# Cohomological Aspects of Noether Theorem for Lagrangians of Classical Mechanics

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*For a given configuration space $M$ and Lie algebra $\mathcal{G}$ whose action is defined on $M$ the space $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians (i.e. Lagrangians whose motion equations left hand sides are $\mathcal{G}$-invariant) is studied. The problem is reformulated in terms of the double complex of Lie algebra cochains with values in the complex of Lagrangians. Calculating the cohomology of this complex using the method of spectral sequences we arrive at the hierarchy in the space $\mathcal{V}_{0.0}$ corresponding to the cohomologies of the algebra $\mathcal{G}$ and configuration space $M$.*

## I Introduction

The cohomology of the symmetries algebra has important consequences for properties of corresponding theory [1,2] and cohomological methods play essential role in many problems of modern fields theory.

In this paper we study relations between Noether identities and related phenomena for global symmetries of Lagrangians and cohomological properties of the algebra of these symmetries.

Our considerations will be carried out for mechanics but the scheme has straightforward generalization on the case of field theory Lagrangians.

The standard statement (Noether 1-st Theorem) is that if the Lagrangian $L$ is invariant under the action of Lie algebra $\mathcal{G}$ of rigid symmetries $\{\delta_i\}$ then to every symmetry $\delta_i$ corresponds the charge $N_i(L)$ which is preserved on the equations of motion [3].

If to $\{\delta_i\}$ corresponds the Lie algebra of vector fields $\{\mathbf{X}_i = X_i^\mu \frac{\partial}{\partial q^\mu}\}$ (infinitesimal transformations of configuration space) then $\delta_i q^\mu \sim X_i^\mu$,

$$\delta_i L = 0 = \mathcal{L}_{\mathbf{X}_i} L = X_i^\mu \mathcal{F}_\mu(L) + \frac{d}{dt}\left(X_i{}^\alpha \frac{\partial L}{\partial \dot q^\alpha}\right) \text{ and}$$

$$N_i = X_i{}^\mu \frac{\partial L}{\partial \dot q^\mu}, \qquad \text{where} \qquad \mathcal{F}_\mu(L) = \frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\mu} \tag{1.1}$$

is the left hand side (l.h.s.) of the equations of motion $\mathcal{F}_\mu = 0$ of the Lagrangian $L$.

The statement of Noether theorem is valid also in a case if under the actions of transformations $\{\delta_i\}$ Lagrangian is preserved up to a full derivative of some functions $\{\alpha_i(q)\}$

$$\delta_i L = 0 \to \delta_i L = d\alpha_i, \text{ then } N_i(L) \to N_i(L) - \alpha_i\,. \tag{1.2}$$

At what extent this full derivative is essential? The redefinition of $L$ on a full derivative $L \to L + df$ changes $\alpha_i$ on $\alpha_i + \delta_i f$. The algebra of symmetries of the Lagrangian can be considered as generalized if $d\alpha_i$ is not equal 0 in (1.2), and it is *essentially generalized* if it cannot be canceled by redefinition of Lagrangian on a full derivative i.e. $\delta_i L = d\alpha_i$ but the equations

$$d(\alpha_i + \delta_i f) = 0\,. \tag{1.3}$$

have no solutions.

Using the basic properties of operators $\delta$ and $d$: $\delta^2 = d^2 = 0$, $d\delta = \delta d$ we obtain from (1.2) that

$$0 = \delta^2 L = \delta d\alpha_i = d\delta\alpha_i, \text{ so } (\delta\alpha)_{ij} = w_{ij} = constant, \tag{1.4}$$

where $(\delta\alpha)_{ij} = \mathcal{L}_i\alpha_j - \mathcal{L}_j\alpha_i - \alpha_k c_{ij}^k$ and $c_{ij}^k$ are structure constants of the symmetries Lie algebra.

It is easy to see that $w_{ij}$ is the cocycle of algebra $\mathcal{G}$ in constants. In a case if $w_{ij}$ is not coboundary one can see that the symmetries are essentially generalized. Indeed if according to (1.3) $\alpha_i = -\delta_i f + t_i$ where $t_i$ are constants then $w_{ij}$ in (1.4) is coboundary in constants: $w_{ij} = (\delta t)_{ij} = -c_{ij}^k t_k$.

Let us consider for example the algebra of space translations. This algebra has 2-cohomology in constants which are represented by antisymmetric tensors $B_{ik}$. (This algebra is abelian, so $\delta B = 0$ and the equation $B = \delta t$ has no solutions in constants.) To obtain Lagrangian which possesses generalized translation symmetries corresponding to these cocycles, we note that for this Lagrangian $\alpha_i = A_{ij}q^j$. By redefinition of a Lagrangian on a full derivative one can reduce $A_{ij}$ to antisymmetric tensor and we come to Lagrangian

$$L = f(\dot{q}) + q^i B_{ij}\dot{q}^j\,. \tag{1.5}$$

If $f(\dot{q}) = \frac{m\dot{q}^2}{2}$ it is the well-known Lagrangian of particle in constant magnetic field.

One can consider an analogous statement for Galilean group: one comes to the Lagrangian of free particle as a unique Lagrangian corresponding to Bargmann cocycle of Lie algebra of Galilean group (See for example [4]).

We see that one of the reasons of generalized symmetries appearing is the existence of 2-cohomology of corresponding Lie algebra [2,4,5]. Of course the situation is more complicated. For example by evident reasons de Rham cohomology of configuration space is responsible for this phenomenon also: If $L_{inv}$ is $\mathcal{G}$-invariant Lagrangian and $L = L_{inv} + A_\mu(q)\dot{q}^\mu$ where $A_\mu(q)dq^\mu$ is a closed differential 1-form which is not exact $(A_\mu(q)dq^\mu \neq df)$ then in general $L$ is not $\mathcal{G}$-invariant. It has the same equations of motion but it differs from $L_{inv}$ on Aharonov-Bohm like effects.

Even in the case if de Rham cohomology is trivial and the cocycle $w_{ij}$ in (1.4) is coboundary the symmetries of Lagrangian can be essentially generalized. The coboundary condition $w_{ij} = -c_{ij}^k t_k$ is necessary but not sufficient for (1.3) to have a solution. It is another cohomologies of symmetries algebra which prevent a Lagrangian to be reduced to $\mathcal{G}$-invariant by redefinition on a full derivative.

Our purpose is to investigate systematically this phenomenon.

We are fixing configuration space $M$ and finite-dimensional algebra $\mathcal{G}$ of its transformations which acts on $M$ via corresponding fundamental vector fields:

$$\mathcal{G} \ni x \xrightarrow{\Phi} \tilde{x} \ (\text{fundamental vector fields}): \quad [\tilde{x}, \tilde{y}] = \widetilde{[x, y]} \,. \tag{1.6}$$

Then we study relations between Lagrangians on $M$ and the cohomologies of algebra $\mathcal{G}$ and $M$ and establish the hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians on $M$. This hierarchy relates the phenomena discussed above with cohomologies groups of the Lie algebra $\mathcal{G}$ and the configuration space $M$.

From considerations above we see that in the phenomena which we are investigating are interplaying two differentials $\delta$ and $d_{E.L}$ where the differential $\delta$ corresponds to the symmetries Lie algebra and $d_{E.L}$ is the prolongation of exterior differential which acts on Lagrangians. It is variational derivative, whose action leads to Euler-Lagrange equation.*

These differentials as well as differentials $d$ and $\delta$ satisfy the conditions: $\delta^2 = d_{E.L}^2 = d_{E.L}\delta - \delta d_{E.L} = 0$. We naturally come to the differential $Q = d_{E.L} \pm \delta$ of the double complex $(E^{*.*}, d_{E.L}, \delta)$:

$$\begin{array}{ccccccc}
\Lambda^0(M) & \xrightarrow{d_{E.L}} & \Lambda^1(M) & \xrightarrow{d_{E.L}} & d_{E.L}\Lambda^1(M) & \xrightarrow{d_{E.L}} & 0 \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
C^1(\mathcal{G}, \Lambda^0(M)) & \xrightarrow{d_{E.L}} & C^1(\mathcal{G}, \Lambda^1(M)) & \xrightarrow{d_{E.L}} & C^1(\mathcal{G}, d_{E.L}\Lambda^1(M)) & \xrightarrow{d_{E.L}} & 0 \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
C^2(\mathcal{G}, \Lambda^0(M)) & \xrightarrow{d_{E.L}} & C^2(\mathcal{G}, \Lambda^1(M)) & \xrightarrow{d_{E.L}} & C^2(\mathcal{G}, d_{E.L}\Lambda^1(M)) & \xrightarrow{d_{E.L}} & 0 \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{1.7}$$

Here $\Lambda^0(M)$ is the space of functions on $M$, $\Lambda^1(M)$ is the space of classical mechanics Lagrangians on $M$, i.e. Lagrangians which depend only on coordinates and velocities. $C^q(\mathcal{G}, A)$ is the space of cochains on the Lie algebra $\mathcal{G}$ which take values in $\mathcal{G}$-module $A$. ($\mathcal{G}$-module structure in corresponding spaces is induced by the action (1.6) of the Lie algebra $\mathcal{G}$ on the configuration space $M$).

The phenomena considered above can be interpreted in terms of this double complex. For example to condition $\delta L = d\alpha$ in (1.2) corresponds the condition $Q(L, \alpha_i) =$

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* If $L$ is Lagrangian which is defined on $q$-dimensional paths $q^\mu(\xi^1, \dots, \xi^q)\colon R^q \to M$, $L \in \Lambda^q$ then $d_{E.L}\colon \Lambda^q \to \Lambda^{q+1}$, $\quad d_{E.L} L\left(q, \frac{\partial q^\mu}{\partial \xi^{\tilde{a}}}, \dots, \frac{\partial q^\mu}{\partial \xi^{\tilde{a}^1}, \dots, \partial \xi^{\tilde{a}^k}}\right) = \mathcal{F}_\mu(L) \frac{\partial q^\mu}{\partial \xi^{q+1}}$ . where $\tilde{\alpha} = (1, \dots, q, q+1), \alpha = (1, \dots, q)$ and $\mathcal{F}_\mu(L)$ are l.h.s. of Euler–Lagrange equations of the Lagrangian $L$, i.e. variational derivatives of the corresponding functional: $\mathcal{F}_\mu(L) = \frac{\delta}{\delta q^a} S_L([q(\xi^\alpha)])$. If $w$ is differential form then $d_{E.L} L_w = L_{dw}$ where $L_w = n! w_{i_1 \dots i_q}(q) \frac{\partial q^{i_1}}{\partial \xi^1} \cdot \ldots \cdot \frac{\partial q^{i_q}}{\partial \xi^q}$. (See in details [6,7].)

$(d_{_{E.L}}L, 0, w = \delta\alpha)$. To redefinition of Lagrangian on a full derivative $L \to L + df$ corresponds the changing of the cochain $(L, \alpha_i)$ on coboundary: $(L, \alpha_i) \to (L, \alpha_i) + Qf = (L + df, \alpha_i + \delta_i f)$.

*It is the cohomology of the differential $Q$ of the complex (1.7) which allow us to reveal the relations between generalized symmetries of Lagrangians and cohomologies of the configuration space and the symmetries Lie algebra.*

We do it in a following way. Using a technique of spectral sequences we calculate first the spectral sequence $\{E_r^{*.*}\}$ of double complex (1.7), i.e. we calculate perturbatively the cohomology of $Q$ via cohomology of $d_{_{E.L}}$ by modulo $\delta$.

Then vice versa we calculate the transposed spectral sequence $\{{}^tE_r^{*.*}\}$ of (1.7), i.e. we calculate perturbatively the cohomology of $Q$ via cohomology of $\delta$ by modulo $d_{_{E.L}}$. Calculating cohomology of operator $Q$ in the first way we come to the spaces $K_s$ which are expressed in terms of cohomologies of Lie algebra and configuration space. On the other hand, calculating the same cohomology in the second way, we come naturally to the space $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians and to its subspaces $\{\mathcal{V}_{s.\sigma}\}$ $(s = 0,1,2,3,4, \sigma = 0,1)$. These subspaces define double filtration in the space $\mathcal{V}_{0.0}$ ($\mathcal{V}_{s+1.\sigma} \subseteq \mathcal{V}_{s.\sigma}$ and $\mathcal{V}_{s.\sigma+1} \subseteq \mathcal{V}_{s.\sigma}$) and every subspace $\mathcal{V}_{s.\sigma}$ is determined by physical properties of weakly invariant Lagrangians belonging to this subspace.

Natural relations which arise between the results of calculations in the first and in the second way lead to the sequence of homomorphisms between the spaces $\{\mathcal{V}_{s.\sigma}\}$ and $\{K_s\}$ which define these spaces in a recurrent way via the kernels of corresponding homomorphisms.

These constructions establish hierarchy in the space of weakly $\mathcal{G}$-invariant Lagrangians making links between the physical properties of Lagrangians and pure mathematical objects: The condition that Lagrangian belongs to some space $\mathcal{V}_{s.\sigma}$ and it *does not belong to the space* $\mathcal{V}_{s+1.\sigma}$ or $\mathcal{V}_{s.\sigma+1}$ in terms of this hierarchy is reformulated to the condition that the value of the corresponding homomorphism on it, is not equal to zero. The problem of analyzing the content of the spaces $\{\mathcal{V}_{s.\sigma}\}$ and their differences is reduced to the problem of calculating the corresponding homomorphisms. For example in the case if the space $K_s$ is trivial, then $\mathcal{V}_{s-1.\sigma} = \mathcal{V}_{s.\sigma}$. In particular if all the spaces $K_s$ are trivial then all weakly invariant Lagrangians are invariant (up to a full derivative).

We present the corresponding calculations and results in the second Section.

We have to note that the phenomena which we study here are actively investigated and used, specially the relations between two cocycles of symmetries Lie algebra and corresponding properties of invariant Lagrangians (see for example [2, 4]). The aim of this paper is to put all weakly invariant Lagrangians on corresponding "floors": to classify the relations between symmetries properties and pure mathematical constructions using the language of spectral sequences. In the third Section we discuss possible applications of our method.

In Appendix we consider two examples of hierarchy diagram. (See in details [7])

## II Calculation of Hierarchy

Now we present briefly the results of our calculations and formulate the basic Theorem. Our calculations are founded on the following remark:

Let $\mathcal{V}_{0.0}$ be the subspace of weakly $\mathcal{G}$-invariant Lagrangians i.e. Lagrangians whose motions equations l.h.s. are $\mathcal{G}$-invariant:

$$\mathcal{V}_{0.0} = \{L\colon L \in \Lambda^1 \text{ and } \delta d_{_{E.L}} L = 0\}\,. \tag{2.1}$$

A cochain $\mathbf{f} = (d_{_{E.L}} L, 0, 0)$ is the cocycle of differential $Q$ in (1.7) iff $L \in \mathcal{V}_{0.0}$. The cohomology class $[(d_{_{E.L}} L, 0, 0)]$ of this cocycle belongs to $H^2(Q)$. If we express the cohomology of differential $Q$ via the stable terms of transposed spectral sequence $\{{}^t E_r^{*.*}\}$, i.e. calculating $H^*(Q)$ in perturbation theory, considering in (1.7) the differential $\delta$ as zeroth order approximation for the differential $Q$, we see that $[d_{_{E.L}} \mathcal{V}_{0.0}]_\infty = {}^t E_\infty^{0.2}$ is the subspace of $H^2(Q)$. On the other hand if we express the cohomology of differential $Q$ via the stable terms of spectral sequence $\{E_r^{*.*}\}$, i.e. calculating $H^*(Q)$ in perturbation theory, considering in (1.7) the differential $d_{_{E.L}}$ as zeroth order approximation, we express $H^2(Q)$ in terms of $\{E_\infty^{p.2-p}\}$. The relations between the space ${}^t E_\infty^{0.2}$ and the spaces $\{E_\infty^{p.2-p}\}$ lead to the relations between the space of weakly $\mathcal{G}$-invariant Lagrangians and cohomologies groups of $\mathcal{G}$ and $M$. (See in details ([7]).)

The calculations give that

$$E_1^{*.*} = H(d_{_{E.L}}, E) = \begin{array}{ccc} \mathbf{R} & H^1(M)\oplus\mathbf{R} & 0 \\ C^1(\mathcal{G}) & C^1(\mathcal{G}, H^1(M)\oplus\mathbf{R}) & 0 \\ C^2(\mathcal{G}) & \cdots & 0 \\ \cdots & \cdots & 0 \end{array}$$

$$E_2^{*.*} = H(\delta, E_1^{*.*}) = \begin{array}{ccc} \mathbf{R} & H^1(M)\oplus\mathbf{R} & 0 \\ H^1(\mathcal{G}) & H^1(\mathcal{G})\otimes H^1(M)\oplus\mathbf{R} & 0 \\ H^2(\mathcal{G}) & \cdots & 0 \\ H^3(\mathcal{G}) & \cdots & 0 \end{array} \tag{2.2}$$

One can show that the spaces $\{E_2^{p.q}\}$ which are of interest for us $(p+q \le 2)$ are stable: $E_2^{p.q} = E_3^{p.q} = \ldots = E_\infty^{p.q}$.

Hence the tables (2.2) establish the relations between the spaces $H^m(Q)$ $(m = 0, 1, 2)$ and the spaces $E_\infty^{p.q}$. These relations define canonical projections

$$p_1\colon \quad H^1(Q) \longrightarrow H^1(M) + \mathbf{R} \tag{2.3}$$

and the isomorphism on the kernel of $p_1$:

$$\iota_1\colon \quad \mathbf{ker} p_1 \longrightarrow H^1(\mathcal{G})\,. \tag{2.4}$$

Correspondingly

$$p_2\colon \quad H^2(Q) \longrightarrow H^1(M)\otimes H^1(\mathcal{G}) \oplus H^1(\mathcal{G}) \tag{2.5}$$

and

$$\iota_2\colon \quad \mathbf{ker} p_2 \longrightarrow H^2(\mathcal{G})\,. \tag{2.6}$$

To (2.3), (2.4) corresponds also the isomorphism

$$H^1(M) \oplus H^1(\mathcal{G}) \oplus \mathbf{R} \longrightarrow H^1(Q): \quad [w] + t + c \longrightarrow [w + c, t + \pi w] \tag{2.7}$$

where $\pi w$ is 1-cochain on $\mathcal{G}$ which corresponds to 1-form $w$:

$$\pi w)(h) = w \rfloor \tilde{h} \tag{2.8}$$

($\tilde{h}$ is the fundamental vector field $\Phi h$ corresponding to the element $h$ of the Lie algebra $\mathcal{G}$ via (1.6).

Now we return again to the complex (1.7) and express the cohomologies of $H(Q)$ in terms of transposed spectral sequence $\{\,{}^tE_r^{*.*}\}$.

For constructing ${}^tE_1^{*.*}$ we have to consider as zeroth order approximation the cohomology of vertical differential $\delta$: $\{\,{}^tE_1^{*.*}\} = H(E^{*.*}, \delta)$.

We arrive at the table

$${}^tE_1^{*.*} = \begin{matrix} \Lambda^0_{inv} & \Lambda^1{}_{inv} & d_{_{E.L}}\mathcal{V}_{0.0} \\ H^1(\mathcal{G}, \Lambda^0(M)) & H^1(\mathcal{G}, \Lambda^1) & \cdots \\ H^2(\mathcal{G}, \Lambda^0(M)) & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \tag{2.9}$$

$\Lambda^0_{inv} = C^0(\mathcal{G}, \Lambda^0(M))$ is the space of the functions on $M$ which are invariant under the action of the Lie algebra $\mathcal{G}$. $\Lambda^1{}_{inv}$ is the space of $\mathcal{G}$-invariant Lagrangians from $\Lambda^1$.

The differential ${}^td_1$ which is generated by $d_{_{E.L}}$ acts in rows of the table ${}^tE_1^{*.*}$ (compare with the table (2.2)). For ${}^tE_2^{*.*} = H(\,{}^tE_1^{*.*},\ {}^td_1)$ we obtain

$${}^tE_2^{*.*} = \begin{matrix} \mathbf{R} & H^1_{inv}(M) \oplus \mathbf{R} & d_{_{E.L}}\mathcal{V}_{0.0}\big/(d_{_{E.L}}\Lambda^1{}_{inv}) \\ {}^tE_2^{1.0} & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \tag{2.10}$$

$H^1_{inv}(M)$ is the space of closed $\mathcal{G}$-invariant differential 1-forms factorised by the differentials of $\mathcal{G}$-invariant functions.

The space ${}^tE_2^{1.0}$ in (2.10) is the subspace of $H^1(\mathcal{G}, \Lambda^0(M))$. It contains the classes $[\alpha]$ from $H^1(\mathcal{G}, \Lambda^0(M))$ for whose the eq. $d\alpha = \delta L$ has the solution. We see that the table (2.10) is not stable in the spaces which we are interesting in, because the differential ${}^td_2$ acting from ${}^tE_2^{1.0}$ in ${}^tE_2^{0.2}$ is not trivial: ${}^td_2[\alpha] = {}^t[d_{_{E.L}}L]_2$. The next table ${}^tE_3^{*.*} = H(\,{}^tE_2^{*.*},\ {}^td_2)$ is stable in the spaces which we are interesting in:

$${}^tE_3^{*.*} = \begin{matrix} \mathbf{R} & H^1_{inv}(M) \oplus \mathbf{R} & \frac{d_{_{E.L}}\mathcal{V}_{0.0}\big/(d_{_{E.L}}\Lambda^1_{1\,inv})}{\mathbf{Im}({}^td_2\,{}^tE_2^{1.0})} \\ {}^tE_3^{0.1} & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \tag{2.11}$$

¿From the general properties of spectral sequences it follows that in (2.11) ${}^tE_3^{0.2} = {}^tE_\infty^{0.2}$ *is the subspace in* $H^2(Q)$ and the space ${}^tE_3^{1.0} = {}^tE_\infty^{1.0}$ (which is the subspace of ${}^tE_2^{1.0}$) is the factorspace of $H^1(Q)$ by the space ${}^tE_3^{0.1} = H_{inv}(M) \oplus \mathbf{R}$. Hence it follows that

$${}^tE_3^{1.0} = (H^1(M) \oplus H^1(\mathcal{G}))/H^1_{inv}(M)\,. \tag{2.12}$$

$H^1_{inv}(M)$ is considered as naturally embedded in $H^1(M)\oplus H^1(\mathcal{G})$. If $w\in H^1_{inv}(M)$ is trivial in $H^1(M)$ then $\delta f\in H^1(\mathcal{G})$ where $w=df$.

In the tables (2.9–2.11) every space ${}^tE_r^{1.0}$ is the subspace of previous one and correspondingly every space ${}^tE_r^{0.2}$ is the factorspace of previous one. We denote by $\Pi_r$ the homomorphism which put in correspondence to every weakly $\mathcal{G}$-invariant Lagrangian its equivalence class in the space ${}^tE_r^{0.2}$:

$$\Pi_r\colon\quad \mathcal{V}_{0.0}\to {}^tE_r^{0.2},\quad \Pi_r(L)={}^t[d_{E.L}L]_r.\quad \forall L\in\mathcal{V}_{0.0}\quad \mathbf{Im}\Pi_3\subseteq H^2(Q). \tag{2.13}$$

Comparing the content of the spaces $\{{}^tE_r^{1.0}\}$ and $\{{}^tE_r^{0.2}\}$ in the transposed spectral sequences (2.9–2.11) with the results above for spectral sequence $\{E_r^{*.*}\}$ we come to

**Proposition**

a) To weakly $\mathcal{G}$-invariant Lagrangians correspond elements in the space ${}^tE_3^{0.2}$, i.e. in $H^2(Q)$. Thus to these Lagrangians via homomorphisms $p_2$ and $\iota_2$ defined by (2.5, 2.6) correspond elements in $E_2^{1.1}$ or in $E_2^{2.0}$.

b) To weakly $\mathcal{G}$-invariant Lagrangians whose image in the space ${}^tE_3^{0.2}$ is equal to zero: $\Pi_3(L)=0$, correspond elements in ${}^tE_2^{0.2}$ which belong to the image of the differential ${}^td_2$. Thus to these Lagrangians correspond elements in ${}^tE_2^{1.0}$ which are defined up to the space ${}^tE_3^{1.0}$ defined by (2.11), which is the kernel of this differential.

c) The space ${}^tE_3^{1.0}$ is related with weakly $\mathcal{G}$-invariant Lagrangians whose image in the space ${}^tE_2^{0.2}$ is equal zero: $\Pi_2(L)=0$.

Now using this Proposition we formulate our main result.

We come to the following hierarchy in the space of $\mathcal{V}_{0.0}$ of weakly $\mathcal{G}$-invariant Lagrangians.

**Theorem**. Let the action of Lie algebra $\mathcal{G}$ on the configuration space $M$ is defined via corresponding fundamental fields (1.6) and $\mathcal{V}_{0.0}$ is the space of weakly invariant Lagrangians defined by (2.1). Then the following relations which establish the classification (hierarchy) in the space $\mathcal{V}_{0.0}$ are satisfied

$$\begin{array}{ccccccccc}
 & & \mathcal{V}_{4.1} & \subseteq & \mathcal{V}_{4.0} & & & & \\
 & & \cap\!| & & \cap\!| & & & & \\
K_4 & \xleftarrow{\phi_4} & \mathcal{V}_{3.1} & \subseteq & \mathcal{V}_{3.0} & \xrightarrow{\overline{\phi}_4} & K_4 & = & H^1(M)/(H^1_{inv}(M))_* \\
 & & \cap\!| & & \cap\!| & & & & \\
K_3 & \xleftarrow{\phi_3} & \mathcal{V}_{2.1} & \subseteq & \mathcal{V}_{2.0} & \xrightarrow{\overline{\phi}_3} & K_3 & = & \dfrac{H^1(\mathcal{G},\Lambda^0(M))}{(H^1(M\oplus H^1(\mathcal{G}))\big/H_{inv}(M)} \\
 & & \cap\!| & & \cap\!| & & & & \\
K_2 & \xleftarrow{\phi_2} & \mathcal{V}_{1.1} & \subseteq & \mathcal{V}_{1.0} & \xrightarrow{\overline{\phi}_2} & K_2 & = & H^2(\mathcal{G}) \\
 & & \cap\!| & & \cap\!| & & & & \\
K_1 & \xleftarrow{\phi_1} & \mathcal{V}_{0.1} & \subseteq & \mathcal{V}_{0.0} & \xrightarrow{\overline{\phi}_1} & K_1 & = & H^1(M)\otimes H^1(\mathcal{G}) \\
 & & & & \Psi\downarrow & & & & \\
 & & & & K_0=H^1(\mathcal{G}) & & & &
\end{array} \tag{2.14}$$

The double filtration $\{\mathcal{V}_{s.\sigma}\}$ is subordinated to the homomorphisms $\{\overline{\phi}_s,\phi_s,\Psi\}$:

$$\mathcal{V}_{s.0}=\mathbf{ker}(\overline{\phi}_s\colon\ \mathcal{V}_{s-1.0}\to K_s),\quad \mathcal{V}_{s.1}=\mathbf{ker}(\phi_s\colon\ \mathcal{V}_{s-1.1}\to K_s),$$

$$\mathcal{V}_{s.1} = \mathbf{ker}(\Psi\colon\ \mathcal{V}_{s.0} \to K_0), \quad \overline{\phi}_s\big|_{\mathcal{V}_{s-1.1}} = \phi_s.$$

We say that (2.14) is the hierarchy diagram $\mathcal{D}([\mathcal{G}, M])$ for the pair $[\mathcal{G}, M]$. ($[\mathcal{G}, M]$ pair is defined by the action of Lie algebra $\mathcal{G}$ on the configuration space $M$ via corresponding fundamental fields (1.6)).

We analyze now the content of the spaces $\mathcal{V}_{s.\sigma}$ in the hierarchy diagram (2.14) and sketch the construction of homomorphisms $\{\Psi, \phi, \bar{\phi}\}$. (See for details [7].)

One can see that for an arbitrary $L$ in $\mathcal{V}$ the condition that $L \in \mathcal{V}_{0.0}$ is equivalent to the condition that the cochain $\delta L$ takes values in closed differential forms + constants:

$$\delta d_{_{E.L}} L = 0 \Leftrightarrow \delta_i L = w_{i\mu}\dot{q}^\mu + c_i \text{ and } dw_i = dc_i = 0\,. \tag{2.15}$$

($\delta_i L$ is the value of the cochain $\delta L$ on the basis vector $e_i$ of the Lie algebra $\mathcal{G}$.)

The value of homomorphism $\Psi$ on $L$ is equal to cohomological class of $c_i$ in $H^1(\mathcal{G})$. The value of homomorphism $\bar{\phi}_1$ on $L$ is equal to cohomological class of cochain $w_{i\mu}dx^\mu$ in $H^1(\mathcal{G}) \otimes H^1(M)$. If

$$\Psi(L) = 0 \tag{2.16}$$

and $\bar{\phi}(L) = 0$ then we come to (1.2). Hence the space $\mathcal{V}_{1.1}$ contains Lagrangians which possess generalized symmetries and have Noether charges with respect to the symmetries algebra $\mathcal{G}$. But in the case if (2.16) is not obeyed ($L \in \mathcal{V}_{0.0}$ and $L \notin \mathcal{V}_{0.1}$) then the corresponding Noether charges depend on time:

$$N_i = X_i{}^\mu \frac{\partial L}{\partial \dot{q}^\mu} - \alpha_i - c_i t \quad (c_i \neq 0)\,. \tag{2.17}$$

The Lagrangians in spaces $\mathcal{V}_{s.1}$ ($s \geq 1$) have time independent Noether charges. If Lagrangian $L \in \mathcal{V}_{1.1}$ then $w_{i\mu}dx^\mu = d\alpha_i$ and the value of homomorphism $\phi_2$ on it is equal to the cohomological class of $\delta\alpha$ in $H^2(\mathcal{G})$. (Compare with (1.4)). If $L \in \mathcal{V}_{2.1}$ ($\phi_2(L) = 0$) then one can see using Proposition that the cohomological class of $\alpha_i$ in $H^1(\mathcal{G}, \Lambda^0(M))$ represents the value of homomorphism $\phi_3(L)$ in $K_3$. The Lagrangians in $\mathcal{V}_{2.1}$ in general possess essentially generalized symmetry in spite of the fact that corresponding cocycle in $H^2(\mathcal{G})$ is trivial. In the case if $\phi_3(L) = 0$ then the Lagrangian $L$ belongs to $\mathcal{V}_{3.1}$ and it differs from $\mathcal{G}$-invariant on a closed form which is defined uniquely up to the exact form (full derivative) and $\mathcal{G}$-invariant closed form. It leads to the homomorphism $\phi_4(L)$. Lagrangians in $\mathcal{V}_{3.1}$ differ from $\mathcal{G}$-invariant on Bohm–Aharonov like effects. Finally the condition $\phi_4(L) = 0, L \in \mathcal{V}_{4.1}$ means that Lagrangian is $\mathcal{G}$-invariant up to a full derivative.

The similar analysis one can perform for spaces $\mathcal{V}_{s.0}$ which correspond to spaces $\mathcal{V}_{s.1}$. The homomorphisms $\bar{\phi}_s$ are similar to $\phi_s$. The weakly $\mathcal{G}$-invariant Lagrangian $L$ belongs to $\mathcal{V}_{1.0}$ if $\delta_i L = d\alpha_i + c_i$. It is easy to see that in this case $\delta\alpha$ is also 2-cocycle as in (1.4). $L \in \mathcal{V}_{2.0}$ if this cocycle is trivial, $L \in \mathcal{V}_{4.0}$ if $\alpha_i = \delta_i f$ and $L \in \mathcal{V}_{3.0}$ if it differs from $\mathcal{V}_{4.0}$ on a closed form. Lagrangians in $\mathcal{V}_{s.0}$ in general have time-dependent Noether currents (2.17).

The hierarchy diagram $\mathcal{D}([\mathcal{G}, M])$ measures the differences in the spaces $\{\mathcal{V}_{s.\sigma}\}$.

We say that weakly $\mathcal{G}$-invariant Lagrangian $L$ is on the floor "$s$" if $L \in \mathcal{V}_{s.0}$ and $L \notin \mathcal{V}_{s+1.0}$. (All Lagrangians from $\mathcal{V}_{4.0}$ are on the 4-th floor.)

We say that weakly $\mathcal{G}$-invariant Lagrangian $L$ is on the floor "$s_+$" if this Lagrangian is on the floor "$s$" and it belongs to $\mathcal{V}_{s.1}$. All other Lagrangians from the floor "$s$" are on the floor "$s_-$".

All Lagrangians which are on the " $+$ "-th floors have time independent Noether charges, except Lagrangians in zeroth floor.

The Lagrangians which are on the floor "$s$" have non-trivial image in the space $K_{s+1}$ in (2.14). The Lagrangian on floor "$s_-$" have also non-trivial image in $K_0$ under homomorphism $\Psi$.

We see that a Lagrangian which belongs to $\mathcal{V}_{s.\sigma}$ and which does not belong to $\mathcal{V}_{s+1.\sigma}$ does have non-trivial image in the space $K_{s+1}$.

The evident but important corollary of the hierarchy diagram is that the floor is empty if the corresponding space $K_s$ is trivial. For example in the case if the first de Rham cohomology of configuration space are trivial then $K_1 = K_4 = 0$ and the zeroth floor and the third floors are empty. In the case if the algebra $\mathcal{G}$ is semisimple only the floors $2_+, 3_+, 4_+$ can be non empty, because in this case $H^1(\mathcal{G}) = H^2(\mathcal{G}) = 0$, hence $K_0 = K_1 = K_2 = 0$.

The hierarchy diagram $\mathcal{D}([\mathcal{G}, M])$ will be called trivial if all the spaces $K_s$ are equal to zero.

In general the inverse statement is not valid. From the fact that the space $K_s$ is not trivial does not follow that the floor "$s-1$" is not empty, because the homomorphisms in (2.14) are not in general surjective. For example homomorphism $\phi_2$ in general is not surjective because the map ${}^t d_2$ which induces this homomorphism is defined on the subspace ${}^t E_2^{0.1}$ of the space $H^1(\mathcal{G}, \Lambda^0(M))$.

The hierarchy diagram can be used to classify weakly invariant Lagrangians for specially choosen symmetries Lie algebra $\mathcal{G}$ and configuration space $M$ [7]. For example one can show that Lagrangian corresponding to the interaction of free non-relativistic particle with Dirac monopole belongs to the second floor of the hierarchy corresponding to the $so(3)$ symmetries (see Appendix and [7]) and Lagrangian of free non- relativistic particle belongs to the first floor of the hierarchy corresponding to the Galilean algebra symmetries [7]. We briefly present in Appendix calculations for two examples.

Now we briefly discuss the properties of the spaces $\mathcal{V}_{s.\sigma}$ in the framework of Hamiltonian mechanics and in a quasiclassical approximation of quantum mechanics.

If the Lagrangian is $\mathcal{G}$-invariant, then to the Noether charges $N_i(L)$ in (1.1) in Hamiltonian framework correspond the charges $N_i^{ham} = X_i^\mu p_\mu$. They generate the $\mathcal{G}$-algebra structure via Poisson brackets

$$\{N_i^{ham}, N_j^{ham}\} = c_{ij}^k N_k^{ham}\,. \tag{2.18}$$

In quasiclassical approximation of quantum mechanics to these charges correspond the operators $X_i^\mu \hat{p}_\mu$ . Their action on quasiclassical wave function in configuration representation is reduced to infinitesimal transformation of wave functions argument:

$$\imath\hat{\delta}_i\Psi = \Psi(q^\mu + \delta_i q^\mu) - \Psi(q^\mu).$$

In the case if symmetries algebra is generalized, one can see that correspondingly to (1.2)

$$N_i^{ham} = X_i^\mu p_\mu - \alpha_i.$$

The corresponding operators act not only on quasiclassical wave functions argument but on its phase too:

$$\hat{\delta}_i \Psi = -\imath X_i^\mu \frac{\partial \Psi(q)}{\partial q^\mu} + \imath \alpha_i(q) \Psi(q^\mu) \,. \tag{2.19}$$

In the case if the Lagrangian is on the first floor i.e. the generalized symmetries lead to non-trivial cocycle, the Lie algebra of Hamiltonian Noether charges $N_i^{ham}$ is the central extension of the Lie algebra $\mathcal{G}$ which corresponds to the cohomology class of the cocycle $w_{ij}$.

$$\{N_i^{ham}, N_j^{ham}\} = c_{ij}^k N_k^{ham} + w_{ij} \,. \tag{2.20}$$

Correspondingly in this case in (2.19) is realized an essentially projective representation of the Lie algebra $\mathcal{G}$.

In the case if Lagrangian belongs to $\mathcal{V}_{2.1}$ then one can choose $\alpha_i$ such that (2.18) is satisfied and the quantum representation (2.19) of $\mathcal{G}$ becomes linear. But if this Lagrangian is on the 2-nd floor then the action of quantum transformation on the phase factor cannot be removed even locally by redefinition $\Psi \to e^{if}\Psi$ of the wave function corresponding to redefinition of Lagrangian on a full derivative. In this case one can say that the linear transformation (2.19) is splitted on a space-like transformation+intrinsic spin-like transformation. Nevertheless if Lagrangian is on the 3-d floor , i.e. it differs of an invariant Lagrangian on Bohm-Aharonov like effects, then the action on phase in (2.19) can be removed locally.

## III Discussions

Of course the problem considered here and the technique which we used to study it can be generalized in a few directions (See [7]). For example it is interesting to investigate in details the content of the second floor of the hierarchy diagram. It also seems to us fruitful to develop the considerations of this paper on the rigid base of investigations of A.M. Vinogradov and his collaborators [8].

But in our opinion, the method considered in this paper maybe is more important than the problem we applied it to.

We give here only three examples, one of them pure mathematical, where the calculations of double comlex cohomology (the method we use in this paper) make a bridge between the corresponding structures.

1. *The calculation of de Rham cohomology in terms of Chech cohomology.*

When manifold $M$ is covered by the family $\{U_\alpha\}$ of open sets one can consider Chech cohomology of this covering. Then one can consider double complex of $q$-forms which are defined on the sets $\{U_\alpha\}$. The differential $Q$ of this complex is the sum of de Rham exterior differential and Chech differential. Considering the differential $Q$ "perturbatively" around Chech differential one arrives naturally at de Rham cohomology of $M$, hence the "perturbative" calculations around de Rham differential lead in general to calculation of spectral sequence which tends to de Rham cohomology of $M$. In the case if the covering is Leray covering, i.e. all the sets and their intersections are convex connected sets then Chech cohomology coincide with de Rham one; the application of Poincaré lemma reduces spectral sequence calculations to trivial resolutions of descent equations. But practically it is more convenient to use for calculations a suitable covering which generally is not a Leray covering. (See for details e.g. [9].).

2. *The relations between Hamiltonian reduction method and BRST cohomology for classical mechanics*

BRST formalism at beginning was formulated in terms of symplectic geometry of phase space expanded by the ghosts and antighosts, then it was understood [10,11,12,13,14] that the language of homological algebra is more deeply related with physical meaning of this formalism: Inclusion of ghosts and antighosts corresponds to the construction of the chain of free modules (free resolvent) on phase space of constrained system where the constrains cannot be resolved in a direct way. The operator corresponding to BRST charge becomes the differential of the complex of these resolvent.

One can say that the relations between these two methods are encoded in the cohomology of double complex differential $Q = \partial + \delta$ in the case if constraints form Lie algebra (so called closed groups.) Here $\partial$ corresponds to Koszul differential of complex generated by constraints and $\delta$ is differential corresponding to Hamiltonian vector fields which are induced by these constraints. Perturbative expansion of $Q$ around $\delta$ leads to standard Hamiltonian methods, and expansion around $\partial$ leads to BRST. In the case if constraints form so called open group, one has to consider the corresponding filtered space instead of this double complex [10,11,13]. This approach seems to be very fruitful.

3. *Local BRST Cohomology*

Considering BRST physical observables as integrals of local functions one comes naturally to differential $Q = s + d$, where $s$ is BRST differential, acting on integrand which is local function and $d$ is the usual de Rham differential. It turns out that the consideration of cohomology of this double complex is a very powerful tool for BRST cohomology investigations in field theory, especially in Lagrangian framework. (See [15,16,17,18] and citations there).

In spite of these examples one has to note that the method of spectral sequences was not used actively in these calculations.

May be at first the method of spectral sequences was applied in physics by J. Dixon in [15] in analysis of local BRST cohomology. In series of works the so called method of descent equations which is in fact a special case reminiscent of this technique was applied successfully to these problems. (See the review [17] and the papers citated there). Nowadays the technique of spectral sequences seems to be not very popular in theoretical physics. We hope to pay attention to importance of this technique which is used here in a simple physical frame. In principal using the method "Deus ex machina" one can formulate the hierarchy (2.14) without using explicitly the method used in this paper which indeed seems to be very tedious. But in our opinion this method is inherent to this problem and it is the adequate technique in other important problems such as constrained dynamics theory; it may have useful applications in future.

## IV Acknowledgments

The work was supported in part by Armenian National Science Foundation and by INTAS-RFBR Grant No 95-0829.

We are grateful to G.Barnich, F.Brandt, R.L. Mkrtchyan and A.P.Nersessian for fruitful discussions and important remarks.

The creative atmosphere of the conference "Secondary Calculus and Comological Physics" which was organized by A.M. Vinogradov's school gave the strong impulse for

this work. We are very grateful to all participants of this conference and especially to N. Maggiore, M. Castrillon and J. Stasheff who encouraged to finish this work.

Finally one of us (O.M.) wants to use opportunity to express his deep gratitude to O.Piguet whose indirect influence on this paper was extremely high.

## Appendix

We consider briefly two examples of hierarchy diagram. The first one is model example, the second one is the example where the second floor of hierarchy diagram is not empty.

Example 1

We consider the following pair $[\mathcal{G}, M]$. Let $\mathcal{G}$ be Lie algebra $\ell_3$ with generators $e_1, e_2, e_3$ such that $[e_1, e_2] = e_3, [e_2, e_3] = [e_3, e_1] = 0$. Let a configuration space $M$ be cylinder: $M = \mathbf{R} \times S^1$ with coordinates $(z, \varphi)$. The homomorphism $\Phi$ (see 1.6) is defined by the relations

$$\Phi e_1 = \tilde{e}_1 = \frac{\partial}{\partial z}, \quad \Phi e_2 = \tilde{e}_2 = z\frac{\partial}{\partial \varphi}, \quad \Phi e_3 = \tilde{e}_3 = \frac{\partial}{\partial \varphi}\,. \tag{$A.1$}$$

This defines the pair $[\ell_3, S^1 \times \mathbf{R}]$. For this pair we calculate the hierarchy diagram $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$. From (A.1) it follows that every $\ell_3$-invariant Lagrangian has the form $F(\dot{z})$ where $F$ is an arbitrary function.

Now we calculate the spaces $\{K_s\}$. $K_0 = H^1(\mathcal{G}) = \mathbf{R}^2$ is generated by the cochains $e^1$ and $e^2$ ($\{e^i\}$ are dual to $\{e_j\}$: $e^i(e_j) = \delta^i_j$). The elements from $H^1(\mathcal{G})$ in components are $t_i = (a, b, 0)$. $H^1(M) = \mathbf{R}$ is generated by 1-form $d\varphi$. Hence $K_1 = \mathbf{R^2}$ is generated by cochains $(d\varphi, 0, 0)$ and $(0, d\varphi, 0)$. $K_2 = H^2(\mathcal{G}) = \mathbf{R}^2$: the cocycles $f_{ij}$ such that $f_{12} = 0$ represent its cohomology class. It is easy to see that $H^1_{inv}(M) = \mathbf{R}$ is generated by 1-form $dz$. The stability subalgebra in every point $(z, \varphi)$ is generated by the vector $e_2 - ze_3$, hence from the result for $H^1(\mathcal{G})$ it follows that $K_3 = 0$. (The explicit calculations give that $H^1(\mathcal{G}, \Lambda^0(M)) = \mathbf{R}^2$ is generated by the cocycles $\alpha_i = (0, az + b, a)$; $d(0, az + b, a) = (0, adz, 0) = \delta_i ad\varphi$, hence ${}^tE^{1.0}_3 = {}^tE^{1.0}_2 = H^1(\mathcal{G}, \Lambda^0(M))$ and $K_3 = 0$.)

The space $K_4 = \mathbf{R}$ is generated by the form $d\varphi$. We come to the result

$$K_0 = K_1 = K_2 = \mathbf{R}^2, \quad K_3 = 0, \quad K_4 = \mathbf{R}. \tag{A.2}$$

We already see that second floor of $\mathcal{D}([\ell_3, S^1 \times \mathbf{R}])$ is empty.

The special analyze of the homomorphism $\phi_2$ leads to the fact that 1-st floor is empty too: the image of $\phi_2$ in $K_2$ is trivial because in this special case the subspaces ${}^tE^{0.2}_\infty$ and $E^{2.0}_\infty$ of $H^2(Q)$ have zero intersection.

Now we consider the following 5-dimensional subspace of formal Lagrangians on $S^1 \times \mathbf{R}$:

$$U = \{L\colon \quad L = a\dot{\varphi} + bz\dot{\varphi} + cz + d\frac{\dot{\varphi}}{\dot{z}} + \frac{q}{2}\frac{\dot{\varphi}^2}{\dot{z}}\}, \tag{$A.3$}$$

where $(a, b, c, d, q)$ are constants.

The straightforward calculations give that for arbitrary Lagrangian from $U$

$$\delta_1 L = \mathcal{L}_{\frac{\partial}{\partial z}} L = bd\varphi + c,\ \delta_2 L = \mathcal{L}_{z\frac{\partial}{\partial \varphi}} L = adz + bzdz + d + qd\varphi,\ \delta_3 L = \mathcal{L}_{\frac{\partial}{\partial \varphi}} L = 0\,. \tag{$A.4$}$$

We consider the restriction of the homomorphisms $\{\Psi, \phi_s, \overline{\phi}_s\}$ on the subspace $U$:

$\phi_2=\overline{\phi}_2=\phi_3=\overline{\phi}_3=0$. $\forall L\in U, \Psi(L)=(c,d,0)\in K_0$. If $c=d=0$ then $L\in U_{0.1}$. $\phi_1(L)=\overline{\phi}_1(L)=(bd\phi, qd\varphi, 0)\in K_1$. If $b=q=0$ then $L\in U_{1.0}$ and if $c=d=b=0$ then $L\in U_{1.1}$. Hence $U_{3.0}=U_{2.0}=U_{1.0}$ and correspondingly $U_{3.1}=U_{2.1}=U_{1.0}$. $\phi_4(L)=\overline{\phi}_4(L)=ad\varphi\in K_4$. If $a=b=q=0$ then we come to $U_{4.0}$. If also $c=d=0$ then we come to $U_{4.1}=0$.

All these homomorphisms except $\phi_2,\overline{\phi}_2$ are surjective. Moreover these homomorphisms are injective on corresponding factor spaces.

From these considerations it follows that for every weakly $\ell_3$-invariant Lagrangian there exists unique Lagrangian in $U$ such that their difference belongs to $\mathcal{V}_{4.1}$:

$$\forall L\in\mathcal{V}_{0.0}\,\exists!(a,b,c,d,q)\colon L=F(\dot z)+\text{full derivative}+a\dot\varphi+bz\dot\varphi+cz+d\frac{\dot\varphi}{\dot z}+\frac{q}{2}\frac{\dot\varphi^2}{\dot z}\,. \quad (A.5)$$

We come to the following gradation in the space $\mathcal{V}_{0.0}$ of weakly $\ell_3$-invariant Lagrangians on $S^1\times\mathbf{R}$:

$$\mathcal{V}_{3.1}=\mathcal{V}_{2.1}=\mathcal{V}_{1.1}=\mathcal{V}_{4.1}\oplus K_4=\mathcal{V}_{4.1}\oplus\mathbf{R},$$
$$\mathcal{V}_{0.1}=\mathcal{V}_{1.1}\oplus K_1=\mathcal{V}_{1.1}\oplus\mathbf{R}^2,\ \mathcal{V}_{s.0}=\mathcal{V}_{s.1}\oplus K_0=\mathcal{V}_{s.1}\oplus\mathbf{R}^2\,.$$

Example 2 $so(3)$ algebra

Let $M=\mathbf{R}^3$ be 3-dimensional linear space with cartesian coordinates $(x^1,x^2,x^3)$. The action of $so(3)$ on $\mathbf{R}^3$ is generated by the standard action of the group $SO(3)$ on $\mathbf{R}^3$: if $\{e_1,e_2,e_3\}$ is a basis in $so(3)$ such that $[e_i,e_j]=\varepsilon_{ijk}e_k$ then $\Phi(e_i)=\tilde L_i=-\varepsilon_{ijk}x^j\partial/\partial x^k$. For the pairs $[so(3),\mathbf{R}^3]$ and $[so(3),S^2]$ the hierarchy diagram is trivial because $SO(3)$ is semisimple compact group.

Alternatively one can see it by the following explicit calculations: From commutation relations it is evident that $H^1(so(3))=H^2(so(3))=0$. Hence $K_0=K_1=K_2=K_4=0$. If $\alpha_i$ is a cocycle with values in functions on $S^2$ then $0=\delta\alpha=\tilde L_i\alpha_k-\tilde L_k\alpha_i-\varepsilon_{ijk}\alpha_k$. Hence $\tilde L^2\alpha_k=\tilde L_k(\tilde L_i\alpha_i)=\tilde L_kF$ and $\alpha_k=\delta\tilde F$ is coboundary where $\tilde F=\sum_l\frac{F^l}{l(l+1)}$. ($F^l$ is defined by the expansion over the spherical harmonics of $F$. The term $F^0=0$ because it leads to cocycle in constants and $H^1(so(3))=0$.) Hence $K_3=0$ also.

All weakly $so(3)$ invariant Lagrangians of classical mechanics on $R^3$ and on $S^2$ are exhausted by $so(3)$-invariant ones (up to a full derivative).

Now we modify little bit this example considering instead the sphere $S^2$ the domain in it, the sphere without North pole (punctured sphere) $S^2\backslash N]$ ($x^3\neq 1$). Thus we come from the pair $[so(3),S^2]$ to the pair $[so(3),S^2\backslash N]$. In the same way we come to the pair $[so(3),\mathbf{R}^3\backslash L_+]$, taking out the ray $L_+$ ($x^1=0,x^2=0,x^3\geq 0$) from $\mathbf{R}^3$.

The essential difference of these pairs from the previous ones is that they cannot be generated by the action of the corresponding Lie group.

It is evident that for the diagram $\mathcal{D}[[so(3),S^2\backslash N]]$ $K_0=K_1=K_2=K_4=0$, also. Now we show that for this diagram $K_3=\mathbf{R}$.

The stability algebra for this pair is one dimensional, hence $K_3=0$ or $K_3=\mathbf{R}$. It remains to prove that $K_3$ is not trivial.

To show it we consider the Lagrangian $L$ which corresponds to the differential form $A=-(1+cos\theta)d\varphi$ on the punctured sphere $S^2\backslash N$. ($\theta,\varphi$ are spherical coordinates.) The two-form $dA = F = sin\theta d\theta d\varphi$ corresponding to its motion equations is $so(3)$-invariant, hence this Lagrangian is weakly $so(3)$-invariant. On the other hand it cannot be reduced to $so(3)$-invariant by redefinition on a full derivative *df* because $so(3)$-invariant 1-form on the sphere is equal to zero. Hence, because all other spaces $K_s$ are equal to zero. this Lagrangian belongs to the floor $2_+$. We come to the result:

$$K_3 = H^1(so(3), \Lambda^0(S^2 \backslash N)) = \mathbf{R} \quad \text{and } \phi_2(A) \in K_3 \neq 0, \qquad (A.6)$$

One can consider the explicit realization of constructions above: We identify the vectors in $\mathbf{R}^3$ with the vectors in the linear space of the Lie algebra $so(3)$ by the linear map $\gamma$: $(x^1, x^2, x^3) \to x^1 e_1 + x^2 e_2 + x^3 e_3$. For any point $x \in S^2$ the corresponding stability subalgebra is generated by $\gamma(x)$. If $\alpha$ is a 1-cocycle with values in functions on the punctured sphere then

$$\alpha(x, \gamma(x)) = x^i \alpha_i(x) \text{ is a constant on the sphere,} \qquad \text{(A.7a)}$$

$$\text{this constant is equal to zero iff this cocycle is a coboundary.} \qquad \text{(A.7b)}$$

We proved that $K_3 = \mathbf{R}$ and all other $K_s$ are equal to zero and presented in (A.6) the Lagrangian with non-trivial image in $K_3$. Hence for the hierarchy diagram $\mathcal{D}([so(3), S^2 \backslash N])$ only fourth and second floor are not empty: every weakly $so(3)$ invariant Lagrangian on the punctured sphere has the form

$$L = L_{inv} + \text{full derivatives} - g(1 + cos\theta)\dot{\varphi}\,. \qquad (A.8)$$

In the case $g \neq 0$ it belongs to the floor $2_+$ of the hierarchy.

The calculations for the diagram $\mathcal{D}[so(3), \mathbf{R}^3 \backslash l_+]$ are analogous and the result is the same: every weakly $so(3)$ invariant Lagrangian on the $\mathbf{R}^3 \backslash l_+$ has the form (A.8).

One can see that in the case if $L_{inv}$ is free particle Lagrangian, then (A.8) corresponds to the Lagrangian which describes the interaction of particle with Dirac monopole.

The explicit calculations for (A.6) give that $\phi_2(L)$ for the Lagrangian (A.8) is equal to the cohomology class in $H^1(so(3), S^2 \backslash N)$ of the following cocycle:

$$\alpha_1 = -gctg\frac{\theta}{2}cos\varphi,\ \alpha_2 = -gctg\frac{\theta}{2}sin\varphi,\ \alpha_3 = g, \quad (\delta L = d\alpha,\ \delta\alpha = 0) \qquad (A.9)$$

and $\alpha_i x^i = -g$.

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