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QUATERNIONIC KÄHLER METRICS AND NILPOTENT ORBITS

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ABSTRACT. The geometry of quaternionic Kähler metrics constructed from nilpotent orbits is surveyed. This includes the relationship with homogeneous twistor spaces, finite-dimensional quotient constructions and the problem of compactification.

1. AUTOMORPHISMS AND HOMOGENEITY

Given a geometric structure on a manifold it is natural to ask what are the symmetries of that structure and what are the examples of such structures with a big group of symmetries. In this talk I wish to start by discussing such questions for quaternionic Kähler manifolds.

Let M be a quaternionic Kähler manifold, that is a manifold whose holonomy is contained in $Sp(n)\,Sp(1)$, the subgroup of $SO(4n)$ acting on $\mathbb{R}^{4n} \cong \mathbb{H}^n$ via $(A, q) \cdot \xi = A\xi\overline{q}$. I will assume that the scalar curvature of M is non-zero, thus ruling out the case when M is locally hyperKähler, but will not necessarily assume that M is compact.

Fibred over M, is the *twistor space* Z [36, 5]. This is a complex-contact manifold with a real structure σ and a (possibly indefinite) Kähler metric. The fibres of $Z \to M$ are rational curves $\mathbb{C}\mathrm{P}(1) \cong S^2$ called *twistor lines* which are preserved by σ and have normal bundle $\nu \cong 2n\mathcal{O}(1)$. I will regard the complex-contact structure as a holomorphic one-form θ taking values in a holomorphic line bundle $\mathcal{L} \to Z$ such that $\theta \wedge (d\theta)^n$ vanishes nowhere and $\sigma^*\theta = \overline{\theta}$. Writing $\mathcal{D} = \ker\theta$, we have an exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow TZ \xrightarrow{\theta} \mathcal{L} \longrightarrow 0.$$

The data $\mathcal{D}$, σ on the complex manifold Z is enough to determine the quaternionic Kähler manifold M via the inverse twistor construction [27, 34]. Thus an automorphism $\check{\phi}$ of M preserving the quaternionic Kähler structure induces a holomorphic automorphism ϕ of Z preserving $\mathcal{D}$ and σ. Salamon [36] and Nitta & Takeuchi [33] showed that there is a correspondence

$$\mathfrak{g}\otimes\mathbb{C}\left\{\begin{array}{c}\text{Infinitesimal}\\ \text{isometries of } M\\ \text{preserving the}\\ \text{quaternionic}\\ \text{structure}\end{array}\right\}\otimes\mathbb{C}\longleftrightarrow\left\{\begin{array}{c}\text{Infinitesimal}\\ \text{holomorphic}\\ \text{complex-contact}\\ \text{automorphisms}\\ \text{of } Z\end{array}\right\}\cong H^0(Z,\mathcal{L})$$

induced by the map $\check{Y}\mapsto\theta(Y)$, where $\check{Y}$ is a vector field on M with lift Y to Z. I have written $\mathfrak{g}$ for the Lie algebra of infinitesimal isometries of M and will use G to denote the corresponding connected, simply-connected Lie group.

This leads to the following two notions of homogeneity:

(A) M is homogeneous under G;
(B) Z is homogeneous under $G^{\mathbb{C}}$.

Case (A) has been well-studied. If G is compact, then in order for it to be the isometry group of a homogeneous quaternionic Kähler manifold, G must be simple and M is then a compact symmetric space uniquely determined by the universal cover of G [40, 1, 2]. We will refer to these spaces M as *Wolf spaces*. For example: $G=Sp(n+1)$ gives $M=\mathbb{HP}(n)$, quaternionic projective space; $G=SU(n)$ gives $M=\mathrm{Gr}_2(\mathbb{C}^n)$, the Grassmannian of complex two-planes in $\mathbb{C}^n$; $G=SO(n)$ gives $M=\widetilde{\mathrm{Gr}}_4(\mathbb{R}^n)$, the Grassmannian of oriented four-planes in $\mathbb{R}^n$; $G=G_2$ gives $M=G_2/SO(4)$. On the other hand, if G is completely solvable, then $M=G$ and the possible groups G are now known [10, 11, 3, 13].

Case (B) does not coïncide with case (A) and is the subject of the next section. Note that the example of the non-compact dual of $\mathbb{HP}(n)$ shows that even if M is homogeneous under G, the group $G^{\mathbb{C}}$ need not act on Z.

2. Homogeneous Twistor Spaces

Let us consider the case when M is a quaternionic Kähler manifold whose twistor space Z is homogeneous under the group $G^{\mathbb{C}}$ of complex-contact automorphisms. Homogeneous complex-contact manifolds have been studied by Lichnerowicz [30] and what is relevant is a complexified version of the symplectification procedure discussed by Arnol'd [4]. Let $\pi\colon\mathcal{U}=\mathcal{L}^*\setminus 0\to Z$ be the total space of the dual of $\mathcal{L}$ with the zero section removed. Define $\alpha=\pi^*\theta$ and let $\omega=d\alpha$. Then ω is a complex-symplectic form on $\mathcal{U}$ and infinitesimal complex-contact automorphisms Y of Z lift to infinitesimal holomorphic automorphisms $\tilde{Y}$ of $\mathcal{U}$ preserving ω.

The map $\mu\colon\mathcal{U}\to\mathfrak{g}_{\mathbb{C}}^*$ defined by

$$\langle\mu(b),\tilde{Y}\rangle=-b(\theta(Y)_{\pi b})$$

is a complex moment map for the action of $G^{\mathbb{C}}$ on $\mathcal{U}$. As Z is a twistor space, we have $\mathcal{L}|_{\mathbb{CP}(1)}\cong\mathcal{O}(2)$ for each twistor line $\mathbb{CP}(1)$ and hence $\theta(Y)\in H^0(Z,\mathcal{L})$ always has a zero. In Lichnerowicz' terminology, this implies that Z is a proper complex-contact manifold, which implies that the image $\mathcal{O} = \mu(\mathcal{U})$ is a coadjoint orbit generated by rays and that μ is a local diffeomorphism.

If we assume that $\mathfrak{g}$ is semi-simple, then we may identify $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{g}^*_{\mathbb{C}}$ equivariantly via an Ad-invariant inner product $\langle \cdot, \cdot \rangle$. Realising $\mathfrak{g}^{\mathbb{C}}$ as a matrix algebra, the orbit $\mathcal{O}$ becomes an orbit of nilpotent matrices, that is $X^r = 0$ for some r. Moreover, via the projectivisation of μ, the twistor space Z is locally diffeomorphic to the complex projectivisation $\mathbb{P}\mathcal{O} \subset \mathbb{P}\mathfrak{g}^{\mathbb{C}}$ and θ is given by $\mu^*\theta_{\mathcal{O}}$, where $\theta_{\mathcal{O}}[X, A] = \langle X, A \rangle$ is the standard complex-contact form on $\mathbb{P}\mathcal{O}$ induced by the Kirillov-Kostant-Souriau complex-symplectic form $\omega_{\mathcal{O}}([X, A], [X, B]) = -\langle X, [A, B] \rangle$ on $\mathcal{O}$.

The question now arises: for which nilpotent orbits $\mathcal{O}$ is $\mathbb{P}\mathcal{O}$ a twistor space?

Since Z is fibred over M we have the following diagram:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathbb{C}^*"] \arrow[dr] & Z \arrow[d, "S^2"] \\ & M \end{tikzcd}$$

and may regard $\mathcal{U}$ as a bundle over M with fibre $\mathbb{H}^*/\{\pm 1\}$. The fact that M is quaternionic Kähler implies that $\mathcal{U}$ admits a hyperKähler metric invariant not only under the action of G, but also under the action of $SO(3) \subset \mathbb{H}^*/\{\pm 1\}$ [39]. Furthermore, this hyperKähler metric is positive definite if and only if M has positive scalar curvature.

By studying appropriate moduli spaces of G-instantons, Kovalev [21] extended work of Kronheimer [26, 25] to show that every adjoint orbit $\mathcal{O}$ of a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ admits a G-invariant hyperKähler metric, where G is the compact real from of $\mathfrak{g}^{\mathbb{C}}$ (see also [6]). An examination [39] of Kronheimer's proof [26] shows that if $\mathcal{O}$ is nilpotent, then $\mathcal{O}$ admits a free action of $\mathbb{H}^*/\{\pm 1\}$ such that $\mathcal{O}/\mathbb{H}^*$ is a quaternionic Kähler manifold of positive scalar curvature with twistor space $\mathbb{P}\mathcal{O}$.

Example 2.1. Let G be a compact, simple, simply-connected Lie group. Let $\mathcal{O}_{\min}$ be the smallest non-trivial nilpotent orbit. Then $\mathcal{O}_{\min}/\mathbb{H}^*$ is the Wolf space with isometry group G (see case (A) above). For example, if $G = SU(n)$, then $\mathcal{O}_{\min} = \{ X \in \mathfrak{sl}(n, \mathbb{C}) : X^2 = 0, \ \operatorname{rank} X = 1 \}$, the $\mathbb{H}^*$-orbit through X is the intersection of $\mathcal{O}_{\min}$ with the three-dimensional subspace of $\mathfrak{sl}(n, \mathbb{C})$ spanned by $\{X, \overline{X}^t, [X, \overline{X}^t] \}$ and the map $X \mapsto \operatorname{span}_{\mathbb{C}^n} \{\operatorname{Im} X, (\ker X)^\perp\}$ identifies $\mathcal{O}/\mathbb{H}^*$ with $\operatorname{Gr}_2(\mathbb{C}^n)$.

Note that the orbits $\mathcal{O}_{\min}$ are the only nilpotent orbits for which the quaternionic Kähler manifolds $\mathcal{O}/\mathbb{H}^*$ are compact.

3. Nilpotent Orbits for the Special Linear Group

The construction of the Kronheimer metric is somewhat indirect. For nilpotent orbits in classical complex Lie algebras there is a more straightforward approach, obtained by interpreting constructions of Kraft & Procesi [24, 23, 22]as hyperKähler quotients.

Here, I will outline the construction for orbits in $\mathfrak{sl}(n,\mathbb{C})$. For the other classical algebras I will only give an example in the next section. Details may be found in [20].

Let $X \in \mathcal{O} \subset \mathfrak{sl}(n,\mathbb{C})$ be nilpotent. Then there is a natural number k such that $X^k = 0$ and $X^{k-1} \neq 0$. I define unitary vector spaces $V_0,\dots,V_k$ by $V_k = \mathbb{C}^n$ and $V_{k-i} = \operatorname{Im} X^i$, for $i > 1$ and let $n_j := \dim_{\mathbb{C}} V_j$. Consider M given by

$$M = \bigoplus_{i=0}^{k-1} \operatorname{Hom}(V_i, V_{i+1}) \oplus \operatorname{Hom}(V_{i+1}, V_i).$$

If we define $\mathbf{j}(\alpha,\beta) = (-\beta^*, \alpha^*)$, where * denotes the unitary adjoint, then M becomes isomorphic to the flat hyperKähler manifold $\mathbb{H}^N$, where $N = n_0 n_1 + \dots + n_{k-1} n_k$. It is convenient to think of a point of M as a diagram

$$\{0\} = V_0 \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} V_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} V_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{k-1}}{\overset{\alpha_{k-1}}{\rightleftarrows}} V_k = \mathbb{C}^n.$$

The action of $\mathbf{G} := U(n_1) \times \dots \times U(n_{k-1})$ on $V_1 \oplus \dots \oplus V_{k-1}$ induces an action of $\mathbf{G}$ on M preserving the hyperKähler structure and this action has hyperKähler moment map $\mu = \mathbf{i}\mu_r + 2\mathbf{k}\mu_c$, where

$$\begin{aligned}
\mu_c &= (\alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}),\\
\mu_r &= (\alpha_i\alpha_i^* - \beta_i^*\beta_i + \beta_{i+1}\beta_{i+1}^* - \alpha_{i+1}^*\alpha_{i+1}).
\end{aligned}$$

The hyperKähler quotient construction [17] now implies that a dense open subset of $\mu^{-1}(0)/\mathbf{G}$ is a hyperKähler manifold. In fact, in this case one can show that $\mu^{-1}(0)/\mathbf{G}$ is a stratified hyperKähler manifold.

The map $M \to \mathfrak{gl}(n,\mathbb{C})$ given by $(\alpha,\beta) \mapsto \alpha_{k-1}\beta_{k-1}$ is $\mathbf{G}$-invariant and induces a homeomorphism $\mu^{-1}(0)/\mathbf{G} \to \overline{\mathcal{O}}$, where $\overline{\mathcal{O}}$ is the closure of the nilpotent orbit containing X. Note that all the $SL(n,\mathbb{C})$-orbits in $\overline{\mathcal{O}}$ are nilpotent: it is these nilpotent orbits that provide the stratification of $\mu^{-1}(0)/\mathbf{G}$. (Incidentally, general considerations [12] show that a hyperKähler quotient $\mu^{-1}(0)/\mathbf{G}$ is always a union of hyperKähler manifolds, but this is weaker than saying that $\mu^{-1}(0)/\mathbf{G}$ is stratified, cf. [15, 37].)

The above construction was used by Nakajima [32] with a non-degenerate value of the moment map to show that the cotangent bundle of the flag manifold associated to $V_1 \leqslant V_2 \leqslant \dots \leqslant V_k = \mathbb{C}^n$ admits complete, $U(n)$-invariant hyperKähler metrics.

The use of the zero value of the moment map above implies that the above construction commutes with the action of $\mathbb{H}^*$ by multiplication on the right and so gives that $\overline{\mathcal{O}}/\mathbb{H}^*$ is a stratified quaternionic Kähler manifold obtained as a quaternionic Kähler quotient of $\mathbb{HP}(N-1)$, cf. [39].

Example 3.1. Let $\mathcal{O} = \mathcal{O}_{\min} \subset \mathfrak{sl}(n, \mathbb{C})$, so points of $M \cong \mathbb{H}^n$ are diagrams

$$\mathbb{C} \rightleftarrows \mathbb{C}^n$$

and $\mathbf{G} = U(1)$. The above construction now describes $\mathrm{Gr}_2(\mathbb{C}^n)$ as a quaternionic Kähler quotient of $\mathbb{HP}(n-1)$ by $U(1)$ and coïncides with the construction given by Galicki [14].

4. Construction of Other Classical Nilpotent Orbits

For the cases of nilpotent orbits in $\mathfrak{o}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$, let us look at the following example.

Let $\mathcal{O}^{O(7)}_{3,2^2}$ be the orbit of elements in $\mathfrak{o}(7, \mathbb{C}) \cong \Lambda^2\mathbb{C}^7$ which are nilpotent and have Jordan type $(3, 2^2)$. Consider the diagram

$$\mathbb{C} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^4 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \mathbb{C}^7. \tag{4.1}$$

Now $\mathbb{C}^7$ has a non-degenerate, symmetric, bilinear form $B_3 = \langle\cdot,\cdot\rangle$ such that $O(7, \mathbb{C})$ consists of the linear automorphisms of $\mathbb{C}^7$ preserving $\langle\cdot,\cdot\rangle$. For $X \in \mathcal{O}^{O(7)}_{3,2^2}$, we have $\operatorname{Im} X = \mathbb{C}^4$ and that $\langle\cdot,\cdot\rangle$ induces a complex symplectic form $B_2 = \omega$ on $\operatorname{Im} X$ [38]. Similarly ω induces a symmetric bilinear form $B_1 = \langle\cdot,\cdot\rangle$ on $\operatorname{Im} X^2 = \mathbb{C}$. We now only consider diagrams (4.1) in which $\beta_i = \alpha_i^\dagger$, the adjoint of α_i with respect to the bilinear forms B_j, and restrict $\mathbf{G}$ to consist of those elements of $U(1) \times U(4)$ preserving $B_1 \oplus B_2$, thus $\mathbf{G} = O(1) \times Sp(2)$. The hyperKähler quotient construction now describes $\mathcal{O}^{O(7)}_{3,2^2}$ as a hyperKähler quotient of $\mathbb{H}^{16}$ by $O(1) \times Sp(2)$.

Note that the group $O(1)$ is isomorphic to $\mathbb{Z}_2$ so the hyperKähler quotient of $\mathbb{H}^{16}$ by just $Sp(2)$ has the same dimension as $\mathcal{O}^{O(7)}_{3,2^2}$. This quotient corresponds to the diagram

$$\mathbb{C}^4 \rightleftarrows \mathbb{C}^8,$$

with $Sp(2)$ acting on $\mathbb{C}^4 \cong \mathbb{H}^2$ and the hyperKähler quotient is the closure $\overline{\mathcal{O}^{O(8)}_{2^4}}$ of the $O(8)$-orbit of Jordan type 2^4. (One should regard the left-hand factor $\mathbb{C}$ in (4.1) has having been moved to the right-hand side and added to $\mathbb{C}^7$ to get $\mathbb{C}^8$.) We thus have a map $\overline{\mathcal{O}^{O(8)}_{2^4}} \to \overline{\mathcal{O}^{O(7)}_{3,2^2}}$ which is generically two-to-one. A closer examination leads to the following diagram:

$$
\begin{tikzcd}
\mathcal{O}^{O(8)}_{2^4} \arrow[r, "2:1"] \arrow[d, dash] & \mathcal{O}^{O(7)}_{3,2^2} \arrow[d, dash] \\
\mathcal{O}^{O(8)}_{2^2,1^4} \arrow[r] \arrow[dr] & \mathcal{O}^{O(7)}_{3,1^4} \arrow[d, dash] \\
& \mathcal{O}^{O(7)}_{2^1,1^3}
\end{tikzcd}
\tag{4.2}
$$

Here a vertical line indicates that the lower orbit is contained in the closure of the upper orbit; the union of the left-hand column is $\overline{\mathcal{O}^{O(8)}_{2^4}} \setminus \{0\}$ and that of the right-hand column $\overline{\mathcal{O}^{O(7)}_{3,2^2}} \setminus \{0\}$. Note that the hyperKähler quotient construction shows that the uppermost map $\mathcal{O}^{O(8)}_{2^4} \to \mathcal{O}^{O(7)}_{3,2^2}$ is a local isometry. This map is also $O(7,\mathbb{C})$ equivariant, although only the compact group $O(7)$ acts isometrically.

Springer & Steinberg [38] show that the nilpotent orbits for $SL(n,\mathbb{C})$, $Sp(n,\mathbb{C})$ and $O(n,\mathbb{C})$ are determined by their Jordan normal forms. This is the reason I have been discussing $O(n,\mathbb{C})$ instead of $SO(n,\mathbb{C})$. In the latter case, the Jordan normal form determines the nilpotent orbit unless it is *very even*, that is all the Jordan blocks are even in size and multiplicity. For each very even Jordan type there are precisely two $SO(n,\mathbb{C})$-nilpotent orbits. For example, $\mathcal{O}^{O(8)}_{2^4}$ splits as $\mathcal{O}^{SO(8)}_{2^4+} \cup \mathcal{O}^{SO(8)}_{2^4-}$. Under this splitting the upper map in (4.2) becomes two isometries $\mathcal{O}^{SO(8)}_{2^4\pm} \xrightarrow{\cong} \mathcal{O}^{O(7)}_{3,2^2} = \mathcal{O}^{SO(7)}_{3,2^2}$. More generally, we have:

Proposition 4.1. *For each $n \geqslant 1$, the three nilpotent orbits $\mathcal{O}^{SO(4n)}_{2^{2n}+}$, $\mathcal{O}^{SO(4n)}_{2^{2n}-}$ and $\mathcal{O}^{SO(4n-1)}_{3,2^{2n-2}}$ are diffeomorphic and carry isometric hyperKähler structures.*

When $n = 1$, the above follows easily from the splitting $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Remark 4.2. Levasseur & Smith [29] found a similar example of equivalence, this time associated to G_2. Let $\pi\colon \overline{\mathcal{O}^{SO(7)}_{2^2,1^3}} \to \mathfrak{g}_2^{\mathbb{C}}$ be the map induced by projection to the first factor in the splitting $\mathfrak{so}(7,\mathbb{C}) \cong \mathfrak{g}_2^{\mathbb{C}} \oplus V_7^{\mathbb{C}}$, where V_7 is the seven-dimensional irreducible representation of G_2. (The representation V_7 may be realised as the space $\operatorname{Im}\mathbb{O}$ of pure imaginary Cayley numbers.) Then $\pi(\mathcal{O}^{SO(7)}_{2^2,1^3})$ is the union of $\mathcal{O}^{G_2}_{16} \cup \mathcal{O}^{G_2}_{12}$ of the two smallest nilpotent orbits in $\mathfrak{g}_2^{\mathbb{C}}$ (the lower indices indicating the real dimensions of these orbits) and π is a birational finite morphism. Note that here we do not have an equivalence of orbits, but rather an equivalence of their closures.

5. Compactifications via Minimal Orbits

Let us return the lower part of (4.2). The orbits $\mathcal{O}^{O(8)}_{2,1^4}$ and $\mathcal{O}^{O(7)}_{2^2,1^3}$ are the smallest non-trivial nilpotent orbits in $\mathfrak{o}(8,\mathbb{C})$ and $\mathfrak{o}(7,\mathbb{C})$ respectively, and the associated quaternionic Kähler manifolds are $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^8)$ and $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$. The lower part of diagram (4.2) shows that $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^8)$ is a compactification of the double cover of the quaternionic Kähler manifold $M = \mathcal{O}^{O(7)}_{3,1^4}$, obtained by adjoining a copy of $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$.

In [19] a similar example was given with an exceptional group. The group G_2 is the group of linear automorphisms of the Cayley numbers $\mathbb{O}$ which preserve Cayley multiplication. The Wolf space $G_2/\,SO(4)$ coïncides with the space of quaternionic subalgebras of $\mathbb{O}$. The group $SU(3)$ may be realised as the subgroup of G_2 consisting of those elements fixing a given pure imaginary unit Cayley number e. A quaternionic subalgebra $\mathbb{H}$ of $\mathbb{O}$ containing 1 and e is determined by a pair of points $\{x, ex\}$ orthogonal to 1 and e. However, left multiplication by e defines a complex structure such that $\mathbb{O} = \langle 1\rangle \oplus \langle e\rangle \oplus \mathbb{C}^3$. As $\{x, ex\}$ spans a complex line in $\mathbb{C}^3$, the set of quaternionic subalgebras containing 1 and e defines a subspace of $G_2/\,SO(4)$ isomorphic to $\mathbb{CP}(2)$. Now $SU(3)$ has centre $\mathbb{Z}_3$ which acts on the complement of $\mathbb{CP}(2)$ freely and isometrically. The quotient $((G_2/\,SO(4)) \setminus \mathbb{CP}(2))/\mathbb{Z}_3$ is isometric to the quaternionic Kähler manifold obtained from the largest nilpotent orbit in $\mathfrak{sl}(3,\mathbb{C})$.

One application of such examples is to the study of harmonic maps in to quaternionic Kähler manifolds. We will not discuss this here, but examples may be found in [31, 18, 19].

The above examples are special cases of the following problem studied on the level of homogeneous complex manifolds by Brylinski & Kostant [8, 9]; namely, when does an open set in a nilpotent orbit $\mathcal{O}' \subset \mathfrak{g}'^{\mathbb{C}}$ cover a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ $G^{\mathbb{C}}$-equivariantly. They prove that if $(\mathcal{O}, \mathfrak{g}^{\mathbb{C}}, \mathcal{O}', \mathfrak{g}'^{\mathbb{C}})$ is a quadruple of such Lie algebras and orbits, then so is $(\mathcal{O}'', \mathfrak{g}^{\mathbb{C}}, \mathcal{O}_{\min}, \mathfrak{g}'^{\mathbb{C}})$, and then go on to find all such quadruples of the latter type, showing in particular how the Lie algebra $\mathfrak{g}'^{\mathbb{C}}$ is determined by the ring of regular functions on $\mathcal{O}''$. In [20] it is shown that if $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}'^{\mathbb{C}}$ are classical then the Brylinski-Kostant covering may be realised quaternionically. Note that this procedure is very different from the Springer resolution which gives a desingularisation of certain nilpotent orbits via cotangent bundles of flag manifolds (see for example [7, 32]).

The known compact examples of quaternionic Kähler manifolds with positive scalar curvature are precisely the Wolf spaces, that is precisely the quaternionic Kähler metrics constructed from the smallest non-trivial nilpotent orbits, Example 2.1. Brylinski & Kostant's result may be used to show that most of the other quaternionic Kähler metrics constructed above from nilpotent orbits are not locally isometric to Wolf spaces. It is therefore intriguing to ask whether these other metrics may be compactified. It is particularly interesting to note that the smallest example not yet known to have a compactification, the quaternionic Kähler manifold associated to the largest nilpotent orbit in $\mathfrak{sp}(2,\mathbb{C})$, is twelve dimensional—precisely the lowest dimension in which the classification of compact quaternionic Kähler manifolds of positive scalar curvature is not yet known [16, 35, 28].

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