Quaternionic Structures in
Mathematics and Physics
SISSA - Trieste, 5-9 September 1994

ABOUT SOME OPERATORS IN QUATERNIONIC ANALYSIS

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Abstract. We define new operators on differential forms on quaternionic manifolds. In the special case of the numerical space **H** of quaternions, they enlight the Fueter holomorphic functions theory.

INTRODUCTION

By analogy to Cauchy-Riemann operator, Fueter defined *regularity* of quaternionic functions. This class leads to fruitfull theory, but fails to contain simple algebraic functions: even identity is not regular. For that reason, Fueter enlarged this class into *holomorphic* functions which are functions whose Laplacian is regular.

As usual, **H** is the algebra of quaternion, V its subspace of purely imaginary. If $(1, i, j, k)$ is the canonical basis, we note $q = x_0 + ix_1 + jx_2 + kx_3$, its quaternionian conjugate is denoted by $q^\star$.

In section 1, differential operators D and T are defined in a *hypercomplex* manifold M. In section 2, we fix our notations for *quaternionic differential forms* and give geometric interpretations of D and T in the special case of $M = \mathbf{H}$, linking them with the Beltrami-Laplace operator. Section 3 is concerned with caracterisation of holomorphy and its relation with bi-harmonicity. In section 4 we give a short construction of the paraboloid model for the hyperbolic quaternionian space.

1. NEW OPERATORS ON HYPERCOMPLEX MANIFOLDS

Let $M(\mathbf{I}, \mathbf{J})$ be a hypercomplex manifold. One of those complex structure $\mathbf{U}$ gives, as usual, a splitting of the complexified exterior algebra into bihomogeneous components of Hodge type (p, q). The exterior derivation decomposes into a $(1, 0)$ and a $(0, 1)$ bihomogeneous operators respectively noted $\partial_\mathbf{U}$ and $\bar\partial_\mathbf{U}$.
Any automorphism φ of TM, thought as real vectorial space, extends to p-form:

$$\varphi\alpha(X_1, ..., X_p) = \alpha(\varphi X_1, ..., \varphi X_p).$$

If q is a non-null quaternion $\mathbf{Q}$, the homothety of rate q is an automorphism of TM, as *real* vectorial space. We will consider following A.Weil the operator $d^q = \mathbf{Q}^{-1} d\mathbf{Q}$, in the special case $u \in V$ we have $d^u = -i(\partial_\mathbf{U} - \bar\partial_\mathbf{U})$. and:

$$dd^u = -d^u d. \qquad (1.1)$$

We now study relations between various d^q.

Lemma 1.1:

If $u, v \in V$ *with* $uv + vu = 0$ *then*

$$d^u d^v = -d^v d^u. \qquad (1.2)$$

Proof:
$d^u d^v = \mathbf{U}^{-1} d\mathbf{U}\mathbf{V}^{-1} d\mathbf{V} = \mathbf{U}^{-1} dd^{vu^{-1}}\mathbf{U} = -\mathbf{U}^{-1} d^{vu^{-1}} d\mathbf{U} = -d^v d^u$
using (1.1) in the fourth equality.

Lemma 1.2:

If $u, v, u', v' \in V$ *with* $uv + vu = 0, u'v' + v'u' = 0$ *and* $uv = u'v'$ *then*

$$d^u d^v = d^{u'} d^{v'}. \qquad (1.3)$$

Proof:
Here we state (1.3) for functions f of class $\mathcal{C}^2$. Choose a local coordinate system $(\frac{\partial}{\partial x_0^1}, ..., \frac{\partial}{\partial x_0^n}, \frac{\partial}{\partial x_1^1}, ..., \frac{\partial}{\partial x_1^n}, \frac{\partial}{\partial x_2^1}, ..., \frac{\partial}{\partial x_2^n}, \frac{\partial}{\partial x_3^1}, ..., \frac{\partial}{\partial x_3^n})$ such that $\mathbf{U}$ and $\mathbf{V}$ have matrix

$$\begin{pmatrix} 0 & -Id & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & 0 & -Id \\ 0 & 0 & Id & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & -Id & 0 \\ 0 & 0 & 0 & Id \\ Id & 0 & 0 & 0 \\ 0 & -Id & 0 & 0 \end{pmatrix}$$

We have

$$d^u d^v f = \sum_{1 \le p,q \le n; r,s \in \{0,1,2,3\}} \frac{\partial^2 f}{\partial x_r^p \partial x_s^q} \mathbf{U} dx_r^p \wedge \mathbf{V} dx_s^q$$

same formula holds for (u', v'). There exists a real t such that $u' = cost.u + sint.v$ and $v' = -sint.u + cost.v$. Formula (1.3) follows from

$$\mathbf{U} dx_r^p \wedge \mathbf{V} dx_s^q + \mathbf{U} dx_s^q \wedge \mathbf{V} dx_r^p = \mathbf{U}' dx_r^p \wedge \mathbf{V}' dx_s^q + \mathbf{U}' dx_s^q \wedge \mathbf{V}' dx_r^p \text{ if } x_r^p \neq x_s^q.$$

and

$$\mathbf{U} dx_r^p \wedge \mathbf{V} dx_r^p = \mathbf{U}' dx_r^p \wedge \mathbf{V}' dx_r^p \text{ for all } (p, r).$$

Definition 1.3:

To $u \in V$ we associate

a) An operator of degree 2

$$D^u = dd^u + d^v d^w. \qquad (1.5)$$

b) An operator of degree 3

$$T^u = -dd^v d^w. \qquad (1.6)$$

where (v, w) are choosen such that $(1, u, v, w)$ is an orthonormal direct basis of $\mathbf{H}$.

Definition 1.4:

$T = d^i d^j d^k$.

Remarks :

(i) Definition 1.3 is consistant since $d^v d^w$ depends only on $u = vw$.

(ii) Applying Lemma 1.2 three times we get for any orthonormal direct basis $(1, u, v, w)$

$$d^u d^v d^w = d^i d^j d^k$$

then T is intrinsec.

Lemma 1.5:

(i) For any $u, u' \in V$ we have
$D^u \circ D^u = 2dT = 2d^{u'} T^{u'}$.

(ii) If u and v anticommutes
$D^u \circ D^v = 0$.

We have :

$$\mathbf{U}d^u = d\mathbf{U}; d^u\mathbf{U} = (-1)^p \mathbf{U}^{-1} d \quad (1.7) \quad \text{on} \quad p\text{-forms}$$

for all $u, v \in V$ with $uv + vu = 0$ we have

$$\mathbf{V}d^u = -d^w \mathbf{V} \qquad (1.8)$$

where $w = uv$. From (1.7) and (1.8) we deduce :

Lemma 1.6:

For u, v orthogonal unit imaginary quaternion.

(a) $[D^u, \mathbf{U}] = 0 \qquad (1.9)$

(b) $\{D^u, \mathbf{V}\} = 0 \qquad (1.10)$

(c) $\mathbf{U}T = T^u \mathbf{U} \qquad (1.11)$.

2. INTERPRETATION OF T AND D

We extend usual formulae for wedging and derivation to the tensor product by $\mathbf{H}$ of the exterior algebra of $\mathbf{H}$, the usual Hodge-de Rham operator, $*$, is extended by $\mathbf{H}$ linearity.

Let us consider the Fueter operators:

$$\begin{aligned} \partial^\star =& \frac{1}{2}(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}) \\ \partial =& \frac{1}{2}(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}) \end{aligned} \qquad (2.1)$$

Δ beeing the ordinary Laplacian, we have

$$\partial\partial^\star = \partial^\star\partial = 4\Delta. \qquad (2.2)$$

In the particular case of $\mathbf{H}$:

$$\begin{aligned} D^u =& \Delta \otimes (dy_0 \wedge dy_1 + dy_2 \wedge dy_3) \\ dT =& \Delta^2 \otimes dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 \end{aligned} \qquad (2.3)$$

dy_l beeing the dual basis of $(1, u, v, w)$.

Definition 2.1 (Fueter):

A function $f : \mathbf{H} \to \mathbf{H}$ is left-regular at $q \in \mathbf{H}$ if it is $\mathbf{R}$-differentiable and $\partial^\star f = 0$ at q.

This class of function has been studied by A.Sudbery.

3. QUATERNIONIC HOLOMORPHIC FUNCTIONS.

Definition 3.1 (Fueter):

A function $f : \mathbf{H} \to \mathbf{H}$ is left holomorphic at q if it is 3 times real- differentiable at q and if Δf is left regular at q.

Holomorphy of f at q is equivalent to $\partial^\star\partial\partial^\star f = 0$ (3.1)

If we call bi-harmonic the solutions of $\Delta^2 f = 0$, left holomorphy implies bi-harmonicity .

Lemma 3.2 (Fueter):

$\mathbf{Z}$-Power functions are holomorphic on their domains.

Lemma 3.3:

Let $f : \mathbf{H} \to \mathbf{H}$ be a 3-times real-differentiable function, then

$$Tf = *d\Delta f \qquad (3.2)$$

Proof:
Let $f : \mathbf{H} \to \mathbf{H}$ be a 3 times real differentiable function by direct calculation :

$$\begin{aligned}
d^k f =& \frac{\partial f}{\partial x_0} dx_3 + \frac{\partial f}{\partial x_1} dx_2 - \frac{\partial f}{\partial x_2} dx_1 - \frac{\partial f}{\partial x_3} dx_0 \\
d^j d^k f =& (\frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial x_1^2}) dx_2 \wedge dx_3 + (\frac{\partial^2 f}{\partial x_1 \partial x_3} + \frac{\partial^2 f}{\partial x_2 \partial x_0}) dx_1 \wedge dx_2 \\
&+ (\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_3 \partial x_0}) dx_2 \wedge dx_0 + (\frac{\partial^2 f}{\partial x_3 \partial x_1} + \frac{\partial^2 f}{\partial x_0 \partial x_2}) dx_3 \wedge dx_0 \, . \\
&+ (\frac{\partial^2 f}{\partial x_0 \partial x_3} - \frac{\partial^2 f}{\partial x_1 \partial x_2}) dx_1 \wedge dx_3 + (\frac{\partial^2 f}{\partial x_3^2} + \frac{\partial^2 f}{\partial x_2^2}) dx_0 \wedge dx_1 \\
Tf =& \frac{\partial \Delta f}{\partial x_0} * dx_0 + \frac{\partial \Delta f}{\partial x_1} * dx_1 + \frac{\partial \Delta f}{\partial x_2} * dx_2 + \frac{\partial \Delta f}{\partial x_3} * dx_3
\end{aligned}$$

Using commutation of $\mathbf{U}$ with the operator $*$ we get the general formula

$$T^u f = *d^u \Delta f \qquad (3.3)$$

Theorem 3.4:

The following statements are equivalent
(a) f is holomorphic at q
(b) $dq \wedge Tf(q) = 0$
(c) $d^u q \wedge T^u f(q) = 0$

Proof :
From formula (3.2) we get
$dq \wedge Tf(q)$
$= (dx_0 + idx_1 + jdx_2 + kdx_3) \wedge (\frac{\partial \Delta f}{\partial x_0} * dx_0 + \frac{\partial \Delta f}{\partial x_1} * dx_1 + \frac{\partial \Delta f}{\partial x_2} * dx_2 + \frac{\partial \Delta f}{\partial x_3} * dx_3)$
$= \partial^\star \Delta f . dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$
Using now formula (3.1) we have the equivalence of (a) and (b). The same holds with formula (3.3) for the equivalence of (a) and (c).

We now turn to relations between quaternionic holomorphy and bi-harmonicity.

Theorem 3.5:

If f is a real bi-harmonic function, on a star-shaped open set U included in $\mathbf{H}$, with continuous partial derivatives of order 4, then there exists at most one holomorphic function φ on U such that $Re(\varphi) = f$.

Proof :

Assuming that U is star-shaped with respect to the origine, the function

$$f(q) = f(0) + 2\int_0^1 \partial f(rq).q dr \qquad (3.5)$$

does the job:

$$\begin{aligned} Re\varphi(q) =& f(0) + 2\int_0^1 Re(\partial f(rq).q)dr \\ =& f(0) + \int_0^1 \frac{d}{dr} f(rq)dr \\ =& f(q) \end{aligned}$$

In this situation, swapping between derivation and integration is allowed, and since for all real differentiable function from $\mathbf{H}$ to $\mathbf{H}$ the relations

$$\begin{aligned} \partial^\star(g(q).q) =& (\partial^\star g(q)).q - (g(q))^\star \\ \partial(g(q).q) =& (\partial g(q)).q + 2Re(g(q)) \end{aligned}$$

hold, we get:

$$\partial^\star \partial \partial^\star \varphi(q) = 2\int_0^1 r^3 \partial^\star \partial \partial^\star \partial f(rq).q - r^2(\partial \partial^\star \partial f(rq))^\star + r^2 \partial^\star \partial \partial^\star f(rq)dr$$

Now as f is bi-harmonic and real, we have $\partial \partial^\star \partial \partial^\star \varphi = 0$. $\partial \partial^\star f$ is also real then $(\partial \partial^\star \partial f(sq))^\star = \partial^\star \partial \partial^\star f(sq)$. We conclude with commutation of the operators ∂ and $\partial^\star$, formula (2.2).

4. PARABOLOID MODEL

By identification of the point $X = (x_1, y_1, z_1, t_1, \dots, x_n, y_n, z_n, t_n)$ of the unit Euclidean ball of $\mathbf{R}^{4n}$ with the point $Z = (z_1, \dots, z_n)$ of the unit ball of $\mathbf{H}^n$, we get a hypercomplex structure. The Riemannian metric

$$g_X(V_1, V_2) = Re\left\{ \frac{1}{(1 - Z.Z^\star)}((1 - Z.Z^\star)W_1.W_2^\star + (W_1.Z^\star)(Z.W_2^\star)) \right\} \quad (4.1).$$

where Z, W_1, W_2 are images of X and of the tangent vectors V_1 and V_2, is hermitian hypercomplex, called hyperbolic metric. The 2-form ω_u for any unit imaginary quaternion u and the fundamental 4-form of Bonan Ω have expressions:

$$\omega_u = \frac{1}{(1-Z.Z^\star)^2}\left(\psi \wedge \mathbf{U}\psi + \mathbf{V}\psi \wedge \mathbf{W}\psi + \frac{1}{2}(1-Z.Z^\star)d\mathbf{U}\psi\right) \quad (4.2).$$

$$\begin{aligned}\Omega = & \frac{1}{(1-Z.Z^\star)^4}\Bigg\{6\psi \wedge \mathbf{U}\psi \wedge \mathbf{V}\psi \wedge \mathbf{W}\psi \\ & + (1-Z.Z^\star)(\psi \wedge \mathbf{U}\psi \wedge d\mathbf{U}\psi + \psi \wedge \mathbf{V}\psi \wedge d\mathbf{V}\psi + \psi \wedge \mathbf{W}\psi \wedge d\mathbf{W}\psi \\ & + \mathbf{U}\psi \wedge \mathbf{V}\psi \wedge d\mathbf{W}\psi + \mathbf{W}\psi \wedge \mathbf{U}\psi \wedge d\mathbf{V}\psi + \mathbf{V}\psi \wedge \mathbf{W}\psi \wedge d\mathbf{U}\psi) \\ & + \frac{(1-Z.Z^\star)^2}{4}((d\mathbf{U}\psi)^2 + (d\mathbf{V}\psi)^2 + (d\mathbf{W}\psi)^2)\Bigg\} \quad (4.3)\end{aligned}$$

where $\psi = \sum_{l=1}^n x_l dx_l + y_l dy_l + z_l dz_l + t_l dt_l$ and $(1, u, v, w)$ is an orthonormal basis of $\mathbf{H}$. Formula (4.2) is obtained from (4.1) by direct computation, (4.3) is straightforward from (4.2)

Lemma 4.1:
The function

$$\begin{array}{cccc} f_0 : & \overset{\circ}{B}_{\mathbf{H}^n} & \to & \mathbf{R} \\ & Z & \mapsto & \frac{1}{16} ln(1-Z.Z^\star) \end{array}$$

is a " potential" for the hyperbolic metric, it satisfies :

$$dTf_0 = \Phi$$

Proof:
As T is intrinsec, we can compute using any orthonormal basis of $\mathbf{H}$, $(1, u, v, w)$. The 2-form $d\mathbf{U}\psi$ is invariant under $\mathbf{U}$, and changed into its opposite under $\mathbf{V}$. We have $d(1-Z.Z^\star) = -2\psi$, now the computation is straightforward :

$$\begin{aligned}16dTf_0 &= d\mathbf{I}^{-1}d\mathbf{I}\mathbf{J}^{-1}d\mathbf{J}\mathbf{K}^{-1}d\mathbf{K}ln(1-Z.Z^\star) \\ &= d\mathbf{I}^{-1}d\mathbf{K}^{-1}d\mathbf{I}^{-1}dln(1-Z.Z^\star) \\ &= d\mathbf{I}^{-1}d\mathbf{K}^{-1}d\Big(\frac{2\mathbf{I}\psi}{(1-Z.Z^\star)}\Big) \\ &= 2d\mathbf{I}^{-1}d\Big(\frac{2\mathbf{K}\psi}{(1-Z.Z^\star)^2} \wedge \mathbf{J}\psi - \frac{1}{1-Z.Z^\star}d\mathbf{I}\psi\Big) \\ &= d\Bigg[\frac{8}{(1-Z.Z^\star)^3}\mathbf{I}\psi \wedge \mathbf{J}\psi \wedge \mathbf{K}\psi \\ &\qquad + \frac{2}{(1-Z.Z^\star)^2}(\mathbf{I}\psi \wedge d\mathbf{I}\psi + \mathbf{J}\psi \wedge d\mathbf{J}\psi + \mathbf{K}\psi \wedge d\mathbf{K}\psi)\Bigg]\end{aligned}$$

with the last differentiation we get formula (4.3).

As $1 - Z.Z^\star = \dfrac{1}{ch^2(\varrho(Z))}$, where $\varrho(Z)$ is the distance to the centre, the function f_0 is invariant under the action of the isotropy group of the centre of the ball, the isometry group acting transitively on the hyperbolic space, we get for any point X a potential invariant under K_X : $f_X(Z) = \frac{-1}{8} ln(ch(dist(Z,X)))$. Letting X converge to a point q of the boundary, we get that one eighth of the Busemann function is a potential, for instance if q is equal to $(1, 0, \dots, 0)$ we get $\frac{-1}{8} ln \frac{|1+z_1|}{(1-\sum_{l=1}^n |z_l|^2)^{\frac{1}{2}}}$

We can now carry out the construction of the paraboloid model :
The Cayley transform

$$\begin{array}{ccccc} \mathcal{C} & : & \overset{\circ}{B}_{\mathbf{H}^n} & \to & \wp_n \\ & & (z_1, \dots, z_n) & \mapsto & (w_1, \dots, w_n) \end{array}$$

where $w_1 = \dfrac{(1+z_1)^\star(1-z_1)}{2 \mid 1+z_1 \mid^2}$ and $w_l = \dfrac{(1+z_1)^\star z_l}{\mid 1+z_1 \mid^2}$ for $l \in \{2...n\}$.
maps the unit ball $\overset{\circ}{B}_{\mathbf{H}^n}$ on $\wp_n = \{W \in \mathbf{H}^n \quad s.t. \quad F(W) = 2Rew_1 - W'.W'^\star > 0\}$ where $W = (w_1, w_2, ..., w_n)$ and $W' = (w_2, ..., w_n)$. The set $\wp_n$ is naturaly hypercomplex.

We choose now the metric so as the Cayley transform becomes an isometry:
The function $\frac{-1}{8} ln \frac{|1+z_1|}{(1-\sum_{l=1}^n |z_l|^2)^{\frac{1}{2}}}$ is a potential for the hyperbolic metric, and

$$F(\mathcal{C}Z) = 2Re\Big(\frac{(1+z_1)^\star(1-z_1)}{2 \mid 1+z_1 \mid^2}\Big) - \frac{(1+z_1)^\star}{\mid 1+z_1 \mid^2} Z'.Z'^\star \frac{(1+z_1)}{\mid 1+z_1 \mid^2} = \frac{1 - Z.Z^\star}{\mid 1+z_1 \mid^2}.$$

therefore $\dfrac{1}{16} lnF$ must be a potential for the hyperbolic metric of $\wp_n$, and $\Phi = dT\frac{1}{16}lnF$ is the fundamental 4-form. We have

$$\begin{aligned} \Phi &= dT\frac{1}{16} lnF \\ &= \frac{1}{4}\Bigg[\frac{-24}{f^4}(dx_1 - \psi') \wedge (dt_1 - \mathbf{K}\psi') \wedge (dz_1 - \mathbf{J}\psi') \wedge (dy_1 - \mathbf{I}\psi') \\ &\quad + \frac{4}{f^3}\Bigg(d\mathbf{K}\psi' \wedge \Big((dx_1 - \psi') \wedge (dt_1 - \mathbf{K}\psi') - (dz_1 - \mathbf{J}\psi') \wedge (dy_1 - \mathbf{I}\psi')\Big) \\ &\qquad + d\mathbf{J}\psi' \wedge \Big((dt_1 - \mathbf{K}\psi') \wedge (dy_1 - \mathbf{I}\psi') + (dx_1 - \psi') \wedge (dz_1 - \mathbf{J}\psi')\Big) \\ &\qquad + d\mathbf{I}\psi' \wedge \Big((dx_1 - \psi') \wedge (dy_1 - \mathbf{I}\psi') - (dt_1 - \mathbf{K}\psi') \wedge (dz_1 - \mathbf{J}\psi')\Big) \\ &\quad + \frac{1}{f^2}(d\mathbf{I}\psi'^2 + d\mathbf{J}\psi'^2 + d\mathbf{K}\psi'^2)\Bigg] \end{aligned}$$

where ψ' have same definition as ψ but with indices running from 2 to n. We now compute the metric applying formula $g_W(dW) = \left(\frac{\Phi_W(dW, \mathbf{I}dW, \mathbf{J}dW, \mathbf{K}dW)}{6}\right)^{\frac{1}{2}}$.

We get :

$$g_W(dW) = \frac{1}{f^2}\Big(((dx_1 - \psi_1')dW)^2 + ((dy_1 - \psi_2')dW)^2 \\ +((dz_1 - \psi_3')dW)^2 + ((dt_1 - \psi_4')dW)^2 + f dW'.dW'^{\star}\Big).$$

$$g_W(dW) = \frac{1}{f^2(W)}\big(f(W)dW'.dW'^{\star} + (dw_1 - dW'.W'^{\star})(dw_1^{\star} - W'.dW'^{\star})\big).$$

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