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WEYL STRUCTURES ON QUATERNIONIC MANIFOLDS

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Abstract. We give a survey on the structure and the topology of a natural class of Einstein-Weyl manifolds, namely the quaternionic manifolds that admit a locally (and non globally) conformal quaternion Kähler metric.

1. Quaternion Hermitian-Weyl manifolds

A *Weyl structure* on a $\mathcal{C}^\infty$ manifold M is a conformal class $[g]$ of Riemannian metrics together with a torsionfree connection D preserving $[g]$. For any representative g of the conformal class $[g]$ there is then an associated 1-form ω: $Dg = \omega \otimes g$, and the associated 1-forms of any two representatives of $[g]$ differ by an exact form. If the symmetric part of the Ricci tensor of D is (pointwise) proportional to the metrics in $[g]$, then the Weyl structure is said to be *Einstein-Weyl*.

We are here interested in the additional datum, on a $4n$-dimensional M^{4n} ($n \geq 2$), of a *quaternion Hermitian structure* (H, g). By H we denote the *quaternionic structure* of M, i.e. a 3-dimensional subbundle of $End(TM)$, locally spanned by almost complex structures I_1, I_2, I_3, satisfying the quaternionic identities and related on the intersections of trivializing open sets by matrices of $SO(3)$. We assume that the subbundle H is 1-integrable, i. e. parallel with respect to some torsionfree connection on M. The metric g is assumed to be Hermitian with respect to the local almost complex structures I_α, $\alpha = 1, 2, 3$. If we endow $(M^{4n}, [g])$ with a structure of Weyl manifold, then it is natural to require the following compatibility condition:

Definition 1.1. *(M^{4n}, H, g) is said to be* quaternion-Hermitian-Weyl *if a torsion-free connection D is given satisfying $DH \subset H,\ Dg = \omega \otimes g$.*

It can be proved [Pe-Po-Sw] that (M^{4n}, H, g) is quaternion-Hermitian-Weyl if and only if g is *locally conformal quaternion Kähler*, i. e. $g_{|U_i} = e^{f_i} g'_i$, where the g'_i are quaternion Kähler over open neighbourhoods $\{U_i\}$ covering M. The connection $D|_{U_i}$

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is in fact the Levi Civita connection of the local quaternion Kähler metric g'_i, and it is related to the global Levi Civita connection ∇^g of g by the formula:

$$D_X Y = \nabla^g_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\}. \tag{1.2}$$

Moreover $\omega|_{U_i} = df_i$. The Einstein property of quaternion Kähler metrics assures that quaternion Hermitian-Weyl manifolds are in particular Einstein-Weyl (we refer to [Fo], [Gau], [Hi], [Pe-Sw] for basic facts on Weyl and Einstein-Weyl manifolds).

The subclass of *hyperhermitian-Weyl manifolds* is then obtained when H is spanned by global D-parallel (and hence integrable) I_α, $\alpha = 1,2,3$. Since the unique torsionfree compatible D is in this case the Obata connection, it follows that on a hypercomplex M^{4n} there is for each hyperhermitian metric g at most one compatible hyperhermitian-Weyl structure.

Examples of quaternion-Hermitian-Weyl manifolds may be obtained in the class of *quaternionic Hopf manifolds* $M = (\mathbb{H}^n - \{0\})/G$, where G is a discrete group acting without fixed points, preserving both the quaternionic structure and the globally conformal hyperkähler metric $g = (\sum_\alpha h_\alpha \overline{h}_\alpha)^{-1} \sum_\alpha dh_\alpha \otimes d\overline{h}_\alpha$ of $\mathbb{H}^n - \{0\}$. The simplest such groups are the dicrete subgroups $G \subset CO^+(4)$ of conformal transformations consisting of $\tilde{x}^i = \rho \sum_j a^i_j x^j$, $a^i_j \in SO(4)$, $\rho > 0$ and not $= 1$ for all the elements in G, with the diagonal action, via the isomorphism $CO^+(4) \cong GL(1,\mathbb{H})\dot{S}p(1)$, on the coordinates of $\mathbb{H}^n$.

The following result shows that the compact quaternion Hermitian-Weyl structures are in fact Ricci-flat-Weyl. (cf. [Gau], [Or-Pi 1]):

Theorem 1.3. *Let $(M, H, [g], D)$ be a compact quaternion Hermitian-Weyl manifold, and assume that no metric in $[g]$ is quaternion Kähler. Then D is Ricci-flat.*

It follows that on such M the local quaternion Kähler metrics, locally conformal related to the global metrics in the conformal class, are locally hyperkähler. This does not imply the existence of a global hypercomplex structure, but suggests the possibility of comparing compact quaternion Hermitian-Weyl manifolds with compact hyperhermitian-Weyl manifolds (cf. n. 3).

2. The fibrations

The fact that the associated 1-forms ω of any quaternion-Hermitian-Weyl manifold are closed allows to apply, for M compact, a result of [Gau] concerning compact Einstein-Weyl manifolds. Accordingly, it is possible to choose a representative g in the conformal class $[g]$ so that the corresponding ω, if not exact, is ∇^g-parallel. The case ω exact corresponds of course to a connection D that is the Levi Civita connection of a quaternion Kähler metric in $[g]$. Hence, apart this case, on compact quaternion-Hermitian-Weyl manifolds there is a 1-dimensional foliation $\mathcal{B}$ generated by the vector field $B = \omega^\sharp$. If we assume $\mathcal{B}$ to be a regular foliation, then the leaf space $P = M/\mathcal{B}$ inherits the structure defined below.

Definition 2.1. *A Riemannian manifold (P, g) is said to be a* locally 3-Sasakian manifold *if a rank 3 vector subbundle $K \subset TP$ is given, locally spanned by an orthonormal triple X_1, X_2, X_3 of Killing vector fields satisfying:*

(i) $[X_\alpha, X_\beta] = 2X_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ and cyclic permutations.

(ii) Any two such triples X_1, X_2, X_3 and X_1', X_2', X_3' are related on the intersections $U \cap U'$ of their definition open sets by matrices of functions with values in $SO(3)$.

(iii) If $F_\alpha = \nabla^g X_\alpha$, $\eta_\alpha = X_\alpha^\sharp$ $(\alpha = 1, 2, 3)$, then $(\nabla^g_Y F_\alpha)\ Z = \eta_\alpha(Z)Y - g(Y, Z)X_\alpha$, for any local vector fields Y, Z.

Then, similarly to a result established in the complex case in [Va 1], the following fact can be proved:

Proposition 2.2. *The class of compact quaternion Hermitian-Weyl manifolds M, that are not quaternion Kähler and with the foliation $\mathcal{B}$ regular, coincides with that of flat principal S^1-bundles over locally 3-Sasakian manifolds $P = M/\mathcal{B}$.*

The above correspondance between quaternion Hermitian-Weyl structures and locally 3-Sasakian structures is described through the projection $\pi : M \longrightarrow P = M/\mathcal{B}$ on the leaf space. The orthonormal triples of local Killing vector fields X_1, X_2, X_3 defining the local 3-Sasakian structure of the leaf space P are obtained by projecting the local vector fields I_1B, I_2B, I_3B of M. Conversely if P is a locally 3-Sasakian manifold, a quaternion Hermitian-Weyl structure $(H, [g], D)$ can be defined on any flat principal S^1-bundle $\pi : M \longrightarrow P$. In fact, if u is a closed 1-form on M giving a flat connection on such S^1-bundle, the metric $g_M = \pi^* g_P + u \otimes u$ and the D defined by formula (1.2) with $\omega = u$, define a Weyl structure on M. A compatible quaternionic structure is then given through the local almost complex structures $I_\alpha Y = -F_\alpha Y - \eta_\alpha(Y)B$, $I_\alpha B = X_\alpha$ (Y horizontal vector field and $B = \omega^\sharp$).

The description (2.2) of compact regular quaternion Hermitian-Weyl manifolds supplements similar correspondances between complex generalized Hopf and Sasakian structures [Va 2], and between hyperhermitian-Weyl and (globally) 3-Sasakian structures [Pe-Po-Sw], [Bo-Ga-Ma].

Another foliation canonically defined on compact quaternion Hermitian-Weyl and non-quaternion Kähler manifolds is the 4-dimensional foliation $\mathcal{D}$ locally spanned by the vector fields B, I_1B, I_2B, I_3B. If all the leaves of $\mathcal{D}$ are compact, the leaf space is an orbifold. Then [Or-Pi 1]:

Theorem 2.3. *Any compact quaternion Hermitian-Weyl manifold, that is not quaternion Kähler and such that all the leaves of $\mathcal{D}$ are compact, fibers over a compact quaternion Kähler orbifold of positive scalar curvature, and the fibers F are conformally flat 4-manifolds. Therefore $F \cong (\mathbb{H} - \{0\})/G$, and G is a discrete subgroup of $GL(1, \mathbb{H}) \cdot Sp(1)$ inducing an integrable quaternionic structure.*

Again the correspondance between the structures is given by projecting metric and quaternionic structure on the leaf space of $\mathcal{D}$. The structure of the fibers is obtained from the vanishing of their Weyl curvature tensor and the expression $S = g - \omega \otimes \omega$ of their Ricci tensor, using a result by [La].

On the other hand also compact locally 3-Sasakian manifolds are endowed with a canonical foliation, namely the the 3-dimensional foliation $\mathcal{F}$ spanned by the local Killing vector fields X_1, X_2, X_3. The leaves of $\mathcal{F}$, if compact, are spherical space forms. It follows:

Corollary 2.4. *Let $(M, H, [g], D)$ be a compact quaternion Hermitian-Weyl manifold that is not quaternion Kähler. Assume that $\mathcal{B}$ is a regular foliation and that all the leaves of the foliations $\mathcal{D}$ on M and $\mathcal{F}$ on $M/\mathcal{B}$ are compact. Then the fibration $M \longrightarrow N = M/\mathcal{D}$ of* thm. 2.3 *factorizes as:*

$$M \longrightarrow P = M/\mathcal{B} \longrightarrow N = P/\mathcal{F} = M/\mathcal{D},$$

and the fibers of $P \to N$ are (generally inhomogeneous) 3-dimensional spherical space forms.

3. The hyperhermitian-Weyl covering

According to thm. 1.3, any compact M with a quaternion Hermitian-Weyl structure, and not quaternion Kähler, admits local compatible I_1, I_2, I_3 satisfying $DI_1 = DI_2 = DI_3 = 0$. The vector bundle $H \longrightarrow M$ over such manifolds is therefore a flat bundle. This implies the flatness also of the vector bundle $K \longrightarrow P$ of compatible Killing vector fields over the locally 3-Sasakian base $P = M/\mathcal{B}$ of M. On the other hand, any local property of (globally) 3-Sasakian manifolds can be trasfered to locally 3-Sasakian manifolds. Thus in particular, due to a result of T. Kashiwada completed in [Bo-Ga-Ma], locally 3-Sasakian manifolds are Einstein manifolds with positive scalar curvature. Myers' theorem allows then to conclude that compact quaternion Hermitian-Weyl manifolds have finite fundamental group. Then, by looking at the universal covering of the locally 3-Sasakian base P of M, one gets [Or-Pi 2]:

Theorem 3.1. *Let M be a compact quaternion Hermitian-Weyl manifold that is not quaternion Kähler, and assume that $\mathcal{B}$ is a regular foliation. Then M admits a finite covering $\overline{M}$ that is hyperhermitian-Weyl and not hyperkähler.*

$\overline{M}$ can in fact be defined as the induced principal S^1 bundle of $M \longrightarrow P$ through the universal covering projection $\overline{P} \longrightarrow P$. The hyperhermitian-Weyl structure is constructed on $\overline{M}$ similarly to how described after prop. 2.2.

Compact hyperhermitian-Weyl manifolds fiber in 1-dimensional complex tori over the twistor space Z of their quaternion Kähler base $N = M/\mathcal{D}$. In this case, the fibers of $M \longrightarrow N$ are complex Hopf surfaces equipped with an integrable hypercomplex structure, and the fibers of $P \longrightarrow N$ are *homogeneous* 3-dimensional spherical space forms. By combining the Gysin sequences of the circle bundles $M \longrightarrow P$, $P \longrightarrow Z$ with well known identities for Betti numbers of compact quaternion Kähler manifolds with positive scalar curvature [Be], [Le-Sa], the following restrictions are obtained:

Proposition 3.2. *Let M^{4n} be a compact hyperhermitian-Weyl manifold with regular foliations $\mathcal{B}$ and $\mathcal{D}$ and let $N = M/\mathcal{D}$ be its quaternion Kähler base. Then the Betti numbers of M and N satisfy:*

$$b_{2p}(M) = b_{2p+1}(M) = b_{2p}(N) - b_{2p-4}(N)$$
$$b_{2n}(M) = 0$$

($0 \leq 2p \leq 2n-2$ and the Poincaré dual equalities for $2n+2 \leq 2p \leq 4n$).

Proposition 3.3. *(Same hypotheses of prop. 3.2). In low dimensions:*

(i) The Betti numbers of a M^{12} are: $b_0 = b_1 = 1$, $b_2 = b_3 = b_4 = b_5 = b_2(N)$, $b_6 = 0$ and Poincaré duals.

(ii) The Betti numbers of a M^{16} are: $b_0 = b_1 = 1$, $b_2 = b_3 = b_6 = b_7 = b_2(N)$, $b_4 = b_5 = b_4(N) - 1$, $b_8 = 0$ and Poincaré duals.

(iii) The Betti numbers of a M^{20} are: $b_0 = b_1 = 1$, $b_2 = b_3 = b_2(N)$, $b_4 = b_5 = b_4(N) - 1$, $b_6 = b_7 = b_6(N) - b_2(N)$, $2b_8 = 2b_9 = -1 + 3b_2(N) + b_4(N) - b_6(N)$, $b_{10} = 0$ and Poincaré duals.

In the homogeneous case, all the above foliations are regular. By looking at the classification of the homogeneous 3-Sasakian manifolds [Bo-Ga-Ma] together with their H^2 the following classification is obtained:

Proposition 3.4. *The only compact homogeneous hyperhermitian-Weyl manifolds are the products $(G/H) \times S^1$, where G/H is any of the globally 3-Sasakian homogeneous manifolds*

$$S^{4n-1},\ \mathbb{R}P^{4n-1},\ SU(m)/S(U(m-2) \times U(1)),\ m \geq 3,$$

$$SO(k)/(SO(k-4) \times Sp(1)),\ k \geq 7,$$

$$G_2/Sp(1),\ F_4/Sp(3),\ E_6/SU(6),\ E_7/Spin(12),\ E_8/E_7,$$

and the unique non trivial principal S^1-bundle over $\mathbb{R}P^{4n-1}$.

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