INTRODUCTION TO THE CONTRIBUTIONS

During the five years, which passed after the first meeting "Quaternionic Structures in Mathematics and Physics " interest in quaternionic geometry and its applications continued to increase. A progress was done in constructing new classes of manifolds with quaternionic structures (quaternionic Kähler, hyper-Kähler, hypercomplex etc.), studying differential geometry of special classes of such manifolds and their submanifolds, understanding relations between the quaternionic structure and other differential-geometric structures, and also in physical applications of quaternionic geometry. Some generalizations of classical quaternionic-like structures (like HKT -structures and hyper-Kähler manifolds with singularities) naturally appeared and were studied. Some of these results are published in this proceedings.

A new simple and elegant construction of homogeneous quaternionic pseudo-Kähler manifolds is proposed by V. CORTES. It gives a unified description of all known homogeneous quaternionic Kähler manifolds as well as new families of quaternionic pseudo-Kähler manifolds and their natural mirror in the category of supermanifolds.

Generalizing the Hitchin classification of $Sp(1)$-invariant hyper-Kähler and quaternionic Kähler 4-manifolds, T. NITTA and T. TANIGUCHI obtain a classification of $Sp(1)^n$-invariant quaternionic Kähler metrics on $4n$-manifold. All of these metrics are hyper-Kähler.

I.G. DOTTI presents a general method to construct quaternionic Kähler compact flat manifolds using Bieberbach theory of torsion free crystallographic groups.

Using the representation theory , U. SEMMELMANN and G. WEINGART find some Weitzeböck type formulas for the Laplacian and Dirac operators on a compact quaternionic Kähler manifold and use them for eigenvalue estimates of these operators. As an application, they prove some vanishing theorems, for example, they prove that odd Betti numbers of a compact quaternionic Kähler manifold with negative scalar curvature vanish.

A hyper-Kähler structure on a manifold M defines a family (J_t, ω_t) of complex symplectic structures, parametrized by $t \in \mathbb{C}P^1$. R. BIELAWSKI gives a generalization of the hyper-Kähler quotient construction to the case when a holomorphic family G_t, $t \in \mathbb{C}P^1$ of complex Lie group is given, such that G_t acts on M as a group of automorphisms of (J_t, ω_t).

The existence of a canonical hyper-Kähler metric on the cotangent bundle T^*M of a Kähler manifold M was proved independently by D.Kaledin and B.Feix. In the present paper D. KALEDIN presents his proof in simplified form and obtains an explicit formula for the case when M is a Hermitian symmetric space.

A toric hyper-Kähler manifold is defined as the hyper-Kähler quotient of the quaternionic vector space $\mathbb{H}^n$ by a subtorus of the symplectic group $Sp(n)$. H. KONNO determines the ring structure of the integral equivariant cohomology of a toric hyper-Kähler manifold.

M. VERBITSKY gives a survey of some recent works about singularities in hyper-Kähler geometry and their resolution. It is shown that singularities in a singular hyper-Kähler variety (in the sense of Deligne and Simpson) have a simple structure and admit canonical desingularization to a smooth hyper-Kähler manifold. Some results can be extended to the case of the hypercomplex geometry.

Hypercomplex manifolds (which is the same as $4n$-manifolds with a torsion free connection with holonomy in $GL(n,\mathbb{H})$) are studied by H. PEDERSEN. He describes three constructions of such manifolds: 1) via Abelian monopoles and geodesic congruences on Einstein-Weyl 3-manifolds , 2) as a deformation of Joyce homogeneous hypercomplex structures on $G \times T^k$ where G is a compact Lie group and 3) as a deformation of the hypercomplex manifold $\mathcal{V}_P(M)$, associated with a quaternionic Kähler manifold M and an instanton bundle $P \to M$ by the construction of Swann and Joyce.

M.L. BARBERIS describes a construction of left-invariant hypercomplex structures on some class of solvable Lie groups. It gives all left-invariant hypercomplex structures on 4-dimensional Lie groups. Properties of associated hyper-Hermitian metrics on 4-dimensional Lie groups are discussed.

D. JOYCE proposes an original theory of quaternionic algebra, having in mind to create algebraic tools for developing quaternionic algebraic geometry. Applications for constructing hypercomplex manifolds and study their singularities are considered.

Properties of hyperholomorphic functions in $\mathbb{R}^4$ are studied by S-L. ERIKSSON-BIQUE. Hyperholomorphic functions are defined as solutions of some generalized Cauchy-Riemann equation, which is defined in terms of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{0,3}) \approx \mathbb{H} \oplus \mathbb{H}$.

Other more general notion of hyper-holomorphic function on a hypercomplex manifold M is proposed and discussed in the paper by ST. DIMIEV , R. LAZOV and N. MILEV.

O.BIQUARD defines and studies quaternionic contact structures on a manifold. Roughly speaking, it is a quaternionic analogous of integrable CR structures.

Generalizing the ideas of A. Gray about weak holonomy groups, A.SWANN looks for G-structures which admit a connection with "small" torsion, such that the curvature of these connections satisfies automatically some interesting conditions, for example, the Einstein equation.

G. GRANTCHAROV and L. ORNEA propose a procedure of reduction which associates to a Sasakian manifold S with a group of symmetries a new Sasakian manifold and relate it to the Kähler reduction of the associated Kähler cone $K(S)$.

The geometry of circles in quaternionic and complex projective spaces are studied by S. MAEDA and T. ADACHI. The main problem is to find the full system of invariants of a circle C, which determines C up to an isometry, and to determine when a circle is closed.

Special 4-planar mappings between almost Hermitian quaternionic spaces are defined and studied by J. MIKEŠ, J. BĚLOHL'AVKOV'A and O. POKORN'A.

Some generalization of the flat Penrose twistor space $\mathbb{C}^{2,2}$ is constructed and discussed by J. LAWRYNOWICZ and O. SUZUKI.

M. PUTA considers some geometrical aspects of the left-invariant control problem on the Lie group $Sp(1)$.

Quaternionic representations of finite groups are studied by G. SCOLARICI and L. SOLOMBRINO.

Quaternionic and hyper-Kähler manifolds naturally appear in the different physical models and physical ideas produce new results in quaternionic geometry. For example, Rozanski and Witten introduce a new invariant of hyper-Kähler manifold as the weights in a Feynman diagram expansion of the partition function of a 3-dimensional physical theory. A variation of this construction, proposed by N. Hitchin and J. Sawon , gives a new invariant of links and a new relations between invariants of a hyper-Kähler manifold X, in particular, a formula for the norm of the curvature of X in terms of some characteristic numbers and the volume of X. These results are presented in the paper by J. SAWON.

The classical Atiyah-Hitchin-Drinfeld-Manin's monad construction of anti-self-dual connections over $S^4 = \mathbb{H}P^1$ was generalized by M. Capria and S. Salamon to any quaternionic projective space $\mathbb{H}P^n$. Using representation theory of compact Lie groups, Y. NAGATOMO extends the monad construction to any Wolf space (i.e. compact quaternionic symmetric space).

A quaternionic description of the classical Maxwell electrodynamics is proposed in the paper by D. SWEETSER and G. SANDRI.

A. PRASTARO applies his theory of non commutative quantum manifolds to the category of quantum quaternionic manifolds and discusses the theorem of existence of local and global solutions of some partial differential equations.

A hyper-Kähler structure on a 4n-manifold M is defined by a torsion free connection ∇ with holonomy group in $Sp(n)$. A natural generalization is a connection ∇ with a torsion $T = (T^j_{kl})$ and the holonomy in $Sp(n)$. If the covariant torsion $\tau = g(T) = (g_{ij}T^j_{kl})$, where g is the ∇-parallel metric on M, is skew-symmetric, then the manifold (M, ∇, g) is called hyper-Kähler manifold with a torsion, or shortly HKT-manifold. If, moreover, the 3-form τ is closed, it is called strongly HKT-manifold. Such manifolds appear in some recent versions of supergravity.

A purely mathematical survey of the theory of HKT-manifolds is given by Y. S. POON.

Some applications of HKT-geometry to physics is discussed in the paper by G. PAPADOPOULUS. He describes a class of brain configurations, which are approximations of solutions of 10 and 11 dimensional supergravitation.

A. VAN PROEYEN gives a review of special Kähler geometry (which can be mathematically defined as the geometry of Kähler manifolds together with a compatible, in some rigorous sense, flat connection), its physical meaning and connections to quaternionic and Sasakian geometry.

Dmitri V. Alekseevsky