example 3.1. Parabolic subgroups of the full triangular group $T^*(p,q,r)$ consist of 8 subgroups. Namely,

1. The trivial subgroup $\{1\}$.
2. Three copies of a cyclic group of order 2 (generated by single element).
3. Dihedral groups of order $2p$, $2q$, and $2r$ (generated by two distinct elements).
4. $T^*(p,q,r)$ itself.

The following observation asserts that the parabolic subgroups of finite order are maximal among the subgroups of finite order in a Coxeter group.

Proposition 1 ([9, Lemma 1.3]). *Let W be a Coxeter group and H its finite subgroup. Then there is a parabolic subgroup W_F of finite order and an element $w \in W$ such that $H \subset wWw^{-1}$.*

4. Euler characteristics

In this section, we introduce the Euler characteristics of groups. First we introduce the class of groups for which the Euler characteristic is defined.

Notation 2. Let Γ be a group. Then $\mathbb{Z}\Gamma$ is the integral group ring of Γ. We regard $\mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module with trivial Γ-action.

Definition 4.1. A group Γ is said to be of *type FL* if $\mathbb{Z}$ admits a free resolution (over $\mathbb{Z}\Gamma$) of finite type. In other words, there is an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

of finite length such that each F_i is a finitely generated free $\mathbb{Z}\Gamma$-module.

Remark 3. If Γ is a group of type FL, then $\operatorname{cd}\Gamma < \infty$ and hence Γ is torsion-free.

Definition 4.2. A group Γ is said to be of *type VFL* if some subgroup of finite index is of type FL.

Now we define the Euler characteristic of a group. Let Γ be a group of type FL, and let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

b a free resolution of finite length. The *Euler characteristic* $e(\Gamma)$ of Γ is defined by

$$e(\Gamma) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}\Gamma} F_i.$$

Let Γ be a group of type VFL. Then its Euler characteristic $e(\Gamma)$ is defined by

$$e(\Gamma) = \frac{e(\Gamma')}{(\Gamma : \Gamma')} \in \mathbb{Q},$$

where Γ' is a subgroup of finite index which is of type FL. The rational number $e(\Gamma)$ is independent of the choice of a subgroup Γ', and we have

Proposition 2. *Let Γ be a group and Γ' a subgroup of finite index. Then Γ is of type VFL if and only if Γ' is of type VFL. If Γ is of type VFL, then*

$$e(\Gamma') = (\Gamma : \Gamma') \cdot e(\Gamma).$$

We give some examples of groups of type VFL and their Euler characteristics.

Example 4.1. Any finite group Γ is of type VFL. Its Euler characteristic is given by

$$e(\Gamma) = \frac{1}{|\Gamma|}.$$

(Take Γ' to be a trivial group $\{1\}$.)

Example 4.2. Let K be a finite aspherical polyhedron. Then its fundamental group $\pi = \pi_1(K)$ is of type FL, and

$$e(\pi) = \chi(K),$$

where $\chi(K)$ is the Euler characteristic of K. The fact that the Euler characteristic of a finite aspherical polyhedron depends only on its fundamental group is the motivation of the definition of Euler characteristics of groups.

For instance, the circle S^1 is aspherical and $\pi_1(S^1) \cong \mathbb{Z}$, hence

$$e(\mathbb{Z}) = \chi(S^1) = 0.$$

Let Σ_g be a closed orientable surface of genus $g > 0$. Then Σ_g is aspherical, proving

$$e(\pi_1(\Sigma_g)) = \chi(\Sigma_g) = 2 - 2g.$$

Example 4.3. If Γ_1, Γ_2 are groups of type VFL, then their free product $\Gamma_1 * \Gamma_2$ and their direct product $\Gamma_1 \times \Gamma_2$ are of type VFL, and

$$\begin{aligned} e(\Gamma_1 * \Gamma_2) &= e(\Gamma_1) + e(\Gamma_2) - 1, \\ e(\Gamma_1 \times \Gamma_2) &= e(\Gamma_1) \cdot e(\Gamma_2). \end{aligned}$$

As a consequence, a free group F_n and a free abelian group $\mathbb{Z}^n$ are of type VFL (in fact type FL), and we have

$$\begin{aligned} e(F_n) &= 1 - n, \\ e(\mathbb{Z}^n) &= 0, \end{aligned}$$

where F_n is the free group of rank n.

Example 4.4. The group $SL(2,\mathbb{Z})$ has a subgroup of index 24 which is isomorphic to the free group of rank 3. Hence $SL(2,\mathbb{Z})$ is of type VFL. Using Example 4.2 and 4.3, one can compute the Euler characteristic of $SL(2,\mathbb{Z})$ as

$$e(SL(2,\mathbb{Z})) = \frac{e(F_3)}{24} = -\frac{1}{12}.$$

Example 4.5. The Euler characteristics of groups are closely related to the Euler characteristics of orbifolds. (See [24] or [21] for the definition of the orbifold Euler characteristics.) Namely, let O be an orbifold such that

1. O has a finite manifold covering $M \to O$ for which M has the homotopy type of a finite complex.
2. The universal cover of O is contractible.

Then the orbifold fundamental group $\pi = \pi_1^{orb}(O)$ of O is of type VFL and one has

$$e(\pi) = \chi^{orb}(O),$$

where $\chi^{orb}(O)$ is the orbifold Euler characteristic of O.

Example 4.6. Let Γ be a full triangular group $T^*(p, q, r)$ of infinite order. Then, as in Example 2.4, Γ is isomorphic to the orbifold fundamental group of the orbifold $O(p, q, r)$. The orbifold $O(p, q, r)$ satisfies the conditions 1 and 2 in Example 4.5. Hence the Euler characteristic of Γ is identified with the orbifold Euler characteristic of $O(p, q, r)$. Using this, one has

$$e(\Gamma) = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right).$$

Finally, we mention two properties of Euler characteristics of groups. Let G be a group of type VFL.

Theorem 1 (Gottlieb-Stallings [12], [23])**.** *If $e(G) \neq 0$, then the center of G is a finite subgroup.*

Theorem 2 (Brown [5])**.** *Let p be a prime. If p^n divides the denominator of $e(G)$, then G has a subgroup of order p^n.*

In view of Example 4.1, Theorem 2 is a generalization of (a part of) Sylow's theorem.

5. Euler characteristics of Coxeter groups (I)

J.-P. Serre [22] proved that Coxeter groups are of finite homological type. In fact he proved that Coxeter groups satisfy a much stronger finiteness condition than finite homological type, called type WFL. He also provided the formulae for the Euler characteristics of Coxeter groups.

The formulae of Euler characteristics of Coxeter groups were simplified by I. M. Chiswell [7], which we now quote. Before doing this, we remark that, if a Coxeter group W is of finite order, then its Euler characteristic is given by $e(W) = 1/|W|$ (Example 4.1). Hence we may assume a Coxeter group W to be of infinite order.

Theorem 3 (Chiswell [7])**.** *The Euler characteristic $e(W)$ of a Coxeter group W of infinite order is given by*

$$e(W) = \sum_{\substack{T \subset S \\ |W_T| < \infty}} (-1)^{|T|} e(W_T) = \sum_{\substack{T \subset S \\ |W_T| < \infty}} (-1)^{|T|} \frac{1}{|W_T|}. \tag{1}$$

Thus the Euler characteristics of Coxeter groups are completely determined their parabolic subgroups of finite order. Since the order of a finite reflection group is easy to compute, so is the Euler characteristic of a Coxeter group.

Serre also obtained in [22] the relation between the Euler characteristic of a Coxeter group and the Poincaré series. Namely, for a Coxeter group (W, S), define

$$g(t) = \sum_{w \in W} t^{l(w)},$$

where $l(w)$ is the minimum of the length of reduced words in S representing w. The function $g(t)$ is known to be a rational function and is called *Poincaré series* of (W, S). Serre proved

$$e(W) = \frac{1}{g(1)}.$$

In general, Poincaré series of arbitrary finitely presented groups may not satisfy this property. See [11].

6. Poset of Parabolic Subgroups of Finite Order

Before continuing the discussion of the Euler characteristics of Coxeter groups, we introduce the simplicial complexes associated with Coxeter groups. Given a Coxeter group (W, S), define $\mathcal{F}(W)$ to be the poset of nontrivial subsets $F \subseteq S$ such that the order of the corresponding parabolic subgroup W_F is of finite order. If there is no ambiguity we write $\mathcal{F}$ instead of $\mathcal{F}(W)$. The poset $\mathcal{F}(W)$ can be regarded as an (abstract) simplicial complex with the set of vertices S.

Example 6.1. If (W, S) is a finite reflection group with $|S| = n$, then any nontrivial subset $F \subset S$ belongs to $\mathcal{F}(W)$ and hence

$$\mathcal{F} = \Delta^{n-1},$$

the standard $(n-1)$-simplex.

Example 6.2. If (W, S) is a full triangular group of infinite order, then

$$\mathcal{F} = \partial\Delta^2,$$

the boundary of the standard 2-simplex (i.e. a triangle).

Example 6.3. The list of Coxeter groups with $\mathcal{F}(W) = \partial\Delta^3$ can be found in [24].

Example 6.4. Let K be a finite simplicial complex. A finite simplicial complex K is called a *flag complex* if K satisfies the following condition: For any subset $V = \{v_0, \cdots, v_n\}$ of vertices of K, if any two element subset $\{v_i, v_j\}$ of V form an edge of K, then $V = \{v_0, \cdots, v_n\}$ spans an n-simplex. A barycentric subdivision $\mathrm{Sd}\, K$ of a finite simplicial complex K is an example of a flag complex.

If K is a flag complex, then there is a Coxeter group W for which $\mathcal{F}(W) = K$. Namely, let S be the set of vertices of K. Define $m : S \times S \to \mathbb{N} \cup \{\infty\}$ by

$$m(s_1, s_2) = \begin{cases} 1 & s_1 = s_2 \\ 2 & \{s_1, s_2\} \text{ forms a 1-simplex} \\ \infty & \text{otherwise.} \end{cases}$$

The resulting Coxeter group (W, S) satisfies $\mathcal{F}(W) = K$. In particular, given a finite simplicial complex K, there is a Coxeter group W with $\mathcal{F}(W) = \mathrm{Sd}\, K$.

Definition 6.1. A Coxeter group (W, S) with all $m(s, t) = 2$ or ∞ for distinct $s, t \in S$ is called *right-angled* Coxeter group.

Coxeter groups constructed in Example 6.4 are examples of right-angled Coxeter groups. Conversely, if W is a right-angled Coxeter group, then $\mathcal{F}(W)$ is a flag complex.

Remark 4. It is not known if there is a Coxeter group W for which $\mathcal{F}(W) = K$ for a given finite simplicial complex K.

7. Euler characteristics of Coxeter groups (II)

Now let us consider the Euler characteristic of W in terms of the structure of $\mathcal{F}(W)$. Proofs of statements of the following three sections will appear in [3]. If (W, S) is a finite reflection group, then

$$|W| \geq 2^{|S|}. \tag{2}$$

The equality holds if and only if W is isomorphic to the elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^{|S|}$ of rank $|S|$. Now let K be a finite simplicial complex. Then the Euler characteristic of any Coxeter group W with $K = \mathcal{F}(W)$ must satisfy

$$1 - \sum_{i:even} \frac{f_i(K)}{2^{i+1}} \leq e(W) \leq 1 + \sum_{i:odd} \frac{f_i(K)}{2^{i+1}}, \tag{3}$$

where $f_i(K)$ is the number of i-simplices of K. This follows from Theorem 3 and the equation (2). As in the following example, the inequality (3) is not best possible.

Example 7.1. Suppose $K = \partial\Delta^3$. Then Coxeter groups W with $\mathcal{F}(W) = K$ are precisely full triangular groups of infinite order. The inequality (3) implies

$$-\frac{1}{2} \leq e(W) \leq \frac{7}{4}.$$

On the other hand, from the formula in Example 4.6 one has

$$-\frac{1}{2} < e(W) \leq 0,$$

which is best possible.

Example 7.1 shows that, for a fixed finite simplicial complex K, Euler characteristics of Coxeter groups with $\mathcal{F}(W) = K$ can vary. However, from the result of M. W. Davis [8], one has:

Theorem 4. *Let W be a Coxeter group such that $\mathcal{F}(W)$ is a generalized homology $2n$-sphere, then*

$$e(W) = 0.$$

Here a generalized homology $2n$-sphere is a simplicial complex K satisfying

1. K has the homology of a $2n$-sphere.
2. The link of an i-simplex of K has the homology of a $(2n - i - 1)$-sphere.

A simplicial complex satisfying the condition 1 and 2 is also called a *Cohen-Macaulay complex.* A triangulation of a homology sphere is an example of a generalized homology sphere.

Note that Davis actually proved that, if W is a Coxeter group such that $\mathcal{F}(W)$ is a generalized homology $2n$-sphere, then, for each torsion free subgroup Γ of finite index in W, there is a closed aspherical $(2n + 1)$-manifold M with $\pi_1(M) \cong \Gamma$ [8, Theorem 10.1]. It follows that

$$e(W) = \frac{e(\Gamma)}{(W : \Gamma)} = \frac{\chi(M)}{(W : \Gamma)} = 0,$$

since M is odd dimensional and has homotopy type of a finite simplicial complex.

We (partially) generalize Theorem 4. A finite simplicial complex K is a PL-triangulation of a closed M if, for each simplex T of K, the link of T in K is a triangulation of $(\dim M - \dim T - 1)$-sphere. If K is a PL-triangulation of a homology sphere, then K is a generalized homology sphere.

Theorem 5 (T. Akita)**.** *Let W be a Coxeter group such that $\mathcal{F}(W)$ is a PL-triangulation of a closed $2n$-manifold, then*

$$e(W) = 1 - \frac{\chi(\mathcal{F}(W))}{2},$$

where $\chi(\mathcal{F}(W))$ is the Euler characteristic of the simplicial complex $\mathcal{F}(W)$.

Remark 5. Given a simplicial complex K, there is a Coxeter group W such that $\mathcal{F}(W)$ agrees with the barycentric subdivision of K (Example 6.4). Hence there are Coxeter groups for which Theorem 5 and Theorem 5 can be applied.

Remark 6. We should point out that the assumptions of Theorem 4 and 5 permit, for instance, K to be an *arbitrary* triangulation of $2n$-sphere. The significance becomes clear if we compare with the case that K is a triangulation of a circle S^1. Indeed, the Euler characteristics of Coxeter groups with $\mathcal{F}(W)$ a triangulation of a circle S^1 can be arbitrary small.

Remark 7. Under the assumption of Theorem 5, $2 \cdot e(W)$ is an integer. On the other hand, given a rational number q, there is a Coxeter group W with $e(W) = q$.

8. Application of Theorem 5

Let K be a flag complex. Let (W, S) be a Coxeter group with $\mathcal{F}(W) = K$ as in Example 6.4. Any parabolic subgroup W_F of finite order is isomorphic to the elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^{|F|}$ of rank $|F|$. Hence the Euler characteristic of W is determined by the number of simplices of K. Explicitly, let $f_i(K)$ be the number of i-simplices of K. Then

$$e(W) = 1 + \sum_i \left(-\frac{1}{2}\right)^{i+1} f_i(K), \tag{4}$$

Using this together with Theorem 5, one obtains

Theorem 6 (T. Akita). *Let K be a PL-triangulation of a closed $2n$-manifold. Assume K is a flag complex. Then*

$$\chi(K) = \sum_i \left(-\frac{1}{2}\right)^{i} f_i(K).$$

In particular, the barycentric subdivision of any finite simplicial complex is a flag complex. Thus

Corollary . *Let K be a PL-triangulation of a closed $2n$-manifold. Let $f_i(\operatorname{Sd} K)$ is the number of i-simplices of the barycentric subdivision $\operatorname{Sd} K$ of K. Then*

$$\chi(\operatorname{Sd} K) = \sum_i \left(-\frac{1}{2}\right)^{i} f_i(\operatorname{Sd} K).$$

In general, if K is a triangulation of a closed n-manifold, then

$$\begin{aligned} \chi(K) &= \sum_i (-1)^i f_i(K) \\ f_{n-1}(K) &= \frac{n+1}{2} f_n(K) \end{aligned} \tag{5}$$

hold. The equality in Theorem 6 is not the consequence of the equalities (5).

For a triangulation K of a sphere S^n (for arbitrary n), the Dehn-Sommerville equations give a set of equations for the $f_i(K)$'s. It would be interesting to investigate the relation between the equation in Theorem 6 and Dehn-Somerville equations.

9. Euler Characteristics of Aspherical Coxeter Groups and the Genus of a Graph

In this section, we consider the Euler characteristics of Coxeter groups W such that $\mathcal{F}(W)$ is a graph (1-dimensional simplicial complex).

Definition 9.1. A Coxeter group (W, S) is called *aspherical* (in [18]) if every three distinct elements of S generate a parabolic subgroup of infinite order.

In view of Example 2.4, a Coxeter group (W, S) is aspherical if and only if for every three distinct elements $s, t, u \in S$,

$$\frac{1}{m_{st}} + \frac{1}{m_{tu}} + \frac{1}{m_{us}} \leq 1$$

holds, where $1/\infty = 0$ by the convention. It is easy to see that a Coxeter group W is aspherical if and only if $\mathcal{F}(W)$ is a graph.

For a graph Γ, let $E(\Gamma)$ be the set of edges of Γ. When (W, S) is an aspherical Coxeter group with $\Gamma = \mathcal{F}(W)$, it follows from Chiswell's formula (1) that

$$1 - \frac{|S|}{2} < e(W) \leq 1 - \frac{|S|}{2} + \frac{|E(\Gamma)|}{4}. \tag{6}$$

One has another inequality for $e(W)$ using the genus of a graph. The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest number g such that the graph Γ imbeds in the closed orientable surface of genus g. For instance, a graph Γ is a planar graph if and only if $\gamma(\Gamma) = 0$.

Theorem 7 (T. Akita)**.** *Let (W, S) be a Coxeter group for which $\mathcal{F}(W)$ is a connected finite graph. Then*

$$e(W) \geq \gamma(\mathcal{F}).$$

Example 9.1. For any non-negative integer n, there is a Coxeter group W satisfying

1. $\mathcal{F}(W)$ is a graph of genus n.
2. $e(W) = n$.

The construction uses the complete bipartite graphs $K_{m,n}$.

Recall that a graph Γ is a *bipartite graph* if its vertex set can be partitioned into two subsets U and V such that the vertices in U are mutually nonadjacent and the vertices in V are mutually nonadjacent. If every vertex of U is adjacent to every vertex of V, then the graph is called *completely bipartite* on the sets U and V. A complete bipartite graph on sets of m vertices and n vertices is denoted by $K_{m,n}$.

Now the genus of the completely bipartite graph $K_{m,n}$ is given by

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

See [13, Theorem 4.5.3]. Now let $S = S_1 \sqcup S_2$ with $|S_1| = m$, $|S_2| = n$. Define $m : S \times S \to \mathbb{N} \cup \{\infty\}$ by

$$m(s,t) = \begin{cases} 1 & s = t \\ 2 & s \in S_i, t \in S_j, i \neq j \\ \infty & \text{otherwise.} \end{cases}$$

Then the resulting Coxeter group (W, S) is right-angled and satisfies $\mathcal{F}(W) = K_{m,n}$. Its Euler characteristic is given by

$$e(W) = \frac{(m-2)(n-2)}{4}.$$

Alternatively, one can construct similar examples by using complete graphs.

10. Cohomology of Coxeter Groups

In this section, we are concerned with the cohomology of Coxeter groups. The content of this section extends the earlier papers [1] and [2]. We restrict our attention to the relation between the cohomology of Coxeter groups and the cohomology of parabolic subgroups of finite order. Let (W, S) be a Coxeter group. Let k be a commutative ring with identity, regarded as a W-module with trivial W-action. Set

$$\mathcal{H}^*(W, k) = \varprojlim{}_{W_F} H^*(W_F, k),$$

where W_F runs all (possibly trivial) parabolic subgroups of finite order. The inverse limit is taken with respect to restriction maps $H^*(W_F) \to H^*(W'_F)$ for $F' \subset F$. Let

$$\rho : H^*(W, k) \to \mathcal{H}^*(W, k)$$

be the ring homomorphism induced by the restriction maps $H^*(W, k) \to H^*(W_F, k)$. The properties of ρ are the main topic of this section.

D. J. Rusin [20], M. W. Davis and T. Januszkiewicz [10] computed the mod 2 cohomology ring of certain Coxeter groups.

Theorem 8 ([20, Corollary 30]). *Let W be a Coxeter group with hyperbolic signature, with all rank-3 parabolic subgroups hyperbolic, and with even exponents $m(s,t)$. Then*

$$H^*(W, \mathbb{F}_2) \cong \mathbb{F}_2[u_r, w_{s,t}] \ (r, s, t \in S)$$

with relations $u_r w_{s,t} = 0$ if $r \neq s$ and $r \neq t$, $w_{r,s} w_{t,u} = 0$ unless $\{r, s\} = \{t, u\}$, and $u_s u_t = 0$ if 4 divides $m(s,t)$ but $u_s u_t = w_{s,t}$ otherwise.

Here we shall not explain the assumptions in Theorem 8. Instead we point out that if all $m(s,t)$ (with s, t distinct) are large enough (compared with the cardinality S), then the resulting Coxeter group has hyperbolic signature and its rank 3 parabolic subgroups are hyperbolic. Such a Coxeter group must be aspherical.

Theorem 9 ([10, Theorem 4.11]). *Let W be a right-angled Coxeter group. Then*

$$H^*(W, \mathbb{F}_2) \cong \mathbb{F}_2[v_1, \cdots, v_m]/I,$$

where I is the ideal generated by all square free monomials of the form $v_{i_1} \cdots v_{i_n}$, where at least two of the v_{i_j} do not commute when regarded as elements of W.

See Definition 6.1 for the definition of right-angled Coxeter group. From their results, one can show that ρ induces an isomorphism

$$H^*(W, \mathbb{F}_2) \cong \mathcal{H}^*(W, \mathbb{F}_2).$$

for a Coxeter group W which satisfies the assumptions in Theorem 8 or 9. Inspired by this observation, we proved

Theorem 10. *Let W be a Coxeter group and k a commutative ring with identity. Let $\rho : H^*(W,k) \to \mathcal{H}^*(W,k)$ be as above. Then the kernel and the cokernel of ρ consist of nilpotent elements.*

A homomorphism satisfying these properties is called an *F-isomorphism* in [19]. Notice that, unlike the famous result of Quillen [19] concerning of the mod p cohomology of groups of finite virtual cohomological dimension, the coefficient ring k can be the ring $\mathbb{Z}$ of rational integers.

Example 10.1. Let W be the full triangular group $T^*(3,3,3)$. Its mod 2 cohomology ring is given by $H^*(W,\mathbb{F}_2) \cong \mathbb{F}_2[u,v]/(u^2)$, where $\deg u = 2$ and $\deg v = 1$ [20, p.52], while $\mathcal{H}(W,\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[w]$ with $\deg w = 1$. Then $\rho(u) = 0$ and hence ρ has nontrivial kernel for $k = \mathbb{F}_2$. This shows the homomorphism ρ may not be an isomorphism in general.

Unfortunately, we do not know whether ρ may have a non-trivial cokernel. We give a sufficient condition for ρ to be surjective.

Theorem 11. *Suppose that W is an aspherical Coxeter group (see Definition 9.1). Then ρ is surjective for any abelian group A of coefficients (with trivial W-action).*

For example, Coxeter groups satisfying the assumptions in Theorem 8 must be aspherical.

In the case $k = \mathbb{F}_2$, there is more to say. By Theorem 10, the homomorphism ρ induces the homomorphism $H^*(W,k)/\sqrt{0} \to \mathcal{H}^*(W,k)/\sqrt{0}$, where $\sqrt{0}$ denotes nilradical. Rusin proved that the mod 2 cohomology ring of any finite Coxeter group (finite reflection group) has no nilpotent elements [20, Theorem 9]. Hence the nilradical of $\mathcal{H}^*(W,\mathbb{F}_2)$ is trivial. From this, together with Theorem 10 and 11 we obtain

Corollary . *For any Coxeter group W, ρ induces a monomorphism*

$$H^*(W,\mathbb{F}_2)/\sqrt{0} \to \mathcal{H}^*(W,\mathbb{F}_2),$$

which is an isomorphism if W is aspherical.

Remark 8. Another study of the relation between the cohomology of aspherical Coxeter groups and their parabolic subgroups of finite order can be found in [18].

Now we turn to our attention to detection by finite subgroups. An element $u \in H^*(W,k)$ is said to be *detected by finite subgroups* if the image of u by the map

$$\prod_H \mathrm{res}^W_H : H^*(W,k) \to \prod_H H^*(H,k)$$

is nontrivial, where H runs all the finite subgroups of W. It would be of interest to know which elements of $H^*(W,k)$ $\hat{H}^*(W,k)$ are detected by finite subgroups. One can reduce this question to the following proposition, which follows from Theorem 10 and Proposition 1.

Proposition 3. *An element $u \in H^*(W,k)$ is detected by finite subgroups if and only if $u \notin \ker \rho$.*

Finally, we give a example of elements of $H^*(W,k)$ which cannot be detected by finite subgroups.

Example 10.2. Let W be the full triangular group $T^*(3,3,3)$. Its mod 2 cohomology ring is given by in Example 10.1. One can check easily that uv^n $(n \geq 1)$ is contained in $\ker \hat{\rho}$. Thus uv^n $(n \geq 1)$ cannot be detected by finite subgroups as elements of $H^{2+n}(W, \mathbb{F}_2)$.

Remark 9. The virtual cohomological dimension of any Coxeter group W is known to be finite [22, p. 107], and its Farrell-Tate cohomology, written $\hat{H}^*(W,k)$, is defined. For the Farrell-Tate cohomology, the analogues of Theorem 10 and 11 and Proposition 3 are valid. See [1] and [2] for detail.

11. Outline of Proof

11.1. **Actions of Coxeter groups.** A suitable complex on which a Coxeter group acts is used in the proof of Theorem 10. We recall how this goes. Let (W,S) be a Coxeter group. Let X be a topological space, $(X_s)_{s \in S}$ be a family of closed subsets of X indexed by S. From these data, one can construct a space on which W acts as follows. Set

$$S(x) = \{s \in S : x \in X_s\},$$

and let $\mathcal{U} = \mathcal{U}(X) = W \times X/\sim$, W being discrete, where the equivalence relation $\sim$ is defined by

$$(w,x) \sim (w',x') \Longleftrightarrow x = x' \& w^{-1}w' \in W_{S(x)}.$$

Then W acts on $\mathcal{U}(X)$ by $w' \cdot [w,x] = [w'w, x]$. The isotropy subgroup of $[w,x]$ is $wW_{S(x)}w^{-1}$.

11.2. **Proof of Theorem 10 (Outline).** Given a Coxeter group (W,S), let X be the barycentric subdivision of $c * \mathcal{F}(W)$, the cone of $\mathcal{F}(W)$ with the cone point c. Define X_s to be the closed star of $s \in S$ (here $s \in S$ is regarded as a vertex of $\mathcal{F}(W)$ and hence a vertex of X). Then one of the main results of M. W. Davis [8, §13.5] asserts that $\mathcal{U}(X)$ is contractible.

Consider the spectral sequence of the form

$$E_1^{pq} = \prod_{\sigma \in \Sigma_p} H^q(W_\sigma, k) \Longrightarrow H^{p+q}(W,k).$$

In the spectral sequence, one can prove that $E_2^{0,*}$ is isomorphic to $\mathcal{H}^*(W,k)$ and the homomorphism ρ is identified with the edge homomorphism $H^*(W,k) \to E_2^{0,*}$. Observe that

1. $E_2^{p,0} = 0$ if $p \neq 0$.
2. There is a natural number $n > 0$ such that $n \cdot E_r^{p,q} = 0$ for all p and $q > 0$.

Together with these observations, Theorem 10 follows from the formal properties of the differentials of the spectral sequence.

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