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TORSION AND THE EINSTEIN EQUATIONS

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1. Introduction

The Einstein equations

$$\mathrm{Ric} = \lambda g$$

are second order partial differential equations in the coefficients of the metric and in general are hard to solve. One approach is to consider first order conditions that lead to solutions. The Killing spinor equations as studied by Friedrich and his coworkers [1] give one such condition. Another comes from consideration of holonomy.

The modern version of Berger's holonomy classification is as follows. Let M be a connected n-dimensional Riemannian manifold with metric g. Assume that M is irreducible, i.e., M is not locally the Riemannian product of lower dimensional manifolds. Let Hol_g denote the restricted holonomy group of M; this is group of linear transformations of T_pM generated by parallel transport around contractible loops based at p. Then locally either (M, g) is isometric to a symmetric space

$$K/\mathrm{Hol}_g \tag{1.1}$$

or Hol_g is one of

$$Sp(n/4)\,Sp(1), \tag{1.2}$$

$$\left.\begin{array}{l} Spin(7) \quad (n=8), \\ G_2 \quad (n=7), \\ Sp(n/4), \\ SU(n/2), \end{array}\right\} \tag{1.3}$$

$$U(n/2), \tag{1.4}$$

$$SO(n). \tag{1.5}$$

From our point of view this list is interesting for two reasons. Firstly, the spaces (1.1)–(1.3) are all Einstein: (1.1) and (1.2) have non-zero scalar curvature, whilst those in (1.3) are scalar-flat. Secondly, the condition that the holonomy is not generic, i.e., that Hol_g is not $SO(n)$, is equivalent to the existence of special geometric structures on M. These structures are encapsulated in a tensor φ on M with

$$\nabla^{\mathrm{LC}}\varphi = 0,$$

where ∇^{LC} is the Levi-Civita connection of g.

For the symmetric spaces, φ is the Riemann curvature tensor R^{LC}. For $Sp(n/4)\,Sp(1)$ and $Spin(7)$, the tensor φ is a four-form. The G_2 case has $\varphi \in \Lambda^3 T^*M$. The remaining cases are special instances of Kähler geometry. Holonomy $U(n/2)$ is a general Kähler manifold and φ is simply the Kähler two-form. The $SU(n/2)$ case is Calabi-Yau geometry which has φ the sum of the Kähler two-form and a complex volume form Ω. This shows that it is convenient to consider tensors that are not of pure type. The $Sp(n/4)$ case is hyperKähler geometry with three parallel Kähler forms ω_I, ω_J, ω_K which we may combine in to one quaternion-valued two-form $\varphi = \omega_I i + \omega_J j + \omega_K k$.

Let G be the Lie group of elements of $SO(n)$ that preserve a given tensor φ at a point p in M. Assume that G is independent of p up to conjugation. Then this defines a G-structure on M. Moreover, the Levi-Civita connection preserves this structure and so is a G-connection. With a little bit of work the holonomy classification implies:

Theorem 1.1. *Let G be a proper subgroup of $SO(n)$ that acts irreducibly on $\mathbb{R}^n$. Suppose $G \neq U(n/2)$ and that (M, g) is an n-dimensional Riemannian manifold with structure group G. If M carries a torsion-free G-connection, then g is Einstein.* □

The aim of this talk/paper is to discuss how the 'torsion-free' condition may be weakened. Much of this is based on the Ph.D. thesis [2] of the first named author written under the supervision of the second named author. We refer the reader to this thesis for a fuller list of references for the material of this paper. Some of these results were presented in [8]. The second named author thanks the organisers of the Workshop on Special Geometric Structures in String Theory, Bonn, 8th–11th September, 2001, for kind hospitality.

2. Intrinsic Torsion

Suppose M is an oriented Riemannian manifold. Using the metric we have an identification of the two-forms with skew-symmetric matrices, which is the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$. We may write

$$\Lambda^2 T^*M \cong \mathfrak{so}(n),$$

where the right-hand side is regarded as a bundle over M as follows. Let $M(SO)$ be the bundle of oriented orthonormal frames in TM. Then $M(SO)$ is a principal $SO(n)$-bundle. Using the actions of $SO(n)$ on $M(SO)$, $u \mapsto u \cdot g$, and on skew-symmetric matrices $\mathfrak{so}(n)$, $A \mapsto gAg^{-1}$, we form the bundle $\mathfrak{so}(n)$ as the quotient of $M(SO) \times \mathfrak{so}(n)$ by $(u, A) \sim (u \cdot g, gAg^{-1})$.

Now suppose that M has a reduction of its structure group to G. This means that we have a principal G-bundle $P \to M$ which is a subbundle of $M(SO)$. In constructing the bundle $\mathfrak{so}(n)$ we can equally well use P instead of $M(SO)$. If G is a proper subgroup of $SO(n)$, we then get a splitting

$$\Lambda^2 T^*M \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

corresponding to the inclusion of the Lie algebra $\mathfrak{g}$ in $\mathfrak{so}(n)$.

In this situation there is a unique G-connection $\tilde{\nabla}$ with the property that the tensor

$$\xi = \nabla^{\mathrm{LC}} - \tilde{\nabla}$$

is an element of $T^*M \otimes \mathfrak{g}^\perp$ inside $T^*M \otimes \Lambda^2 T^*M$: to construct $\tilde{\nabla}$, choose any G-connection ∇ and then add to ∇ the $\mathfrak{g}$-part of $\nabla^{\mathrm{LC}} - \nabla$. The connection $\tilde{\nabla}$ has torsion

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = -\xi_X Y + \xi_Y X,$$

which in general is non-zero. Note that $\tilde{T}$ uniquely determines $\xi \in T^* \otimes \mathfrak{g}^\perp$. The connection $\tilde{\nabla}$ is then seen to be the G-connection whose torsion has minimal norm. We will call both $\tilde{T}$ and ξ the *intrinsic torsion* of the G-structure.

The simplest case of this construction is when ξ is identically zero. We then have $\tilde{\nabla} = \nabla^{\mathrm{LC}}$ and so ∇^{LC} is a torsion-free G-connection. This corresponds exactly to the situation where Hol_g is a subgroup of G.

Of current interest in string theory, is the situation where one has a G-connection with totally skew torsion. There are some intriguing geometries whose *intrinsic* torsion has this totally skew property.

Example 2.1. Suppose M is a manifold of real dimension 6. An $SU(3)$-structure on M corresponds to a choice of metric g, almost complex structure I and complex three-form Ω so that

$$g(I\cdot, I\cdot) = g, \quad \Omega \in \Lambda^{3,0} T^*M, \quad \text{and} \quad \|\Omega\|^2 = 1.$$

Let $\omega = g(I\cdot, \cdot)$ be the corresponding two-form and write $\Omega = \Omega_1 + i\Omega_2$ as a sum of real three-forms. Under the action of $SU(3)$ we have

$$\Lambda^3 T^*M = 2\mathbb{R} + [\![\Lambda^{1,0}]\!] + [\![S^{2,0}]\!],$$

where $[\![V]\!]$ denotes the real module such that $[\![V]\!] \otimes \mathbb{C} \cong V \oplus V^*$ and $\Lambda^{1,0} = \mathbb{C}^3$ as the standard representation of $SU(3)$. The intrinsic torsion ξ lies in

$$\begin{aligned} T^* \otimes \mathfrak{su}(3)^\perp &= [\![\Lambda^{1,0}]\!] \otimes \left(\mathbb{R} \oplus [\![\Lambda^{2,0}]\!]\right) \\ &= 2\mathbb{R} + 2\,\mathfrak{su}(3) + 2\,[\![\Lambda^{1,0}]\!] + [\![S^{2,0}]\!]\,. \end{aligned}$$

Note that each module in $\Lambda^3 T^*$ arises in the decomposition of $T^* \otimes \mathfrak{g}^\perp$. However, the requirement that ξ be totally skew is stronger than that ξ should have no component in the missing module $2\,\mathfrak{su}(3)$. Indeed, computation in standard bases shows that

$$\left(T^* \otimes \mathfrak{su}(3)^\perp\right) \bigcap \Lambda^3 T^* = 2\mathbb{R},$$

and we deduce that M^6 has totally skew intrinsic torsion if and only if

$$\xi = a\Omega_1 + b\Omega_2,$$

for some functions $a, b \in C^\infty(M)$.

In this case we may compute ξ using the Levi-Civita connection and ω as follows:

$$\nabla^{\mathrm{LC}}\omega = \tilde{\nabla}\omega + \xi\omega = \xi\omega,$$

where $(\xi\omega)(X, Y, Z) = \omega(\xi_X Y, Z) + \omega(Y, \xi_X Z)$. We see that if ξ is totally skew then $\xi\omega \in \Lambda^3 T^* M$. Since ∇^{LC} is torsion-free we deduce that this is equivalent to

$$\nabla^{\mathrm{LC}}\omega = 3d\omega,$$

which is the condition that (M, g, I) be *nearly Kähler*. If $\nabla^{\mathrm{LC}}\omega \neq 0$ then Gray [4] showed that g is Einstein with positive scalar curvature. So we have a first example of a condition on the intrinsic torsion that leads to Einstein metrics.

Remark 2.2. We could have considered the above example with the larger structure group $G = U(3)$. Then the geometry would have been specified by g and I alone. However, if ξ is non-zero and totally skew then the conclusion $\xi\omega = \nabla^{\mathrm{LC}}\omega = 3d\omega$ still holds. We may now use $d\omega$ and $*d\omega$ to define a complex volume Ω and a reduction of the structure group to $SU(3)$, and thus return to the situation of the above example.

Remark 2.3. One geometry that arises in string theory is KT geometry. The data here is a metric and an *integrable* complex structure I, both preserved by a connection whose torsion is totally skew. The integrability of I, is exactly the condition that the intrinsic torsion ξ has no component of type $(3, 0)$. But if ξ is also totally skew, then our computations above say that $\xi = \alpha + \overline{\alpha}$ for an element $\alpha \in \Lambda^{3,0} T^* M$, so ξ is forced to be zero and we have a Kähler structure. In this way, the condition of totally skew intrinsic torsion is seen to be 'orthogonal' to KT geometry. Similar remarks apply to HKT and QKT geometries.

Example 2.4. As a second example consider a seven-manifold M with a G_2-structure. This structure identifies T_xM with the imaginary octonions $\operatorname{Im}\mathbb{O}$. The Lie group G_2 is by definition the group that preserves octonion multiplication $(X,Y)\mapsto X.Y$. Let V_7 denote $\operatorname{Im}\mathbb{O}$ as a representation of G_2. We have $\Lambda^2V_7=\mathfrak{g}_2\oplus V_7$, so $\mathfrak{g}_2^\perp=V_7$, and find

$$\left(T^*M\otimes\mathfrak{g}_2^\perp\right)\bigcap\Lambda^3T^*M=\mathbb{R}$$

is spanned by the invariant three-form $\varphi(X,Y,Z)=g(X.Y,Z)$. Thus if the intrinsic torsion is totally skew, then $\xi=f\varphi$ for some function f. We now have

$$\nabla^{\mathrm{LC}}\varphi=f\varphi\varphi=f*\varphi.$$

This equation gives $d\varphi=\frac{1}{4}f*\varphi$, which implies that f is constant. In the terminology of Gray [3], M has weak holonomy G_2. If $f\neq0$, Gray showed that this implies that g is Einstein of positive scalar curvature.

3. Invariant Intrinsic Torsion

The two examples given in the previous section have the common property that the intrinsic torsion ξ lies in a trivial G-submodule of $T^*M\otimes\mathfrak{g}^\perp$, so pointwise ξ is invariant under the structure group G. In this section, we will discuss how the condition of invariant intrinsic torsion interacts with curvature and in particular the Einstein equations.

The curvatures R^{LC} of ∇^{LC} and $\tilde R$ of $\tilde\nabla$ are related schematically as follows

$$R^{\mathrm{LC}}=\tilde R+(\tilde\nabla\xi)+(\xi^2),$$

where $(\tilde\nabla\xi)$ and (ξ^2) denote terms determined by linear algebraic relations from the covariant derivative $\tilde\nabla\xi$ and from $\xi^2=\xi\otimes\xi$, respectively. Since R^{LC} is the curvature of g we have that R^{LC} lies in $S^2\Lambda^2T^*M$, however we have the splitting $\Lambda^2T^*M=\mathfrak{g}\oplus\mathfrak{g}^\perp$, so we may write

$$R^{\mathrm{LC}}=R^{\mathfrak{g}}+R^{\mathrm{m}}+R^\perp,$$

corresponding to the decomposition

$$S^2(\Lambda^2T^*M)=S^2\mathfrak{g}+\mathfrak{g}\vee\mathfrak{g}^\perp+S^2(\mathfrak{g}^\perp).$$

As $\tilde\nabla$ takes values in $\mathfrak{g}$, the components R^{m} and $R^\perp$ are algebraically determined by ξ^2 and $\tilde\nabla\xi$.

The first component $R^{\mathfrak{g}}$ of R^{LC} may be further split as

$$R^{\mathfrak{g}}=R^{\mathfrak{g}}_0+R^{\mathfrak{g}}_1$$

according to

$$S^2\mathfrak{g}=\mathcal{K}(\mathfrak{g})\oplus\mathcal{K}(\mathfrak{g})^\perp,$$

where $\mathcal{K}(\mathfrak{g})$ is the kernel of the Bianchi map $\mathbf{b}\colon S^2\,\mathfrak{g} \to S^2\Lambda^2T^*M \to \Lambda^4T^*M$. Thus, $\mathcal{K}(\mathfrak{g})$ is the space of algebraic curvature tensors with holonomy algebra contained in $\mathfrak{g}$. The remaining component $R^{\mathfrak{g}}_1$ is now uniquely determined by $R^{\mathfrak{g}}_0$, $\tilde{\nabla}\xi$ and ξ^2 via the condition $\mathbf{b}(R^{\mathrm{LC}}) = 0$

The Einstein equations say that the trace-free part Ric_0 of the Ricci tensor Ric is zero. Writing V for the G-module T^*M we have $\mathrm{Ric}_0 \in S^2_0V$. So for g to be Einstein, it is sufficient that R^{LC} has no components in G-modules isomorphic to S^2_0V or any of its submodules.

Suppose that the intrinsic torsion ξ lies in a trivial module $\mathbb{R}$. Then we have $\xi^2 \in \mathbb{R}$ and $\tilde{\nabla}\xi \in V$ and the above analysis now gives

$$R^{\mathrm{LC}} \in \mathbb{R} + V + \mathcal{K}(\mathfrak{g}).$$

We therefore deduce that when $\xi \in (T^*M \otimes \mathfrak{g}^{\perp})^G$, the following three conditions are sufficient to imply that g is Einstein:

(a) $(S^2_0V)^G = \{0\}$,

(b) $(V \otimes S^2_0V)^G = \{0\}$ and

(c) $(\mathcal{K}(\mathfrak{g}) \otimes S^2_0V)^G = \{0\}$.

These representation theoretic conditions have interesting interpretations. Firstly, condition (a) implies that V is an irreducible representation. The irreducibility of V together with condition (b) imply that the G-invariant elements of $T^*M \otimes \Lambda^2T^*M$ are totally skew, since the complement of Λ^3T^*M in this space is contained in $S^2T^*M \otimes T^*M$. So (a) and (b) imply that ξ is an element of Λ^3T^*M. Thus the Einstein equations lead to consideration of manifolds with totally skew intrinsic torsion.

Condition (c) is rather more complex. However, Schwachhöfer's recent algebraic proof [7] of the holonomy classification yields much information about $\mathcal{K}(\mathfrak{g})$. Indeed, $\mathcal{K}(\mathfrak{g})$ equals $\mathcal{K}(\underline{\mathfrak{g}})$, where $\underline{\mathfrak{g}} = \{R_{X,Y} : R \in \mathcal{K}(\mathfrak{g}),\ X, Y \in V\}$ is the *Berger algebra* of $\mathfrak{g}$. If $\mathfrak{g}$ acts irreducibly on V then one can show that $\underline{\mathfrak{g}}$ does too unless $\mathcal{K}(\mathfrak{g}) = \{0\}$. But the Berger algebras $\underline{\mathfrak{g}}$ that act irreducibly on V are exactly the holonomy algebras of (1.1–1.3). Pushing the analysis of the previous section further, one finds that there are very few holonomy representations that admit invariant three-forms. Two cases are those of G_2 on $\mathbb{R}^7$ and $SU(3)$ on $\mathbb{R}^6$. The remaining cases are where $V \cong \mathfrak{g}$, however the three-forms in this case can not represent intrinsic torsion of a G-connection. Thus if we have non-zero invariant intrinsic torsion and conditions (a) and (b) are satisfied then either we have the 6

K	G	dim	
$Sp(n)$	$SO(n)\,Sp(1)$	$\frac{3}{2}(n+2)(n-1)$	$n \geqslant 3$
$SO(4n)$	$Sp(n)\,Sp(1)$	$3(2n+1)(n-1)$	$n \geqslant 3$
$Sp(2)$	$Sp(1)$	7	†
$Sp(7)$	$Sp(3)$	84	
$Sp(10)$	$SU(6)$	175	
$Sp(16)$	$Spin(12)$	462	
$Sp(28)$	E_7	1463	
$SO(14)$	$Sp(3)$	70	
$SO(16)$	$Spin(9)$	84	
$Spin(7)$	G_2	7	†
G_2	$SU(2)$	11	
F_4	$G_2\,Sp(1)$	35	
E_7	$G_2\,Sp(3)$	98	
E_7	$F_4\,Sp(1)$	78	
E_8	$F_4\,G_2$	182	

TABLE 1. The isotropy irreducible spaces K/G satisfying the hypotheses of Theorem 3.1. Note that cases † have weak holonomy G_2.

and 7 dimensional representations of G_2 and $SU(3)$ and $\mathcal{K}(\mathfrak{g})$ is known to only contain Ricci-flat metrics, or $\mathcal{K}(\mathfrak{g}) = \{0\}$. In all cases, condition (c) is automatically satisfied.

When $\mathcal{K}(\mathfrak{g}) = \{0\}$, the curvature is fully determined by ξ^2 and $\tilde{\nabla}\xi$. As $\tilde{\nabla}\xi$ lies in V, only ξ^2 can contribute to the scalar curvature of the metric. If ξ is locally a function times a parallel tensor, then the Einstein condition implies that the scalar curvature is constant and forces ξ itself to be parallel. Such a situation occurs when the space of invariant three-forms is one-dimensional over either $\mathbb{R}$ or $\mathbb{C}$ and one obtains the following results.

Theorem 3.1. *Let (M, g) be a Riemannian manifold with a reduction of its structure group to G. Suppose the intrinsic torsion ξ is invariant under the structure group and non-zero. If the cotangent representation $V = T^*M$ satisfies conditions* (a) *and* (b) *and* $\dim_{\mathbb{R}}(\Lambda^3 V)^G = 1$, *then M is Einstein. If the scalar curvature is non-zero then either M has weak holonomy G_2 or M is locally isometric to one of the spaces in Table 1.*

Theorem 3.2. *Let (M, g) be a Riemannian manifold with a reduction of its structure group to G. Suppose the intrinsic torsion ξ is invariant under the structure*

K	G	dim
$SO(8)$	$SU(3)$	20
G_2	$SU(3)$	6 ‡
F_4	$SU(3)^2$	36
E_6	$SU(3)$	54
E_7	$SU(6)\,SU(3)$	90
E_8	$SU(9)$	168
E_8	$E_6\,SU(3)$	162

Table 2. The isotropy irreducible spaces K/G satisfying the hypotheses of Theorem 3.2. Note that case ‡ is nearly Kähler.

group and non-zero. If the cotangent representation $V = T^*M$ is of complex type, satisfies conditions (a) *and* (b) *and* $\dim_{\mathbb{C}}(\Lambda^3 V)^G = 1$*, then M is Einstein. If the scalar curvature is non-zero then either M has weak holonomy $SU(3)$ or M is locally isometric to one of the spaces in Table 2.*

In this theorem, V is of complex type if $V \otimes \mathbb{C} = W \oplus \overline{W}$ with $W \ncong \overline{W}$.

The spaces in the tables are homogeneous spaces K/G such that G acts irreducibly on $\mathfrak{k}/\mathfrak{g}$. In fact any homogeneous Riemannian manifold $M = K/G$ has a reductive decomposition

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{m},$$

(see [5]), and a canonical G-connection whose torsion is $c(X, Y, Z) = B([X, Y], Z)$, where $-B$ is the Killing form of $\mathfrak{k}$. Suppose B is positive definite on $\mathfrak{m}$. Then B induces an invariant Riemannian metric on M. The G-structure has c as its intrinsic torsion if $(\mathfrak{g} \otimes \mathfrak{m})^G = \{0\}$. If $c = 0$, then M is a symmetric space. So non-zero intrinsic torsion together with condition (a), implies that M is isotropy irreducible. However, Theorems 3.1 and 3.2 do not exhaust all these spaces. In Table 3, we list those that satisfy conditions (a–c). They all share the property that $\dim_{\mathbb{R}}(\Lambda^3 V)^G = 2$, but V is not of complex type.

4. An Example in Dimension 77

Let us consider one particular geometry from Table 3: the model space $SO(14)/G_2$. The isotropy representation V has dimension 77 and dominant weight $0\Rrightarrow 3$. We may build other homogeneous spaces with such a $(G_2, 0\Rrightarrow 3)$-structure as follows. Let W be the Lie algebra $\mathfrak{g}_2$ of G_2 regarded as a representation of $U(1)$ for any circle in G_2. As G_2 has rank two, W contains a trivial $U(1)$-module and we may

K	G	dim
$SO(42)$	$Sp(4)$	825
$SO(128)$	$Spin(16)$	8008
$SO(14)$	G_2	77
$SO(26)$	F_4	273
$SO(52)$	F_4	1274
$SO(78)$	E_6	2925
$SO(133)$	E_7	8645
$SO(248)$	E_8	30380

TABLE 3. Strongly isotropy irreducible spaces satisfying conditions (a–c) but not the hypotheses of Theorems 3.1 or 3.2.

write $W = \mathbb{R} + U$. Now

$$V = \Lambda^2 W \ominus W = \Lambda^2 U \oplus U \ominus W = \Lambda^2 U \ominus \mathbb{R}.$$

Now $\dim U = 13$ and $U(1)$ acts preserving an inner product on U. We may thus regard $U(1)$ as a subgroup of $SO(13)$. We have $\Lambda^2 U \cong \mathfrak{so}(13)$ and $V = \Lambda^2 U \ominus \mathbb{R}$ is the isotropy representation of $SO(13)/U(1)$. As this construction holds for any circle subgroup of G_2 we thus obtain a countably infinite family of manifolds $SO(13)/U(1)$ carrying $(G_2, 0\Rrightarrow 3)$-structures.

Do these $(G_2, 0\Rrightarrow 3)$-structures have invariant intrinsic torsion? Suppose M is a 77-dimensional manifold with a $(G_2, 0\Rrightarrow 3)$-structure with invariant intrinsic torsion. Note that for this representation the space of algebraic curvature tensors is trivial $\mathcal{K}(\mathfrak{g}_2, 0\Rrightarrow 3) = \{0\}$. This implies that the Riemannian curvature is algebraically determined by $\tilde{\nabla}\xi$ and ξ^2. Examination of the contribution from $\tilde{\nabla}\xi$ shows that this component is non-trivial in both $\mathfrak{g}_2 \wedge \mathfrak{g}_2^{\perp}$ and $S^2\mathfrak{g}_2$. However, $\tilde{\nabla}\xi$ lives in a module isomorphic to $0\Rrightarrow 3$ and $0\Rrightarrow 3$ does not occur as a submodule of

$$S^2\mathfrak{g}_2 = \mathbb{R} + S^2_0 V_7 + (2\Rrightarrow 0)$$

(even though $2\Rrightarrow 0$ also happens to have dimension 77). Thus $\tilde{\nabla}\xi$ must be zero. We now have that $\tilde{R}$ is algebraically determined by ξ^2 and is $\tilde{\nabla}$-parallel. In other words $\tilde{\nabla}$ is a connection whose torsion and curvature are $\tilde{\nabla}$-parallel. This means that $\tilde{\nabla}$ is an Ambrose-Singer connection. As $0\Rrightarrow 3$ is irreducible the theory of infinitesimal models may now be used to show that M is locally isometric to the homogeneous space $SO(14)/G_2$. Thus the family $SO(13)/U(1)$ gives no new Einstein metrics.

It is expected that the ideas of this example will lead to generalisations of Theorems 3.1 and 3.2. This is being pursued in ongoing work.

References

[1] H. Baum, Th. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, B. G. Teubner Verlagsgesellschaft, Stuttgart, Leipzig, 1991.

[2] R. Cleyton, *G-structures and Einstein metrics*, Ph.D. thesis, University of Southern Denmark, Odense, 2001, `ftp://ftp.imada.sdu.dk/pub/phd/2001/24.PS.gz`.

[3] A. Gray, *Weak holonomy groups*, Math. Z. **123** (1971), 290–300.

[4] ______, *The structure of nearly Kähler manifolds*, Math. Ann. **223** (1976), 233–248.

[5] O. Kowalski and J. Szenthe, *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*, Geom. Dedicata **81** (2000), no. 1-3, 209–214, see also [6].

[6] ______, *Erratum: "On the existence of homogeneous geodesics in homogeneous Riemannian manifolds" [Geom. Dedicata* **81** *(2000), no. 1-3, 209–214]*, Geom. Dedicata **84** (2001), no. 1-3, 331–332.

[7] L. J. Schwachhöfer, *On the classification of holonomy representations*, Habilitationsschrift, Unvirsität Leipzig, 1998.

[8] A. F. Swann, *Weakening holonomy*, Proceedings of the Second Meeting on Quaternionic Structures in Mathematics and Physics, Roma, September 6–10, 1999 (Singapore) (S. Marchiafava, P. Piccinni, and M. Pontecorvo, eds.), World Scientific, 2001, electronic `http://www.univie.ac.at/EMIS/proceedings/QSMP99/contents.html`, pp. 405–415.

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