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Instantons on Quaternion-Kähler Manifolds

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Abstract. On $4n$-dimensional quaternion-Kähler manifolds, self-dual connections are defined which is the same as self-dual connections in 4-dimensional Riemannian geometry in the case $n = 1$. Every simply connected quaternion-Kähler manifold with non-zero scalar curvature admits only one standard self-dual connection in the case $n \geqq 2$. As for anti-self-dual connections, we have vanishing theorems on the Salamon twistor spaces. ADHM-construction are generalized on the quaternionic projective spaces and 2-plane complex Grassmannians. A few recent results are stated in the last section.

0. Introduction

On quaternion-Kähler manifolds, Self-Dual connections are defined by Nitta [Ni] and Mamone Capria-Salamon [M-S]. These correspond to Self-Dual connections in the 4-dimensional case.

The purpose of this note is to give classification theorems for self-dual and anti-self-dual bundles and to describe the moduli spaces. This note consists of two parts. In the former part, the rigidity theorem for self-dual connections is proved. In the case of the quaternionic projective space, the extended BPST-instanton is defined in a similar way, because the standard metric of S^{4n+3} gives the canonical connection to the Hopf fibration $S^{4n+3} \longrightarrow \mathbb{H}P^n$. However, any deformation preserving self-duality does *not* exist except $n = 1$. It is essential that the curvature form of an arbitrary self-dual connection is parallel in higher dimentional case. Consequently, it is easy to understand the holonomy algebra, which is a Lie algebra $\mathfrak{sp}(1)$. Then holonomy theorem yields our theorem.

In the latter part, we treat anti-self-dual connections and so the Salamon twistor space Z. In ADHM-construction on the 4-dimensional sphere and Buchdahl's construction on the complex projective space, the following vanishing theorem plays an essential role.

$$H^1(Z, \tilde{F}(-2)) = 0,$$

where $\tilde{F}$ is the pull back bundle of an anti-self-dual bundle and $\mathcal{O}(-2)$ is a line bundle which is associated with the complex contact structure of the Penrose twistor space Z. In §3, the above vanishing theorem is generalized to an arbitrary compact quaternion-Kähler manifold with positive scalar curvature. We establish an analogue of the Penrose transform between the cohomology group $H^1(Z, \tilde{F}(-2))$ and the solution space of quaternionic operator which is a second-order operator. The positivity of this operator implies our vanishing theorem.

In §4, we use the above vanishing theorem to generalize ADHM-construction to an arbitrary quaternionic projective space. A monad method is available for our purpose as in the 4-dimensional case. To obtain a monad, we use Beilinson's spectral sequence on the complex projective spaces. Then, further vanishing theorems are needeed. Our vanishing theorem combined with the Hirzebruch-Riemann-Roch formula gives sufficient conditions for a monad construction. The main idea is to intertwine even dimensional complex projective space between twistor spaces of quaternionic projective spaces.

In §5, a vanishing theorem $H^1(Z, \tilde{F}(-2))$ is extended. Mamone Capria and Salamon showed that there are elliptic complexes on quaternion-Kähler manifolds, and related them to Dolbeault complexes on the twistor space [M-S]. Baston extended these quaternionic complexes to include second order operators [B]. In this way, we have that there are ono-to-one correspondence between the Dolbeault cohomology groups on the twistor space and the cohomology of the corresponding elliptic complex on a quaternion-Kähler manifold. This elliptic complex is composed of quaternionic operators. We define new operators on a quaternion-Kähler manifold to obtain relations between quaternionic operators. The Weitzenböck formula for our operators yields vanishing theorems.

In §6, a complex Grassmannian manifold of 2-planes are the base space. We generalize Buchdahl's classification along the almost same line as in §4. Main difference are the following two points. Beilinson type spectral sequence is not known explicitly, and so we make spectral sequence for holomorphic vector bundles on the twistor space. On the other hand, the ring structure of the cohomology of the twistor space is complicated, compared with complex projective space. Consequently, to get further vanishing theorems, we prepare two divisors on the twistor space.

In §7, it is showed that the moduli space of "1-instantons" on $Gr_2(\mathbb{C}^{n+2})$ is identified with an open cone over $\mathbb{P}(\wedge^2\mathbb{C}^{n+2})$. This result is a generalization of Buchdahl and Donaldson. The monad which we obtain in §6 translates our problem into linear algebraic terms. The vertex of this moduli space represents a reducible connection. A point in the boundary manifold $\mathbb{P}(\wedge^2\mathbb{C}^{n+2})$ represents an ideal instanton whose support is a quaternionic submanifold $\mathbb{H}P^i$ $(i = 0, \cdots, [\frac{n}{2}])$ of $Gr_2(\mathbb{C}^{n+2})$.

In §8, recent results about instanton bundles are stated.

The results in §5,§6 and §7 are due to joint works with Professor Takashi Nitta.

1. Preliminaries

Let (M, g) be a compact quaternion-Kähler manifold such that $dim_{\mathbb{R}} M = 4n$.

The vector bundle $\wedge^2 T^*M$ has the following holonomy invariant decomposition:

$$\wedge^2 T^*M = S^2\mathbb{H} \oplus S^2\mathbb{E} \oplus (S^2\mathbb{H} \oplus S^2\mathbb{E})^{\perp},$$

where $\mathbb{H}$ and $\mathbb{E}$ are (local) vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. (This decomposition can also be explained in terms of the Hodge $*$-operator as in the 4-dimensional case. See [G-P].)

Definition 1.1. [Ni, M-S, G-P] An $\omega \in \Gamma(\wedge^2 T^*M) = \Omega^2(M)$ is called a *Self-Dual* (SD) form or an *Anti-Self-Dual* (ASD) form if

$$\omega \in \quad \Gamma(S^2\mathbb{H}) \text{ or } \Gamma(S^2\mathbb{E}),$$

respectively.

We shall treat metric connections on a complex vector bundle E equipped with a hermitian metric h over M.

Definition 1.2. [Ni, M-S, G-P] A connection ∇ is called *SD* or *ASD* if its curvature 2-form R^{∇} is a SD form or an ASD form, respectively..

Theorem 1.3. [Ni, M-S, G-P] *Every SD (or ASD) connection is a Yang-Mills connection.*

Remark. Moreover, if M is compact, then an arbitrary SD or ASD connection minimizes the Yang-Mills functional [M-S, G-P].

2. Rigidity of Self-Dual connections

First of all, we state a result about Self-Dual connections. Here, the dimension of M is assumed to be greater than or equal to 8 (*i.e* $n > 1$.) We give a few examples of SD connections.

Example 2.1. *A quaternion-Kähler manifold M is an Einstein manifold. Using this fact, we can show that the induced connection D on $S^2\mathbb{H}$ is a SD connection in the same way as 4-dimensional case.*

Example 2.2. *The quaternionic projective space $\mathbb{H}P^n$ is a typical example of a compact quaternion-Kähler manifold. The Hopf fibration $S^{4n+3} \to \mathbb{H}P^n$ is a principal $Sp(1)$-bundle. The canonical connection induced by the standard metric on S^{4n+3} is a SD connection.*

A key property of SD connections is as follows.

Lemma 2.3. *Let E be a vector bundle over M with a connection ∇. If S is a section of $End(E) \otimes S^2\mathbb{H}$ which satisfies the Bianchi's identity $d^{\nabla}S = 0$, then S is parallel with respect to the induced connection ∇.*

From the definition of SD connection, we get

Corollaly 2.4. *Any Self-Dual connection is a parallel Yang-Mills connection.*

Let P be a reduced principal $Sp(n) \cdot Sp(1)$-bundle associated to the tangent bundle TM. If we define a $SO(3)$-bundle $P' = P/Sp(n)$, then $P' \times \mathfrak{so}(3)$ is nothing but $S^2\mathbb{H}$. A principal bundle P' has a connection induced by the Riemannian connection D of M, which is denoted by φ. The next theorem asserts that φ is the only one SD connection.

Theorem 2.5. *Let M be a simply-connected quaternion-Kähler manifold of dimension $4n > 4$ which has a non-zero scalar curvature. Assume that (E, h) is a rank r vector bundle with a metric h. If (E, h) admits a metric non-flat Self-Dual connection ∇, then (E, ∇) can be expressed as follows.*

- **(i)** *Suppose that the structure group of the bundle $S^2\mathbb{H}$ can be lifted to $Sp(1)$. In other words, there exists a principal $Sp(1)$-bundle $\tilde{P}'$ whose adjoint bundle is the bundle $S^2\mathbb{H}$. (See Example 2.2.) In this case, we have a representation ρ of $Sp(1)$ such that*

$$E \cong \tilde{P}' \times_{\rho} \mathbb{C}^r \quad \textit{and} \quad \nabla = \rho_* \tilde{\varphi},$$

where $\tilde{\varphi}$ denotes the connection in $\tilde{P}'$ induced from the connection φ in P',

(ii) *Otherwise, there exists a representation ρ of $SO(3)$ such that*

$$E \cong P' \times_\rho \mathbb{C}^r \quad \textit{and} \quad \nabla = \rho_* \varphi.$$

Remark 1. Salamon showed that a compact quaternion-Kähler manifold which has a positive scalar curvature is simply connected [S].

Remark 2. Combined with a result of Salamon [S], a compact quaternion-Kähler manifold with a positive scalar curvature can not admit any $Sp(1)$-Self-Dual connections except the quaternionic projective space.

For more details, see [Na1].

3. Vanishing Theorem

Next, we focus attention on ASD connections. Salamon showed that there is a twistor fibration $\pi : Z \to M$ over an arbitrary quaternion-Kähler manifold M [S]. The total space Z with compact fibre $\mathbb{P}^1$ has naturally a complex structure in the same way as 4-dimensional case. For example, a complex projective space $\mathbb{P}^{2n+1}$ is the twistor space of $\mathbb{H}P^n$. To construct and classify ASD bundles E on $\mathbb{H}P^1 \cong S^4$, Atiyah, Drinfeld, Hitchin and Manin make use of the twistor space $\mathbb{P}^3$. The crucial step in this work is the vanishing of the sheaf cohomology group

$$H^1(\mathbb{P}^3, \tilde{E}(-2)) = 0,$$

where $\tilde{E}$ is the pull-back bundle on $\mathbb{P}^3$. (We do not distinguish between a vector bundle and the associated locally free sheaf.) First, the above vanishing theorem was proved by Drinfeld-Manin [D-M] and Rawnsley [R] in the case of S^4. Hitchin gave a generalization to a compact half-conformally-flat manifold with positive scalar curvature [H].

In higher dimensional case, Nitta and Mamone Capria-Salamon showed the following.

Theorem 3.1 [Ni, M-S]. *Any ASD bundle on M pulls back to a holomorphic bundle on Z.*

So, if we attempt to generalize ADHM-construction, it is natural to ask whether certain cohomology groups vanish. Here the author introduces a summary of [Na2], in which the following theorem is proved.

Theorem 3.2. *Let M be a compact quaternion-Kähler manifold with positive scalar curvature and E be an ASD bundle on M with a unitary structure. If the twistor space of M is denoted by Z and the pull-back bundle of E on Z is denoted by $\tilde{E}$, then we have*

$$H^1(Z, \tilde{E} \otimes L^{-2}) = 0,$$

where the line bundle L, which is introduced by Salamon [S]*, restricted to an arbitrary fiber (or real line) is the hyperplane bundle on $\mathbb{P}^1$.*

outline of proof. We identify $H^1(Z, \tilde{E}\otimes L^{-2})$ with the corresponding Dolbeault cohomology group. If we fix a base point $x \in M$, we get a transformation $T : H^{0,1}(Z, \tilde{E}\otimes L^{-2}) \to \Gamma(E)$

in the following way.

$$\begin{aligned} H^{0,1}(Z, \tilde{E} \otimes L^{-2}) &\underset{\text{restriction}}{\longrightarrow} H^{0,1}(\mathbb{P}_x, \mathcal{O}(-2)) \otimes E_x \\ &\underset{\text{Serre duality}}{\sim} H^{1,0}(\mathbb{P}_x, \mathcal{O}(2))^* \otimes E_x \\ &\cong H^0(\mathbb{P}_x, \mathcal{O})^* \otimes E_x \\ &\cong E_x. \end{aligned}$$

It can be showed that the horizontal part of the element of the above Dolbeault cohomology group is completely determined in terms of vertical part, from the complex structure of the twistor space Z. Hence, a transformation T is injective.

An element of $H^{0,1}(Z, \tilde{E} \otimes L^{-2})$ is denoted by ω. We put

$$T(\omega) = s \in \Gamma(E).$$

Then $\overline{\partial}\omega = 0$ implies that

$$(\nabla^*\nabla + 4nct)s = 0,$$

where $\nabla^*\nabla$ is the Laplace operator, c is a positive constant depending only on n ([S]) and t is the scalar curvature. From the hypothesis of theorem 3.1, $\nabla^*\nabla + 4nct$ is positive as an operator. So, the equation $(\nabla^*\nabla + 4nct)s = 0$ has only a trivial solution. □

Remark. The above operator is explained in terms of a *Dirac operator* and a *twistor operator* in 4-dimensional case [H]. In higher dimensional case, we can define a *Dirac-like operator* and a *twistor operator* (see also [S-2] and [B]). Moreover, the above operator $\nabla^*\nabla + 4nct$ is also closely related to these operator via the Weitzenböck formula. Using these operators, we can show various vanishing theorems which will be explained in the section 5.

Since $\tilde{E}$ restricted to an arbitrary fibre $\mathbb{P}^1$ is holomorphically trivial, we obtain the following.

Corollary 3.3. *If we put $M = \mathbb{H}P^n$ in Theorem 3.2, then*

$$H^1(\mathbb{P}^{2n+1}, \tilde{E}(k)) = 0, \qquad k \leqq -2,$$

where $E(k)$ means $E \otimes \mathcal{O}(k)$.

4. k-instantons on $\mathbb{H}P^n$

In this section, we shall confine ourselves exclusively to a quaternionic projective space $\mathbb{H}P^n$. We treat a bundle E on $\mathbb{H}P^n$ satisfying the following.

(1) E has an ASD $Sp(r)$-connection ($r \geqq 2$), and
(2) its total Chern clss $c(E) = (1 - x)^{-k}$, where $k \in \mathbb{Z}_+$ and x is the standard positive generator of $H^4(\mathbb{H}P^n, \mathbb{Z})$.

Remark. When $r = n$ and $k = 1$, the pull-back bundle $\tilde{E}$ on $\mathbb{P}^{2n+1}$ is the null correlation bundle. In general, if $r = n$, $\tilde{E}$ is called a mathematical instanton bundle [O-S, S-T, A-O]. On the other hand, in the case $n = 1$, E is a k-instanton bundle on $\mathbb{H}P^1 \cong S^4$.

Then, the following vanishing theorem holds (See [K-N]).

Theorem 4.1.
$$H^i(\mathbb{P}^{2n+1}, \tilde{E}(p)) = 0 \quad \textit{for } 2 \leqq i \leqq n \textit{ and } p \in \mathbb{Z},$$

Remark 1. Applying Serre duality, from a symplectic structure of E, we know
$$H^i(\mathbb{P}^{2n+1}, \tilde{E}(p)) = 0 \quad \textit{for } 2 \leqq i \leqq 2n-1 \textit{ and } p \in \mathbb{Z},$$

Remark 2. Theorem 4.1 is proved under a weaker condition with respect to its total Chern class such as
$$c_2(E) = k, \qquad c_4(E) = \frac{1}{2}k(k+1),$$
because we make use of the Hirzebruch-Riemann-Roch theorem only on $\mathbb{P}^4$.

Combined with Beilinson's spectral sequence (see [O-S-S p.240]), our vanishing theorem (Corollary 3.3 and Theorem 4.1) yields the next classification theorem [K-N].

Theorem 4.2. *$\tilde{E}$ is the cohomology bundle of a monad of the type:*
$$W(-1) \longrightarrow V \longrightarrow W^*(1),$$
where $W^* = H^1(\mathbb{P}^{2n+1}, \tilde{E}(-1))$, $V = H^1(\mathbb{P}^{2n+1}, \tilde{E} \otimes \Omega^1)$ (see [C-S],or[O-S]).
$$\textit{and} \quad dim_{\mathbb{C}} H^1(\mathbb{P}^{2n+1}, \tilde{E}(-1)) = k.$$

Remark 1. A "monad" is a complex of vector bundles
$$A \xrightarrow{a} B \xrightarrow{b} C,$$
with homomorphisms a and b between them, such that a is injective and b is surjective. The quotient bundle
$$E = Kerb/Ima$$
is called the cohomology of the monad.

Remark 2. If the base space is $\mathbb{H}P^1 \cong S^4$, the monad in Theorem 4.2 corresponds to one used in ADHM-construction (See [A] or [B-H]).

To prove Theorem 4.1, we employ an algebraic geometrical technique and induction with respect to the dimension of the base space. The main idea is to intertwine the twistor spaces with even dimensional complex projective space. Explicitly, we consider the following diagram,

$$\begin{array}{ccc} \mathbb{P}^{2n-1} & \longrightarrow & \mathbb{P}^{2n+1} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{H}P^{n-1} & \longrightarrow & \mathbb{H}P^n \end{array}$$

and the sequence $\mathbb{P}^{2n-1} \subset \mathbb{P}^{2n} \subset \mathbb{P}^{2n+1}$, where $\mathbb{P}^{2n-1}$ and $\mathbb{P}^{2n+1}$ are the corresponding twistor spaces, respectively. Making use of the standard sequence of sheaves
$$0 \longrightarrow \mathcal{O}(-1)|_{\mathbb{P}^{2n}} \longrightarrow \mathcal{O}|_{\mathbb{P}^{2n}} \longrightarrow \mathcal{O}|_{\mathbb{P}^{2n-1}} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}(-1)|_{\mathbb{P}^{2n+1}} \longrightarrow \mathcal{O}|_{\mathbb{P}^{2n+1}} \longrightarrow \mathcal{O}|_{\mathbb{P}^{2n}} \longrightarrow 0,$$
we get imformation on $\mathbb{H}P^n$ from one on $\mathbb{H}P^{n-1}$.

5. Vanishing Theorem II

Baston showed that there are elliptic complexes on compact quaternion-Kähler manifolds with positive scalar curvature ([B] and see also [S-2]). The elliptic complexes are related to the Dolbeault complexes on the twistor spaces. Here we state vanishing theorems for these elliptic complexes, which are generalizations of Theorem 3.2.

We denote by $S^k\mathbb{H}$ and $\wedge^i\mathbb{E}$ kth-symmetric product of $\mathbb{H}$ and i-th skew-symmetric product of $\mathbb{E}$, respectively. Recall the Clebsch-Gordan formula

$$S^k\mathbb{H}\otimes\mathbb{H}\cong S^{k-1}\mathbb{H}\oplus S^{k+1}\mathbb{H}.$$

The tensor product $S^k\mathbb{H}\otimes\wedge^i\mathbb{E}\otimes\mathbb{H}\otimes\mathbb{E}$ involves the two components $S^{k-1}\mathbb{H}\otimes\wedge^{i+1}\mathbb{E}$ and $S^{k+1}\mathbb{H}\otimes\wedge^{i+1}\mathbb{E}$. Using the contraction $Cont: S^k\mathbb{H}\otimes\wedge^i\mathbb{E}\otimes\mathbb{H}\otimes\mathbb{E}\to S^{k-1}\mathbb{H}\otimes\wedge^{i+1}\mathbb{E}$, and the symmetrization $Sym: S^k\mathbb{H}\otimes\wedge^i\mathbb{E}\otimes\mathbb{H}\otimes\mathbb{E}\to S^{k+1}\mathbb{H}\otimes\wedge^{i+1}\mathbb{E}$, We define two types of differential operators on M (cf. [S-2] and [B]).

Definition 5.1. Quaternionic operators D and $\mathcal{D}$ are defined by

$$\begin{aligned} D_k =& Cont\nabla:\Gamma(S^k\mathbb{H}\otimes\wedge^i\mathbb{E})\to\Gamma(S^{k-1}\mathbb{H}\otimes\wedge^{i+1}\mathbb{E}),\\ \mathcal{D}_k =& Sym\nabla:\Gamma(S^k\mathbb{H}\otimes\wedge^i\mathbb{E})\to\Gamma(S^{k+1}\mathbb{H}\otimes\wedge^{i+1}\mathbb{E}). \end{aligned}$$

Remark 1. When no confusion can arise, we will omit the subscript k.

Remark 2. In case of $n=1$, the above operators D and $\mathcal{D}$ are nothing but the Dirac operator and the twistor operator, respectively.

As quaternionic analogues of (complex) Kähler geometry, we define

$$L: S^k\mathbb{H}\otimes\wedge^i\mathbb{E}\to S^k\mathbb{H}\otimes\wedge^{i+2}\mathbb{E}$$

as $L(\phi)\underset{\text{def.}}{=}\phi\wedge\omega_{\mathbb{E}}$, where $\omega_{\mathbb{E}}$ is the section of $\wedge^2\mathbb{E}$ determined by an invariant symplectic form [S]. The operator $\Lambda: S^k\mathbb{H}\otimes\wedge^{i+2}\mathbb{E}\to S^k\mathbb{H}\otimes\wedge^i\mathbb{E}$ denotes the formal adjoint operator of L.

Baston proved the following theorem.

Theorem 5.2. ([B] and see also [S-2]) *There exist elliptic complexes on a compact quaternion-Kähler manifold M such that*

$$\begin{aligned} B_k: 0\to S^{k-2}\mathbb{H}\underset{D}{\to}S^{k-3}\mathbb{H}\otimes\mathbb{E}\underset{D}{\to}\cdots\underset{D}{\to}\wedge^{k-2}\mathbb{E}\\ \underset{\triangle}{\to}\wedge^k\mathbb{E}\underset{\mathcal{D}}{\to}\mathbb{H}\otimes\wedge^{k+1}\mathbb{E}\underset{\mathcal{D}}{\to}\cdots\underset{\mathcal{D}}{\to}S^{2n-k}\mathbb{H}\to 0, \end{aligned}$$

where

$$\triangle=-D\mathcal{D}+4ctL.$$

Moreover, if E is an ASD bundle on M, we can couple these complexes with E.

From now on, the author present a joint work with T.Nitta [N-N1]. To investigate these complexes, we denote by D^* and $\mathcal{D}^*$ the formal adjoint operator of D and $\mathcal{D}$ respectively. Then we have the following formulae.

Proposition 5.3.

$$\begin{aligned} D_{m+1}\mathcal{D}_m + \frac{m+2}{m+1}\mathcal{D}_{m-1}D_m &= 2ct(m+2)L, \\ [\Lambda, L] &= (n-i)id., \\ [L, \mathcal{D}^*] &= -D, \\ [\mathcal{D}, \Lambda] &= -D^*, \\ [D, \Lambda] &= \mathcal{D}^*, \\ [L, D^*] &= \mathcal{D}. \end{aligned}$$

Using these formulae, we get the following.

Theorem 5.4. *Let M be a compact quaternion-Kähler manifold $(dim M = 4n)$ with positive scalar curvature and F be a unitary anti-self-dual bundle on M. Then*

$$H^i(\Gamma(M, F \otimes B_k)) = 0 \quad \text{for} \quad i < k-2 \quad \text{or} \quad (i,k) = (0,2),\ (1,3) \quad \text{and} \quad i \leqq n-1.$$

Combined with the Penrose transform by Baston ([B]) which yields $H^i(\Gamma(M, E \otimes B_k)) = H^{i+1}(Z, \tilde{E} \otimes L^{-k})$, Theorem 5.4 implies vanishing theorems for the Dolbeault cohomology groups.

Corollaly 5.5. *Let M be a $4n$-dimensional compact quaternion-Kähler manifold with positive scalar curvature and Z be the twistor space of M. If $\tilde{E}$ is the pull back bundle of E on M which has a unitary structure and an anti-self-dual connection, then we have*

$$H^i(Z, \tilde{E} \otimes L^{-k}) = 0, \quad \text{for} \quad 1 \leqq i \leqq n \quad \text{and} \quad i + k + 1 < 0.$$

and

$$H^2(Z, \tilde{E} \otimes L^{-3}) = 0 \ (\text{if} \quad n \geqq 2).$$

Applying Serre duality, we obtain

Corollary 5.6. *Under the same assumption in Corollary 5.5, the following vanishing theorems hold:*

$$H^i(Z, \tilde{E} \otimes L^{-k}) = 0, \quad \text{for} \quad n+1 \leqq i \leqq 2n \quad \text{and} \quad i + k > 0$$

and

$$H^{2n-1}(Z, \tilde{E} \otimes L^{-2n+1}) = 0 \ (\text{if} \quad n \geqq 2).$$

Remark. Under the same argument, vanishing theorems can be obtained for M with *negative* scalar curvature. Combined with the Penrose transform due to Mamone Capria-Salamon [M-S], these results generalize M.Thornber's vanishing theorem [T].

Theorem 5.7. *If M is a compact quaternion-Kähler manifold with negative scalar curvature* $(\dim M = 4n)$ *and $\tilde{E}$ is the pull-back of a unitary anti-self-dual bundle on M, then*

$$H^i(Z, \tilde{E} \otimes L^{-k}) = 0 \quad \textit{for} \quad k + i \geqq 1 \quad \textit{and} \quad i \leqq n - 1.$$

Remark. Horan extends independently Thornber's theorem [Ho].

6. k-instantons on $Gr_2(\mathbb{C}^{n+2})$

The twistor space of $Gr_2(\mathbb{C}^{n+2})$ is a flag manifold

$$F_n \underset{\text{def.}}{=} \{(l, V) | 0 \in l \subset V \subset \mathbb{C}^{n+2},\, dim_{\mathbb{C}} l = 1,\, dim_{\mathbb{C}} V = n + 1\},$$

(see [S]). Here we work on $Gr_2(\mathbb{C}^4)$, because no big change arises in higher dimensional case and see [N-N2] for more details.

An F_2 has double (holomorphic) fibrations to $\mathbb{P}^3$ and $\mathbb{P}^{3*}$ such that

$$\begin{aligned} p_1 &: (l, V) \longrightarrow [l] \\ p_2 &: (l, V) \longrightarrow [V]. \end{aligned}$$

We denote by x and y pull-back elements of $H^2(F_2, \mathbb{Q})$ of the standard positive generators of $H^2(\mathbb{P}^3, \mathbb{Z})$ and $H^2(\mathbb{P}^{3*}, \mathbb{Z})$ respectively. Then the cohomology ring $H^*(F_2, \mathbb{Q})$ is generated by x and y. On the other hand, the cohomology ring $H^2(G_2(\mathbb{C}^4), \mathbb{Q})$ is regarded as a subring of $H^2(F_2, \mathbb{Q})$ by the twistor fibration. Hence, an element of $H^2(Gr_2(\mathbb{C}^4), \mathbb{Q})$ may be written with x and y.

We consider a bundle E over $Gr_2(\mathbb{C}^4)$ satisfying the following.

(1) E has an ASD $Sp(r)$-connection $(r \geqq 1)$, and
(2) $c_2(E) = k(x^2 + y^2)$ and $c_4(E) = k^2 x^2 y^2$, where $k \in \mathbb{Z}$.

Remark. When $n = 1$, the bundle E is a k-instanton bundle on $\mathbb{P}^2$ which Buchdahl classified by using a monad on F_1 [Bu].

In the case of $\mathbb{H}P^n$, a monad can be obtained via vanishing theorems and Beilinson's spectral sequence on $\mathbb{P}^{2n+1}$ in section 4. Instead of Beilinson's spectral sequence, we have a Beilinson-type spectral sequence on F_n. To represent this spectral sequence, we define vector bundles on F_n.

Definition 6.1. A vector bundle Q_1 denotes the quotient bundle of $p_1^* \Omega_{\mathbb{P}^{n+1}}$ by $\mathcal{O}(-1, -1)$, where $\Omega_{\mathbb{P}^{n+1}}$ is the holomorphic cotangent bundle on $\mathbb{P}^{n+1}$ and in general, $\mathcal{O}(p, q)$ is the line bundle $p_1^* \mathcal{O}(p) \otimes p_2^* \mathcal{O}(q)$. The quotient bundle Q_2 is defined in a similar way.

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(-1, -1) \longrightarrow & p_1^* \Omega_{\mathbb{P}^{n+1}} \longrightarrow Q_1 \longrightarrow 0 \\ 0 \longrightarrow \mathcal{O}(-1, -1) \longrightarrow & p_2^* \Omega_{\mathbb{P}^{n+1*}} \longrightarrow Q_2 \longrightarrow 0. \end{aligned}$$

Proposition 6.2. *For an arbitrary holomorphic vector bundle S on F_n, there exists a spectral sequence $E_r^{p,q}$ converging to*

$$E_\infty^{p,q} = \begin{cases} \sum_{p=0}^{2n+1} E_\infty^{-p,p} = S, & \text{if } p+q=0 \\ 0, & \text{otherwise.} \end{cases}$$

The E_1-terms satisfy exact sequences

$$\cdots \to \sum_{r=0}^{p-1} H^q(F_n, \wedge^r Q_1^* \otimes S(-p,0)) \otimes \wedge^{p-1-r} Q_2^*(0,-p)$$
$$\to E_1^{-p,q}$$
$$\to \sum_{r=0}^{p} H^q(F_n, \wedge^r Q_1^* \otimes S(-p,0)) \otimes \wedge^{p-r} Q_2^*(0,-p) \to \cdots .$$

where, for example, $S(p,q)$ means $S \otimes \mathcal{O}(p,q)$ as usual.

Using a similar (but more complicated) argument as Theorem 4.2, we have the classification theorem.

Theorem 6.3. *$\tilde{E}$ is the cohomology bundle of the following monad,*

$$W_1(-1,0) \oplus W_2(0,-1) \to V \to W_1^*(1,0) \oplus W_2^*(0,1),$$

whrere $W_1^* = H^1(F_2, \tilde{E}(-1,0))$, $W_2^* = H^1(F_2, \tilde{E}(0,-1))$ and V is a trivial bundle of rank $4k+2r$. ($dim_{\mathbb{C}} H^1(F_2, \tilde{E}(-1,0)) = dim_{\mathbb{C}} H^1(F_2, \tilde{E}(0,-1)) = k$).

Remark. If the base space is $Gr_2(\mathbb{C}^3) \cong \mathbb{P}^{2*}$, the monad in Theorem 6.3 is what is used in Buchdahl's construction (See [Bu]).

7. Moduli of 1-instantons on $Gr_2(\mathbb{C}^{n+2})$

For simplicity, we will restrict our attention to 1-instantons on $G_2(\mathbb{C}^4)$. Then, the monad given in section 6 is

$$\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to V \to \mathcal{O}(1,0) \oplus \mathcal{O}(0,1).$$

Note that the rank of V is $4+2r$. However, it is proved that the condition $H^0(F_2, \tilde{E}) = 0$ implies that $4+2r \leqq 8$. If $\tilde{E}$ has a non-trivial section, it is proved that E contains a trivial summand, so E can not have $Sp(r)$-connection. Moreover, when $r=1$, there does not exist a monad of the above type. Consequently, the value of r must be 2. For this argument and a proof of the following theorem, see [N-N3].

Theorem 7.1. *The moduli of 1-instanton bundles on $G_2(\mathbb{C}^4)$ is identified with a (real) open cone over $\mathbb{P}^5$ (i.e. not containing the boundary mainfold $\mathbb{P}^5$). Hence, the (real) dimension of the moduli equals to 11.*

The vertex of this cone represents a direst sum $E_1 \oplus E_2$ (*i.e.* a reducible connection). These bundles are the cohomology bundles of monads:

$$\tilde{E}_1 : \mathcal{O}(0,-1) \to V_1 \to \mathcal{O}(1,0),$$
$$\tilde{E}_2 : \mathcal{O}(-1,0) \to V_1^* \to \mathcal{O}(0,1).$$

We also have isomorphisms

$$\tilde{E}_1 \cong Q_1(1,0) \quad \text{and} \quad \tilde{E}_2 \cong Q_2(0,1).$$

Remark. By Bott-Borel-Weil theorem,

$$H^1(F_2, End(\tilde{E}_1 \oplus \tilde{E}_2)) = \wedge^2\mathbb{C}^4 \oplus \wedge^2\mathbb{C}^{4\,*}.$$

On the other hand, we choose a point p of the boundary of the moduli. The vertex of the cone and p are linked by a straight line l with a parameter $0 \leqq t \leqq 1$ and $l(0) =$the vertex. Then, in the limit $(t \to 1)$, two cases occur depending on p.

(1) The curvature density converges to a one point in $Gr_2(\mathbb{C}^4)$.
(2) The curvature density converges to a *quaternionic submanifold* $\mathbb{H}P^1$ of $Gr_2(\mathbb{C}^4)$.

In higher dimensional cases, a quaternionic submanifold $\mathbb{H}P^k$ of $G_2(\mathbb{C}^{n+2})$ appears as the *limiting manifold* in the above sense.

8. Final Remarks

Remark 1. We can determine irreducible homogeneous bundles with anti-self-dual canonical connections on compact quaternion symmetric spaces G/K in terms of weights, where G is the compact real form of the simply connected complex simple Lie group. Moreover, a monad can be obtained from representation spaces of G. Consequently, ADHM-construction is generalized from the viewpoint of the representation theory. In other words, one of the way of the deformation of the canonical connection is given on every compact quaternion symmetric space. In the simplest cases including "1-instantons on $\mathbb{H}P^n$ and $Gr_2(\mathbb{C}^{n+2})$", we can describe moduli spaces explicitly using the Bott-Borel-Weil theorem. (For more details, see [Na3].)

Remark 2. On $Gr_2(\mathbb{C}^{n+2})$, we have another type of instanton bundles. The structure group of this bundle is $SU(n+1)$. In the case $n=1$, these are the same as bundles in §6, because of the isomorphism $SU(2) \cong Sp(1)$. The moduli of these $SU(n+1)$-instanton is identified with an open cone over $\mathbb{P}(\mathbb{C}^{n+2})$, where $\mathbb{C}^{n+2}$ is the standard representation space of $SU(n+2)$ [Na4].

Remark3. Making use of a result of LeBrun-Salamon [L-S] and the Bott-Borel-Weil theorem, we can classify complex line bundles with an anti-self-dual connection over compact quaternion Kähler manifolds. In fact, unless the base manifold is $Gr_2(\mathbb{C}^{n+2})$, the only trivial complex line bundle can admit an anti-self-dual connection, which is the product connection. An anti-self-dual connection on a complex line bundle over $Gr_2(\mathbb{C}^{n+2})$ is necessaly the canonical connection on some homogeneous line bundle.

Remark4. The vector bundles E_1 and E_2 in §7 can admit the only one anti-self-dual connections, respectively. These are also the canonical connections.

References

[A] M. F. Atiyah, *Geometry of Yang-Mills Fields*, Lezioni Fermiane, Scuola Normale Superiore, Pisa, 1979.

[A-O] V. Ancona and G. Ottaviani, *Stability of special instanton bundles on* $\mathbb{P}^{2n+1}$, Trans. A.M.S. **341** (1994), 677–693.

[B-H] W. Barth and K. Hulek, *Monads and moduli of vector bundles*, Manuscripta Math. **25** (1978), 323–347.

[B] R. J .Baston, *Quaternionic complexes*, J.Geom.Phys. **8** (1992), 29–52.

[Bu] N. P. Buchdahl, *Instantons on* $\mathbb{C}P^2$, J.Diff.Geom. **24** (1986), 19–52.

[D-M] V. G. Drinfeld and Yu. I. Manin, *Instantons and sheaves on* $\mathbb{C}P^3$, Funk.Analiz. **13** (1979), 59–74.

[G-P] K. Galicki and Y. S. Poon, *Duality and Yang-Mills fields on quaternionic Kähler manifold*, J.Math.phys. **32** (1991), 1263–1268.

[H] N. J. Hitchin, *Linear field equations on self-dual spaces*, Proc.R.Soc.A. **370** (1980), 173–191.

[Ho] R.Horan, *A rigidity theorem for quaternion-Kähler manifolds*, Diff.Geom.Appl. **6** (1996), 189–196.

[K-N] Y. Kametani and Y. Nagatomo, *Construction of* c_2*-self-dual Bundles on a Quaternionic Projective space*, Osaka.J.Math. **32** (1995), 1023-1033.

[L-S] C. LeBrun and S. M. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent.Math **118** (1994), 109–132.

[M-S] M. Mamone Capria and S. M. Salamon, *Yang-Mills fields on quaternionic spaces*, Nonlinearity **1** (1988), 517–530.

[Na1] Y. Nagatomo, *Rigidity of* c_1*-self-dual connections on quaternionic Kähler manifolds*, J. Math. Phys. **33** (1992), 4020–4025.

[Na2] Y. Nagatomo, *Vanishing theorem for cohomology groups of* c_2*-self-dual bundles on quaternionic Kähler Manifolds*, Diff.Geom. Appl. **5** (1995), 79–95.

[Na-3] Y. Nagatomo, *Representation theory and ADHM-construction on quaternion symmetric spaces*, a preprint.

[Na-4] Y. Nagatomo, *Another type of of instanton bundles on* $Gr_2(\mathbb{C}^{n+2})$, to appear in Tokyo J. Math..

[N-N1] Y. Nagatomo and T. Nitta, *Vanishing theorem for quaternionic complexes*, to appear in Bull. London. Math. Soc. **29** (1997), 359-366.

[N-N2] Y. Nagatomo and T. Nitta, *k-instantons on* $G_2(\mathbb{C}^{n+2})$ *and stable vector bundles*, to appear in Math. Z.

[N-N3] Y. Nagatomo and T. Nitta, *Moduli of 1-instantons on* $G_{2,n+2}(\mathbb{C})$, to appear in Diff. Geom. Appl. **7** (1997), 115-122.

[Ni] T. Nitta, *Vector bundles over quaternionic Kähler manifolds*, Tohoku.Math.J. **40** (1988), 425–440.

[O-S-S] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces* Progress in Math.3, Birkhäuser, Boston, 1980.

[O-S] C. Okonek and H. Spindler, *Mathematical instanton bundles on* $\mathbb{P}^{2n+1}$, J.Reine.Angew. Math. **364** (1986), 35–50.

[R] J. H. Rawnsley, *On the Atiyah-Hitchin-Drinfeld-Manin vanishing theorem for cohomology groups of instanton bundles*, Math.Ann. **241** (1979), 43–56.

[S] S. M. Salamon, *Quaternionic Kähler Manifolds*, Invent.Math. **67** (1982), 143–171.

[S-2] S. M. Salamon, *Differential geometry of quaternionic manifolds*, Ann. SC. Ec. Norm. Sup. **19** (1986), 31–55.

[S-T] H. Spindler and G. Trautmann, *Special instanton bundles on* $\mathbb{P}^{2n+1}$*, their geometry and their moduli*, Math.Ann. **286** (1990), 559–592.

[T] M. Thornber, *Vanishing theorem for quaternionic Kähler Manifolds*, D. Phil. Thesis, State University of New York (1989).

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