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SYMMETRIC SPACES AND QUATERNIONIC STRUCTURES

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Abstract. We apply our geometric theory of symmetric spaces (still under construction) to the classification of the twistor spaces [Br] (which are more general than those over quaternionic spaces) and to that of the simple graded Lie algebras of the second kind [K]. Certain results on the cohomology rings will also be mentioned.

1. Introduction.

We are interested in geometric facts which are closely related to the twistor spaces of quaternionic Kählerian manifolds. We will discuss them from a broader angle within the framework of the compact symmetric spaces in a simple and unified way. We will begin our geometric story on them from scratch in Section 1.

A substantial part is our explanation of R. Byant's classification of the twistor spaces Z over compact 1-connected symmetric spaces N.

Our method will be first to take a compact 1-connected group G, then draw a circle c (a closed geodesic) in G which passes through the unit element 1 and finally to make the G-orbits through the midpoint $q := c\,(1/2)$ and the antipodal point $p = c\,(1)$. We have the conclusion: the orbit $G(q) = \{bc\,(1)\,b^{-1} \mid b \in G\}$ is the twistor space of $G(p)$; every M is either some $G(p)$ or its twistor space is trivial (i.e. $Z = N$).

One will obtain more information by starting with a compact symmetric space M as follows. We draw a circle c in $M = G/K$ and make the orbits $K(q)$ and $K(p)$, where K is the isotrophy subgroup at $c(0)$. Since we can take G so that M is a subspace (2.3.) of G and $c(0)$ is 1, (see 2.4.), c is also a circle in G and we make the orbits $G(q)$ and $G(p)$ as before. Thus we have a "two-tiered" fibration (Theorem 3.2.)

$$\begin{array}{ccccccc} G_{ev}\,(q) & \longrightarrow & G\,(q) & \xrightarrow{\pi} & G\,(p) & & \text{in G} \\ \uparrow & & \uparrow & & \uparrow & & \\ K_{ev}\,(q) & \longrightarrow & K\,(q) & \longrightarrow & K\,(p) & & \text{in M,} \end{array}$$

where π is the projection (carrying q into p), $G_{ev}\,(q)$ is the fibre $\pi^{-1}(p)$ ($K_{ev}(q)$ is similarly the fibre $\pi^{-1}(p) \cap M$) and the vertical arrows are the inclusions of subspaces. A remarkable fact is that q acts on $G(p)$, fixes p and rotate $K(p)$ around p and carrying $K(p)$ onto the subspace $K(p)^{\perp}$ whose tangent space $T_pK(p)^{\perp}$ at p is the orthogonal complement of $T_pK(p)$ in $T_pG(p)$; in particular, q defines a complex structure on $T_pG(p)$ (3.2.). Since π is G-equivariant, every point of $G(q)$ has a similar property.

The "two tiered" fibrations give classifications of several other objects such as the simple graded Lie algebras ℓ of the second kind:

$$\ell = \ell_{-2} \oplus \ell_{-1} \oplus \ell_0 \oplus \ell_1 \oplus \ell_2,$$

satisfaying $[\ell_p, \ell_q] \subset \ell_{p+q}$, the R-spaces (3.4.), the C- spaces (3.4.) and so forth.

As another application, we are thinking about possibility of the "Clifford-Kählerian" structures which generalize the Kählerian and the quaternionic Kählerian structures and are defined on certain exceptional symmetric spaces.

Acknowledgements. This is partly a joint-work with M. S. Tanaka [NT]. In developing our geometric theory of symmetric spaces, I have worked together with her, B. Y. Chen and S. Kobayashi and had cooperations of J. H. Cheng, M. Clancy, M. Takeuchi and others in expanding and improving it. I feel grateful to them.

2. Preliminaries from the theory.

Definition 2.1. A (Riemannian) *symmetric space* M is a Riemannian manifold M with a *point-symmetry* s_x at every point x of M such that s_x is an isometric involution ($s_x \cdot s_x =$ the identity 1_M) and x is an isolated fixed point of s_x.

Remark 2.1a. The point symmetry s_x acts on the tangent space T_xM as -1 times the identity. Hence a tensor field of odd degree is zero if it is left invariant by all the point symmetries. In particular, the covariant derivative ∇R of the curvature (R^h_{ijk}) vanishes; this is a condition which characterizes symmetric spaces locally. By the same token, an affine connection is unique without torsion and an almost complex structure is necessarily integrable (defining a complex structure on M) if these are invariant under all the point symmetries.

Remark 2.1b. More general symmetric spaces will be mentioned later in 3.5b..

Definition 2.2. A *homomorphism* f of a symmetric space into another is a smooth map which commutes with the point-symmetries ($f \circ s_x = s_{f(x)} \circ f$).

Remark 2.2a. The identity map 1_M of M and the composite $f \circ g$ of two homomorphisms f and g are obviously homomorphisms; hence the symmetric spaces make a category. This contains the category of the compact Lie groups, which are all symmetric spaces with $s_x : y \longrightarrow xy^{-1}x$.

Remark 2.2b. In case M is connected, a smooth map $f : M \longrightarrow N$ between symmetric spaces is a homomorphism if and only if f is totally geodesic (that is, f carries the geodesics into geodesics, preserving the affine parameters). A geodesic c in M is a homomorphism of the real line $\mathbb{R}$ into M. If c is not injective, the image $c(\mathbb{R})$ is a circle S^1 (a closed geodesic without singularity like a self-intersection) or a single point.

Remark 2.2c. The automorphism group $\mathrm{Aut}(M)$ is a Lie transformation group which contains the isometry group $I(M)$; in particular, the point symmetries are automorphisms.

$I(M)$ is transitive on M, if M is connected; in fact, given two points o, x and a geodesic arc c joining o to x, take the midpoint m of c, and s_m carries o into x (in particular, M is complete). Since the composite $s_m \circ s_o$ also carries o into x, the identity component $I(M)_{<1>}$ is transitive as well. One can prove that

2.2D. a homomorphism $f : M \longrightarrow N$ of a connected M gives rise to a homomorphism $F : I(M)_{<1>}\hat{} \longrightarrow I(N)_{<1>}$ of a covering group $I(M)_{<1>}\hat{}$ of $I(M)_{<1>}$ which makes f equivariant, as mentioned in (1.5.) of [N88]. Thus the theory of the symmetric spaces is closely related to that of Lie groups and Lie algebras.

DEFINITION 2.3. If a homomorphism $f : M \longrightarrow N$ is injective, then we call the image $f(M)$ a *subspace* or a *symmetric subspaces* of N.
One sees that the image $f(M)$ and the inverse image $f^{-1}(y)$ of a point y are subspaces (of N and M respectively) for any homomorphism f.

REMARK 2.3A. For an automorphism t of M, the fixed point set $F(t, M) := \{x \in M \mid t(x) = x\}$ and its connected components are subspaces of M. If p is a point on $F(t, M)$, $F(t \circ s_p, M)$ is also a subspace; its component $F(t \circ s_p, M)_{<p>}$ through p has the tangent space at p which is the orthogonal complement to $T_p F(t, M)$ in $T_p M$. We call these spaces the *orthogonal spaces* of each other at p.

2.3B. We are mostly interested in compact spaces and will denote by M a compact connected symmetric space. The isometry group $I(M)$ is then compact; the identity component $I(M)_{<1>}$ of $I(M)$ equals $\mathrm{Aut}(M)_{<1>}$. The isotropy subgroup K of $I(M)$ at a point o is an open (and compact) subgroup of $F(\mathrm{ad}(s_0), I(M))$ and M is a homogenous manifold $I(M)/K = G/K \cap G$, where G denotes $I(M)_{<1>}$. Conversely, if G is a compact connected Lie group with an involutive automorphism σ and K denotes an open subgroup of $F(\sigma, G)$, then $M := G/K$ becomes a symmetric space with the point symmetry $s_K : \mathrm{b}K \longrightarrow \sigma(\mathrm{b})K$ and with any G-invariant Riemannian metric on M; G acts on M as $I(M)_{<1>}$. However, the projection $\pi : G \longrightarrow M : \mathrm{b} \mapsto \mathrm{b}K$ is not a homomorphism. For a remedy, E. Cartan found the next fact, one could say.

PROPOSITION 2.4. (i) The mapping $Q = Q_0 : M \longrightarrow I(M) : x \mapsto s_x \circ s_0$ is a homomorphism for a given point o. Furthermore,
(ii) Q is a covering homomorphism onto its image $Q(M)$;
(iii) Q is $I(M)$-equivariant with respect to the action of $\mathrm{b} \in I(M) : I(M) \longrightarrow I(M) : \mathrm{a} \mapsto \mathrm{b}\mathrm{a}\,\sigma(\mathrm{b})^{-1}$, where σ denotes $\mathrm{ad}(s_0)$; and
(iv) Q has a lift $Q\hat{} : M \longrightarrow I(M)\hat{}$ which is a momomorphism into a finite covering group $I(M)\hat{}$ of $I(M)$.

One can prove (iv) by connectedness of $F(\sigma, G\tilde{})$ for every involution group σ of a 1-connected group $G\tilde{}$ (E. Cartan).

REMARK 2.4A. Q is called the Cartan immersion for (M, o). In case M is a torus $T^n = S^1 \times \cdots \times S^1 \cong SO(2)^n \cong U(1)^n$ and o is the unit element 1, Q carries every point x into x^2; Q could also be called the quadratic transformation for (M, o).

2.5. It follows from 2.4. that every M is a subspace of the orthogonal group $O(n)$ for some n. The classification of all the compact connected symmetric spaces is known

(see the end of this section 2.31.). But their local classes are more convenient to consider sometimes; M is called *locally isomorphic* with another space if they share the universal covering space $M^{\sim}$ and $\ll M \gg$ will denote the class of the locally isomorphic spaces with M. The compact symmetric spaces are locally isomorphic with the product of a torus and simple (or irreducible) spaces, just as the compact Lie groups are so with that of a torus and simple Lie groups.
This decomposition agrees with the de Rham decomposition defined by the holonomy group, which happens to be the 1-component of the isotropy subgroup of $I(M)$ for a symmetric space M. We call M *semisimple* similarly if $\ll M \gg$ contains a product of simple spaces (without torus as a factor).

2.6. If M is semisimple, $\ll M \gg$ consists of a finite number of spaces, there is a unique space called *the bottom* (or *adjoint*) space, denoted by $M^{\%}$ in this article, such that every space in $\ll M \gg$ is a covering space of $M^{\%}$ and the fundamental group $\pi_1(M^{\%})$ is abelian. $M^{\%}$ is then a product of simple bottom spaces and Q for $M^{\%}$ is a monomorphism. If M is simple, M is an Einstein manifold and its Ricci form is positive definite; hence $M^{\sim}$ is also compact.

2.7. Now that the classification of the compact connected symmetric spaces is known, the determination of homomorphisms $f : L \longrightarrow M$ is one of the fundamental problems in the theory of the symmetric spaces. It was done by E. Cartan for the case of $L = \mathbb{R}$, a line, and by Dynkin for the local account of the case of 2 or 3 dimensional space L. Working with B. Y. Chen ([CN78], [N88]), we determined those defined on the space $\{0, 1\}$ of two points; it amounts to determining the fixed point set $F(s_0, M)$ for a point o of M. We also determined the 2-number of M, i.e. the maximal cardinality of finite sets L of which every point symmetry is the identity map 1_L and from which there is a monomorphism: $L \longrightarrow M$; the 2-number is a generalization of the 2-rank of a group defined by A. Borel and J. Serre and, for instance, gives an obstruction to the existence of monomorphism of M into another space [CN88].

Definition 2.8. Each connected component, $F(s_0, M)_{<p>}$ through a point p, of $F(s_0, M)$ is called a *polar* of o in M, denoted by M^+ or $M^+_{<p>}$.
Its orthogonal space $F(s_0 \circ s_p, M)_{<p>}$ at p is called the *meridian* which corresponds to $M^+_{<p>}$ at p, denoted by M^- or $M^-_{<p>}$. M^- passes through o. A polar $M^+_{<p>}$ of o in $M = G/K$, $K(o) = \{o\}$, is an orbit $K(p) = K_{<1>}(p)$. When $M^+_{<p>}$ is a singleton $\{p\}$, it is also called a *pole* of o in M. The poles of o in M constitute the inverse image $Q^{-1}(1)$ of the identity 1 of $I(M)$ under the Cartan immersion Q. Thus,

2.9. for a pole p of o , M admits a double covering morphism $\pi : M \longrightarrow M''$ such that $\pi(o) = \pi(p)$.

2.10. For example, the orthogonal spaces $F(t, M)$ and $F(t \circ s_p, M)$ in (2.3a.) meet again in polars of p in both spaces most often.

The polar and the meridian make basic building blocks of M; for instance,
2.11. a single pair $(M^+_{<p>}, M^-_{<p>})$ determines M completely (1.15. in [N92]); in particular,

2.12. a 1-connected M is Kählerian if and only if $M^{+}_{<p>}$ and $M^{-}_{<p>}$ are (2.30. in [N92]). (An analogous is false for $\mathbb{H}$-Kählerian spaces).

The polars and the meridians are of course (symmetric) subspaces of M.
The polars in M are related to the topology of M; for a simplest example,

2.13. M is orientable if and only if all the polars are even-dimensional [NT].
A geodesic is a circle if and only if it meets a polar M^{+} of a point o on it ($M^{+} \neq \{o\}$). Since a compact Riemannian manifold (of dim > 0) contains a closed geodesic, there exists a polar $M^{+} \neq \{o\}$) in every M (of dim > 0). (A closed geodesic is a circle in M.)

2.14. A homomorphism f carries the polars of o into those of $f(o)$. The converse does not quite obtain; for instance, the double covering homomorphism f of the sphere S^n onto the projective space $\mathbb{R}P^n$ carries the equator S^{n-1} onto a polar $\mathbb{R}P^{n-1}$, but S^{n-1} is not a polar in S^n. S^{n-1} is an example of the centriole we are about to define.

Definition 2.15. Each component of the set of the midpoints of the geodesic arcs joining a point o to a pole p is called a *centriole* for the pair (o, p).

2.16. If a double covering homomorphism f which carries a pole p of o onto $f(o)$, then f maps every centriole for (o, p) onto a polar $f(o)$ in the image $\operatorname{Im} f$. Conversely, the inverse image of a polar of $f(o)$ is a polar, a centriole or a disjoint union of two of them. In particular a centriole is a subspace.

Remark 2.17. One can generalize the concepts of the polar and the meridian to a disconnected space L. Then one sees that every M is a polar or a centriole in a compact group L (which acts on M transitively as an isometry group). M is of inner type if and only if M can be a polar or a centriole of the unit element 1 in $L_{<1>}$. If M is not of inner type (but of outer type), then the corresponding meridian does not pass through 1.
Anyway, with the knowledge of groups, the classification of compact symmetric spaces amounts to the determination of the polars in the groups.

The importance of the meridian is seen in the next fact too.

Proposition 2.18.

(i) Every meridian in M contains a maximal (with respect to inclusions) torus in M.

(ii) Every maximal torus through a point o meets all the polars and the centrioles of o in M.

(iii) The maximal tori in M are congruent with each other (i.e. carried into one another by $I(M)_{<1>}$).

Definition 2.18a. The dimension of a maximal torus in M is called the *rank* of M and denoted by $r(M)$.

Remarks 2.19. Every meridian in M has the rank equal to $r(M)$. Another consequence is that the induced homomorphism: $\pi_1(M^{-}) \longrightarrow \pi_1(M)$ on the fundamental group is surjective; in particular, M is 1-connected if some meridian M^{-} is.

Remarks 2.20. The following conditions are equivalent to each other:

(i) a simple $M = G/K$ is of inner type;

(ii) $r(K) = r(G)$;

(iii) the Euler number $\chi(M) > 0$;

(iv) M is congruent with a polar or a centriole in (a finite covering group of) G;

(v) $\chi(M)$ equals the sum of the Euler numbers of the polars of a point in M, (see [N88], [NT]).

The sum of the Euler numbers of the polars of 1 in a compact connected group G equals $2^{r(G)}$ (= the Lefschetz number of s_1). These topological facts can be used in several ways. For instance, since the rank of G equals that of any covering group $\tilde{G}$ of G, the sum of the Euler numbers of the polars in $\tilde{G}$ equals the correspoinding sum for G, which property will be of some use in checking the result on determing their polars.
The signature (or the index defined with the cohomology ring) also equals the sum of the signatures of the (appropriately oriented) polars of a point in M, of which we still lack a good proof.

2.21. Now fix a maximal torus A in M. Let o be a point on A. A tangent vector $H \in T_oA$ is the initial tangent to a geodesic c in M; c satisfies the equation $\nabla_{c'} c' = 0$ and lies in A. The Jacobi equation for c is a second order linear differential equation with constant coefficients

2.22. $$\nabla_{c'} \nabla_{c'} v + R(v, c') c' = 0,$$

constant (in terms Fermi coordinates) because of $\nabla R = 0$. Besides the symmetric linear operator: $v \mapsto R(v, c') c'$ has nonnegative eigenvalues $\alpha(H)^2$, where α is a linear form on T_oA.

2.23. Every solution v of the Jacobi equation is a vector field (restricted to c) in the Lie algebra $\mathcal{L} I(M)$, another beauty of the symmetric space ("variational completeness").

2.24. To explain more, we write $\mathbf{g}$ for the Lie algebra $\mathcal{L} G = \mathcal{L} I(M)$. Then $\mathbf{g}$ is the direct sum of the eigenspaces $\mathbf{g}(\alpha)$ for the eigenvalue $\alpha(H)^2$, $H \in \mathbf{a} := \mathcal{L} A = T_0 A$ (this identification comes from $c(t) = \exp(tH)(o)$ or $\mathbf{a} = \mathcal{L} Q(A)$). Since the point symmetry s_o preserves the curvature R, the automorphism $\operatorname{ad}(s_0)$ stabilizes $\mathbf{g}(\alpha)$ which is the direct sum $\mathbf{k}(\alpha) + \mathbf{m}(\alpha)$ of the eigenvalues ± 1 of $\operatorname{ad}(s_0)$; similarly one has $\mathbf{g} = \mathbf{k} + \mathbf{m}$. α is called a *root* of M with respect to (A, o) if $\alpha \neq 0$. Let $R(M)$ denote the totality of the roots.

We obtain the direct sums:

2.25. $\mathbf{k} = \mathbf{k}(0) + \sum_{R(M)} \mathbf{k}(\alpha)$ and $\mathbf{m} = \mathbf{a} + \sum_{R(M)} \mathbf{m}(\alpha)$. One has $\mathbf{g}(-\alpha) = \mathbf{g}(\alpha)$, but this ambiguity does not matter, and it can be eliminated. For every $u \in \mathbf{m}(\alpha)$, there is some $v \in \mathbf{k}(\alpha)$ such that one has

2.26. $[H, u] = \alpha(H) v$ and $[H, v] = -\alpha(H) u$ for every $H \in \mathbf{a}$ by the formulas $R(w, H) H = -[H, [H, w]]$ at o for $w \in \mathbf{g}$, $[\mathbf{m}, \mathbf{m}] \subset \mathbf{k}$, and so forth. It follows that the value of v at $c(t) = \exp(tH)(o)$ is the image of $-\sin(\alpha(H) t) u(o) + \cos(\alpha(H) t) v(o) = -\sin(\alpha(H) t) u(o)$ under (the differential of) $\exp(tH)$. In particular, assuming $v \neq 0$, one has

2.27. $v(c(t)) = 0$ if and only if $\sin(\alpha(H) t) = 0$. A point $c(t)$ on a geodesic c (or strictly speaking, the value t of the parameter) is a conjugate point of $c(0)$ if there is a nontrivial solution of the Jacobi equation which vanishes at $c(t)$ and $c(0)$. $c(t)$ is not a conjugate point of $c(0)$ if c on $[0,t]$ is one of the shortest curves that join $c(0)$ to $c(t)$.

2.28. Here are basic examples for later use. Let c be a circle in a semisimple space M, given by $c(t) = \exp(tH)(o)$ satisfying $c(2) = c(0)$ and $c(t) \neq c(0)$ for $0 < t < 2$. First one notes that $2\alpha(H)$ is an integral multiple of π. Thus the set $N^{<c>} = N^{<c>}(M) := \{\,|\,\alpha(2H)\,|\,/\pi|\ \alpha \in R(M),\ \alpha(H) \neq 0\}$ consists of integers > 0. With $N^{<c>}$ one can locate the conjugate points of o. The antipodal point $c(1)$ is a conjugate point of o if and only if $N^{<c>}$ contains 2 (e.g. $M = S^n$). But if M is the projective space $\mathbb{R}P^n$, then $N^{<c>} = \{1\}$, although $c(1)$ lies in the polar $\mathbb{R}P^{n-1}$. One has $N^{<c>} = \{2\}$ if and only if $c(1)$ is the only conjugate point of o on the interval $(0,2)$. And $N^{<c>}$ is contained in $\{1,2\}$ if and only if there is no conjugate point of o on the interval $(0,1)$.

2.29. It is important that $R(M)$ is a root system in the sense of Lie algebra theory; i.e. every root α carries every root β into another root $\beta - 2\,(<\beta,\alpha>\,/<\alpha,\alpha>)\,\alpha$ by the reflection with respect to α and this coefficient of α is an integer. The isomorphism class of $R(M)$ is independent of A and o. In contrast to the case of groups, roots of different lenghts may have different multiplicities ≥ 1, where the multiplicity is $m(\alpha) := \dim \mathbf{m}(\alpha)$. $R(M)$ together with m may be the thought of as a contracted expression of the curvature R, which characterizes $\ll M \gg$. $\ll M \gg$ is in a one-to-one correspondences with the quotient groups of the abelian groups $\pi_1(M^{\%})$ and depends on $R(M)$ only (not on m). Some people call $R(M)$ the restricted root system of M.

2.30. Using $R(M)$, one can easily list up all the candidates of the meridians $M^- = G^-/K^+$ in $M = G/K$, just as to list up all the maximal subgroups of a compact connected group G and of rank $= r(G)$, [N92]. Actual occurence of G^-/K^+ as M^- depends on the position of M in the class $\ll M \gg$ ([CN78], [N88]). If $M^- = G^-/K^+$ is a meridian, then the corresponding polar is isomorphic with K/K^+. With this line of ideas, one can determine all the polars and meridians of every M; they are completely known.

Symbols and notations 2.31. We first list more or less standard symbols for compact symmetric spaces as in [H]. The orthogonal group $O(n)$ has the identity component $SO(n)$, called the special (meaning det =1) orthogonal group. $O(2n)$ and $SO(2n)$ contain the unitary group $U(n)$ and $SU(n)$ respectively, whose orthogonal spaces at 1 are $O\mathrm{III}(n) = O(2n)/U(n)$ and $D\mathrm{III}(n)$. $U(2n)$ and $SU(2n)$ contain the symplectic group $S\mathrm{p}(n)$ with the orthogonal spaces $U\mathrm{II}(n)$ and $A\mathrm{II}(n)$. $O(n)$ and $SO(n)$ are subgroups of $U(n)$ and $SU(n)$ with the orthogonal spaces $U\mathrm{I}(n)$ and $A\mathrm{I}(n)$. Also $C\mathrm{I}(n)$ denotes $S\mathrm{p}(n)/U(n)$, the orthogonal space of $U(n)$. We denote by $G_p(F^n)$ the Grassmann manifold of the p-dimensional linear subspaces of an n-dimensional vector space F^n for $F =$ the real numbers $\mathbb{R}$, the complex numbers C and the quaternion algebra H. The projective space $G_1(F^{n+1})$ is denoted by PF^n. For example, the polars of 1 in $U(n)$ are the conjugate classes of the involutive members; thus the polar through b is $G_p(\mathbb{C}^n)$ if the eigenvalues of b has multiplicity p. $G_p(\mathbb{R}^n)\tilde{}$ consists of the oriented planes and is denoted by $G_p^o(\mathbb{R}^n)$ in the Table.

The spinor group $SO(2r)\tilde{}$ has the center $\{1,\epsilon,\delta_r,\delta_r\,\epsilon\} \equiv \mathbb{Z}_2\times\mathbb{Z}_2$ if r is even; $SO(2r)\tilde{}/\{1,\epsilon\}$ is $SO(n)$. The semispinor groups $SO(2r)\tilde{}/\{1,\delta_r\}$ and $SO(2r)\tilde{}/\{1,$
$\delta_r\,\epsilon\}$ are isomorphic by an outer involution and both denoted by $SO(2r)^{\#}$. There are

spaces denoted by $G_r(\mathbb{R}^{2r})^{\#}$, since $G_r(\mathbb{R}^{2r})$ has the same root system as $SO(2r)$.

The dot product $M \cdot N$ as in $U(n) = T \cdot SU(n)$ and $SO(4) = Sp(1) \cdot Sp(1)$ means the orbit space $M \times N/\mathbb{Z}_k$ for $\mathbb{Z}_k$ acting on both M and N as a subgroup of the centers of both $I(M)_{<1>}$ and $I(N)_{<1>}$ acting on $M \times N$ diagonally; this description is good enough in almost all cases.

There are five exceptional simple groups G_2, F_4, E_6, E_7 and E_8 which are compact and 1-connected; the lower indices denote their ranks. Other exceptional spaces are $G\mathrm{I} = G_2/SO(4)$, $F\mathrm{I} = F_4/Sp(1) \cdot Sp(3)$, $F\mathrm{II} = F_4/SO(9)\tilde{}$, $E\mathrm{I} = E_6/Sp(4)^{\%}$, $E\mathrm{II} = E_6/Sp(1) \cdot SU(6)$, $E\mathrm{III} = E_6/T \cdot SO(10)\tilde{}$, $E\mathrm{IV} = E_6/F_4$, $E\mathrm{V}=$ $E_7/(SU(8)/\mathbb{Z}_2)$, $E\mathrm{VI} = E_7/Sp(1) \cdot SO(12)\tilde{}$, $E\mathrm{VII} = E_7/T \cdot E_6$, $E\mathrm{VIII} = E_8\ /SO(16)^{\#}$ and $E\mathrm{IX} = E_8/Sp(1) \cdot E_7$.

3. The theorems and comments.

3.1. We begin with the setting in this section. Consider a closed geodesic $c : \mathbb{R} \longrightarrow M$ in a compact connected (simple) symmetric space $M = G/K$; the image $c(\mathbb{R})$ is a circle S^1. Let o be a point $c(0)$ on c and let p be the antipodal point $c(1)$ of o on c (1 is the positive minimum of $c^{-1}(p)$); thus s_o fixes p. We may assume that K fixes o, M is a subspace of G and o coincides with the unit element 1 of G. We will study the orbits $K(c(t))$ and $G(c(t))$ for the adjoint action; $b \in G$ carries $x \in G$ into $\mathrm{ad}(b)(x) := \mathrm{b}x\mathrm{b}^{-1}$. Of course $K(c(0)) = \{o\} = \{1\} = G(c(0))$ and $K(c(1))$ is polar $M^+_{<p>}$ along with $G(c(1)) = G^+_{<p>}$. Most of the orbits are not subspaces but just submanifolds.

Theorem 3.2. In this setting,

(i) the orbit $K(q)$ through the midpoint $q := c(1/2)$ is a homogenous fibre bundle over the polar $K(p)$;

(ii) similarly, $G(q)$ is a homogenous bundle over the polar $G(p)$;

(iii) $\mathrm{ad}(q)$ fixes p, acts as a complex structure on the tangent space $T_pG(p)$ (i.e. $(\mathrm{ad}(q))^2$ or its differential is -1 times the identity on $T_pG(p)$) and carries $K(p)$ onto its orthogonal space $K(p)^{\perp}$ at p in $G(p)$; and

(iv) similarly, the fibre $K_{ev}(q)$ of the bundle $K(q)$ can be rotated around q onto its orthogonal space within the fibre $G_{ev}(q)$ of the bundle $G(q)$.

Proof. (ii) Since $q^2 = p$ in G, the isotropy subgroup G_0 of G at q fixes p; thus G_0 is a subgroup of the isotropy subgroup G_{ev} of G at p. Hence one has a fibration $G(q) = G/G_0 \longrightarrow G/G_{ev} = G(p)$ with fibre $G_{ev}/G_0 = G_{ev}(q)$.
(i) follows from the above, because the adjoint action of K restricts to the given on M (cf. 2.4.; but it is not intricate here). One has a fibration $K_{ev}/K_0 = K_{ev}(q) \longrightarrow K(q) = K/K_0 \longrightarrow K/K_{ev} = K(p)$. (iii) $\mathrm{ad}(q)$ fixes $q^2 = p$. And $\mathrm{ad}(p)$ acts on $T_pG^+(p)$ as -1, the negative identity, since $\mathrm{ad}(p)$ equals $s_p \circ s_1$ by $p^2 = 1$ (see 2.2a.). Hence $(\mathrm{ad}(q))^2 = \mathrm{ad}(q) = -1$ and $\mathrm{ad}(q)$ is a complex structure of $T_pG^+(p)$. One has to show that $\mathrm{ad}(q)$ carries $T_pM^+(p)$ onto its orthogonal complement $T_pG^+(p)$.
Let τ denote an involution of G which defines M as $F(\tau \circ s_1, G)_{<1>}$. One has $\tau(q) = q^{-1}$, while τ fixes p as well as every point on the polar $M^+(p)$; hence $T_pM^+(p) = F(\tau, T_pG^+(p))$. (Also $M^+(p) = K^+(p)$ if $F(\tau, G) \supset K$ is connected; cf. 2.4..) By

$\tau(q) = q^{-1}$, one has $\tau \circ \mathrm{ad}(q) = \mathrm{ad}(q)^{-1} \circ \tau = -\mathrm{ad}(q) \circ \tau$; thus $\mathrm{ad}(q)$ exchanges the two eigenspaces of τ (for the eigenvalues ± 1) in $T_pG^+(p)$. That is, $\mathrm{ad}(q)$ carries $T_pM^+(p)$ onto its orthogonal complement in $T_pG^+(p)$. Naturally $\mathrm{ad}(q)$ stabilizes $G^+(q)$. Therefore $\mathrm{ad}(q)$ carries $K(p)$ onto the orthogonal space $K(p)^\perp$ at p.
(iv) The point p is a pole of 1 in the meridian $G^-_{<p>}$ and the fibre $G_{ev}(q)$ is a centriole for $(1,p)$ in $G^-_{<p>}$. We use the double covering homomorphism $\pi : G^-_{<p>} \longrightarrow G^-_{<p>}/\{1,p\}$ (see 2.9.) in order to apply (iii). $\pi(G_{ev}(q))$ is then a polar in $\pi(G^-_{<(p>)})$ and (iii) implies (iv).
Alternatively, one can prove (iv) by using $\mathrm{ad}(c)(1/4)$.

$\square$

REMARK 3.2A. Every point of $G(q)$ defines a complex structure on the tangent space to $G(p)$ at its projection.

REMARK 3.2B. The bundle projections: $K(q) \longrightarrow K(p)$ and $G(q) \longrightarrow G(p)$ become Riemannian submersions if $K(q)$ and $G(q)$ are given induced metrics multiplied by a certain constant. Since fibres are totally geodesic, the projections are harmonic maps by a theorem of Eells-Sampson. We have studied their stability with M. S. Tanaka (Paper in preparation).

To proceed further, we will restrict the circle in two steps by posing conditions on conjugate points.

THEOREM 3.3. In the same setting, assume that there is no conjugate point of $c(0)$ on the subarc of c corresponding to the interval $(0, 1/2]$ and that the isotropy subgroup G_0 of G at q in $G(q) = G/G_0$ is connected.
Then
(i) the manifold $G(q) = G/G_0$ is homogenous Kählerian;
(ii) the subspace $G_{ev}(q)$ is also homogenous Kählerian; and
(iii) the subspace $K_{ev}(q)$ is totally real in $G_{ev}(q)$; i.e. the complex structure of $G_{ev}(q)$ rotates each tangent space $T_xK_{ev}(q)$ onto its orthogonal complement.

Proof. By the assumptions, G_0 fixes c pointwise. Hence G_0 is the centralizer of the circular subgroup $c(\mathbb{R})$. Hence G/G_0 is homogenous Kählerian by a theorem of H. C. Wang; and so is $K_{ev}(q)$. (iii) follows from this and (iv) of (3.2.).

$\square$

REMARK 3.3A. If G_0 is disconnected, the covering space $G/G_{0\,<1>}$ of $G(p)$ is homogenous Kählerian and a fibre bundle over $G(p)$ naturally.

REMARK 3.3B. For G in the theorem, the set $N^{<c>}(G)$ defined in (2.28.) may contain numbers greater than 2 as in A1 through A5 in the Table at the end.

3.4. Here we explain a few terms to state the next theorems. A *twistor space* Z of N is (in our context and in the sense of [Br]) a bundle over N such that Z is Kählerian manifold whose points define complex structures of the tangent spaces to N at their projections like $G(q) \longrightarrow G(p)$ (see 3.2a.) and that the *horizontal bundle* $\sharp TZ$ (the subbundle of TZ which restrict to the normal bundle of each fibre) is holomorphic.

A *graded Lie algebra* ℓ *of the second kind* has a vector space gradation

$$\ell = \ell_{-2} \oplus \ell_{-1} \oplus \ell_0 \oplus \ell_1 \oplus \ell_2,$$

satisfaying $[\ell_p, \ell_q] \subset \ell_{p+q}$ and $\ell_1 \neq 0 \neq \ell_2$. It is *of the first kind* if $\ell_1 = 0$ or $\ell_2 = 0$. An *R-space* (or a totally-real subspace) M of a Kählerian space M^C is a subspace such that the complex structure J at each tangent space $T_x M^C$ carries $T_x M$ onto its orthogonal complement in $T_x M^C$; M^C is a complexification of M . Finally, a *C-space* M of a quaternionic Kählerian space M^H is a locally Kählerian subspace (i.e. some space in $\ll M \gg$ is Kählerian) which satisfies a similar condition. More precisely, consider the quaternion structure as a set of complex structures (of the tangent spaces); the set may be thought of as S^2 for each point, and M is a C-space if some complex structure J carries $T_x M$ onto its orthogonal complement in $T_x M^H$ and another complex structure coincides with that of M.

THEOREM 3.5. Still in the same setting (3.1.), assume that there is no conjugate point of $c(0)$ on the subarc of c corresponding to the open interval (0,1) and that the isotropy subgroup G_0 at q in $G_{ev}(q) = G_{ev}/G_0$ is connected. Then

- (i) one has 2-tired fibrations of $G(c(t))$, $0 < t < 2$, and $B := \bigcup_{0<t<2} G(c(t))$ over $G(p)$ obviously; and
- (ii) the list of all the 2-tiered fibrations satisfying the conditions above and the condition $N^{<c>}(G) = \{1, 2\}$ (see Table) gives classifications of
- (ii.1) the twistor spaces of 1-connected symmetric spaces [Br], of
- (ii.2) the simple graded algebras of the second kind [K], [KA], and of
- (ii.3) the C-spaces ([C], [Tk86]); and
- (iii) the list of all those that satisfy $N^{<c>}(G) = \{1\}$ or $\{2\}$ gives a classification of the R-spaces ([N65], [KN], [Tk65]).

No proof is given (see [NT]), but we will try some explanations.
(i) is obvious indeed; the projections are obtained with c and its images under the actions of G and K. The condition $N^{<c>}(G) = \{1, 2\}$ in (ii) implies the existence of some $H \in \mathbf{g}$ such that the eigenvalues of $-\mathrm{ad}(H)^2$ are 0, 1^2, and 2^2; take a renormalized initial tangent $c'(0)$. Now we switch to the noncompact form $\ell := \mathbf{k} + i\mathbf{m}$ of $\mathbf{g} = \mathbf{k} + \mathbf{m}$ (2.24.), $i^2 = -1$. There $\mathrm{ad}(z)$, $z := iH$, has the eigenvalues 0, ± 1 and ± 2. Denoting the eigenspace $\subset \ell$ for the eigenvalue p by ℓ_p, one obtains the graded algebra of the second kind (3.4.); in fact the condition $N^{<c>}(G) = \{1, 2\}$ is equivalent to nonvanishing of $\dim G_{ev}(q) = \dim G_{ev}/G_0$, the dimension of the fibre (3.2.), and $\dim G(p) = \dim G/G_{ev}$. As to the twistor space, one notes that the homogenous Kählerian manifold $G(q)$ is a homogenous space G^C/P of the complexification G^C of G with $\mathcal{L}P$ isomorphic with the complexification $\mathbb{C} \oplus (\ell_0 \oplus \ell_1 \oplus \ell_2)$. The tangent space $T_q G(q)$ may be identified with $\mathbb{C} \oplus (\ell_{-2} \oplus \ell_{-1})$ and $T_q G_{ev}(q)$ with $\mathbb{C} \oplus \ell_{-2}$. P acts on the space $C \oplus \ell_{-1}$ as a complex linear group. Hence $\sharp TG_{ev}(q)$ is holomorphic. The converse is proven with the existence of z (see [K]). A classification of the twistor spaces needs the bundle $G(q)$ of the upper tier only; indeed Bryant classified the simple graded Lie algebras of the second kind over $\mathbb{C}$. The classification of the C-spaces is equivalent to that of the simple graded Lie algebras of the second kind with $\dim \ell_2 = 1$

or, equivalently, that of the 2-tiered fibrations with the fibre $G_{ev}(q) = S^2$, in which case $K(p)$ is a C-space in the $\mathbb{H}$-Kählerian space $G(p)$ by (iii) of (3.2.). Now we turn to (iii). The condition on $N^{<c>}(G)$ means that either $\dim G_{ev}/G_0$ or $\dim G(p)$ equals zero; that is, either $G(p)$ or $G(q)$ is a Kählerian subspace of G. Hence $K(p)$ or $K(q)$ is an R-space by (3.2.) and its proof; in fact $G(q)$ is a Real space (in Atiyah's sense [A]). That is, there $\tau \circ s_1$ is an anti-holomorphic involution of $G(q)$, and $K(q)$ is (a connected component of) the fixed point set $F(\tau \circ s_1, G(q))$. Moreover $\tau \circ s_1$ acts on the homogenous bundle $\sharp TG(q)$ anti-holomorphically. The converse will also obtain, since a Kählerian subspace of G is a polar or a centriole in G by (2.20.); see Section 5 to find all the R-spaces and their complexifications.

Remark 3.5a. on (i) We do not know too much about it. But we like to mention that the closure (which is one-point compactification) of the bundle $B := \bigcup_{0<t<2} G(c(t))$ is a Pontryagin cycle in certain special cases; Mashimo proved that they have the smallest volume among the compact varifolds in their homology classes (To appear). H. Urakawa studied stability of the projection: $K(c(t)) \longrightarrow K(p)$ as a harmonic map when t tends to 1. It is well known that if $f : N \longrightarrow G(q)$ is a harmonic map (resp. a minimal immersion) and "horizontal", then its composite with the projection: $G(q) \longrightarrow G(p)$ is harmonic (resp. minimal). Calabi used the fibration 4 ($n = 2r+1$) in Table to prove the converse when $N = S^2$ and the given map is "full".

Remark 3.5b. on the graded Lie algebras of the second kind. They are substantial in various research areas. For a geometric formulation, one needs to generalize the concept of the symmetric space by dropping the Riemannian metric and assuming the existence of an affine connection ∇ which all the point symmetries leave invariant. ∇ is unique by (2.1a.). (And the minor discrepancy between $\mathrm{Aut}(M)$ and $I(M)$ disappears.)

Remark 3.5c. on the R-spaces. Their history goes back to a pioneering work of E. Cartan, as mentioned in [KN64]. [N65] was motivated by the conformal transformation group of S^n and the projective transformation group of $P\mathbb{R}^n$. They are Lie groups L which contain $I(M)$, $\dim L > \dim I(M)$. We wanted to determine all M which admit such an L. We found that $\ll M \gg$ is exactly the local class of an R-space M and that L is the subgroup of the holomorphic transformation group of the complexification M^C which leaves M invariant. The analogue of $z = iH$ in the proof of (3.5.) is a gradient of a zonal spherical function f (hence f is a K-invariant eigenfunction of the Laplacian Δ). f is a Bott-Morse function and its critical manifolds are exactly the polars in the R-space M. f (or rather $z = \mathrm{grad}\, f$) was used by several authors to study stability of harmonic maps, etc.

4. Problems.

4.1. We want more applications of the 2-tiered fibrations. One of their features is the existence of concrete rotations (by the right angle). This might be used to understand the exceptional groups and spaces. On the other hand, we want to generalize the Kählerian and the $\mathbb{H}$-Kählerian structures. A space is a "Clifford- or Spin (n)-Kählerian" by definition if the honolomy group contains Spin (n) or the semi-spinor group as a normal subgroup; thus it is Kählerian for $n = 2$ and $\mathbb{H}$-Kählerian for $n = 3$; $\mathrm{Spin}(n) = S\mathcal{O}(n)$ for $n > 2$.

There must be a few manifolds with this structure for $n > 3$, but one has a sequence of these Spin$\,(n)$ - Kählerian subspaces:

$$S^2 \subset S^4 \subset S^8 \subset F\mathrm{II}^{16} \subset E\mathrm{III}^{32} \subset E\mathrm{VI}^{64} \subset E\mathrm{VIII}^{128}$$

of dimension indicated by the upper indices for $n = 2, 3, 8, 9, 10, 12$ and 16. The existence of the Clifford structure is witnessed by $G_{ev}/G_0 = G_2(\mathbb{R}^n)\tilde{}$, which consists of the relevant $\mathbb{C}$-structures ($\mathrm{ad}\,(q)$ and its conjugates) on the tangent space T_pG^+. There exists an 8-form η on EVIII such that η is invariant (hence $\nabla\eta = 0$), the exterior power η^{16} is not zero (hence its integral over EVIII does not vanish) and its pullback to S^8 through the above inclusion maps is not zero either. We suspect this η may serve the part of the fundamental 2- and 4-forms on the Kählerian and $\mathbb{H}$-Kählerian manifolds respectively.

4.2. For the R-spaces M, the essential property of the group $L \supset I(M)$ is not known, at least not to me. What kind of differential equations will characterize L?

5. The table.

We list up the 2-tiered fibrations in (3.3.):

$$\begin{array}{ccccccc} G_{ev}/G_0 & \longrightarrow & G/G_0 & \longrightarrow & G/G_{ev} & = G^+ & (\subset \; G) \\ \uparrow & & \uparrow & & & \uparrow & \uparrow \\ K_{ev}/K_0 & \longrightarrow & K/K_0 & \longrightarrow & K/K_{ev} & = M^+ & (\subset \; M), \end{array}$$

where the upward arrows are the restrictions of the monomorphism of M into G. The spaces in lower fibration are obtained from the spaces above as the fixed point sets of $\tau \circ s_1$. For each, we indicate G_{ev}/G_0, G/G_{ev}, G, K_{ev}/K_0, K/K_{ev} and M. Those numbered 1 through 32 satisfy the condition $N^{<c>}(G) = \{1, 2\}$, while $N^{<c>}(G)$ contains 3 for nos. A1 through A5. For simplicity, the information is only local; we chose just one from each local class $\ll G \gg$ and, to make the symbols reasonably simple, some of the spaces $G_{ev}/G_0, \cdots, M$ are not quite correctly denoted. Let us add these equalities: $\dim G_{ev}/G_0 = 2\dim K_{ev}/K_0$ and $\dim G/G_0 = \dim K/K_0$.

The R-spaces appear in nos. 1 through 7, 12, 17, A3 and A4 except for the Kählerian spaces N which are R-spaces in $N \times \kappa N$, where κN is the complex conjugate of N; of course, they appear on the lower tier and their complexification above them.

Both the twistor bundles over the $\mathbb{H}$-Kählerian spaces and the C-spaces appear in nos. $1\,(r = 2)$, $4\,(r = 2)$, $5\,(r = 1)$, $7\,(r = 1)$, 11, 13, 15, 19, 21, 23, 25, 27, 29 and 32; $G\,(p)$ is $\mathbb{H}$-Kählerian if and only if the fibre G_{ev}/G_0 is S^2, or, equivalently, $(G_{ev}/G_0,\, K_{ev}/K_0) = (S^2,\, S^1)$. The point-symmetry s_1 acts on the twistor space Z as an involution which restricts to the fibre S^2 as the antipodal map.

More general "Clifford-Kählerian" spaces do in nos. $1\,(r = 4, s = 2)$, $2\,(r = 2, s = 1)$, $3\,(r = 2)$, $4\,(r = 2)$, $5\,(r = 2)$, $6\,(r = 1)$, $7\,(r = 2)$, $8\,(n = 5)$, 12, 14, 16, 17, 24, 26, 28, 30, and 31.

Table

No.	$\mathbf{G_{ev}/G_0}$	$\mathbf{G/G_{ev}}$	$\mathbf{G}$	$\mathbf{K_{ev}/K_0}$	$\mathbf{K/K_{ev}}$	$\mathbf{M}$
1	$G_s(\mathbb{C}^r)$	$G_r(\mathbb{C}^n)$	$SU(n)$	$G_s(\mathbb{R}^r)$	$G_r(\mathbb{R}^n)$	$\mathrm{AI}(n)$
2	$G_{2s}(\mathbb{C}^{2r})$	$G_{2r}(\mathbb{C}^{2n})$	$SU(2n)$	$G_s(\mathbb{H}^r)$	$G_r(\mathbb{H}^n)$	$\mathrm{AII}(n)$
3	$G_r(\mathbb{C}^{2r})$	$G_{2r}(\mathbb{C}^n)$	$SU(n)$	$U(r)$	$G_{p-r}(\mathbb{C}^p)\times G_r(\mathbb{C}^q)$ $p+q=n,\ r\le p\le q$	$G_p(\mathbb{C}^n)$
4	$\mathrm{DIII}(r)$	$G_{2r}(\mathbb{R}^n)$	$SO(n)$	$SO(r)$	$G_{p-r}(\mathbb{R}^p)\times G_r(\mathbb{R}^q)$ $p+q=n,\ r\le p\le q$	$G_p(\mathbb{R}^n)$
5	$\mathrm{CI}(r)$	$G_r(\mathbb{H}^n)$	$S_p(n)$	$\mathrm{UI}(r)$	$G_r(\mathbb{C}^n)$	$\mathrm{CI}(n)$
6	$\mathrm{CI}(2r)$	$G_{2r}(\mathbb{H}^n)$	$S_p(n)$	$S_p(r)$	$G_{p-r}(\mathbb{H}^p)\times G_r(\mathbb{H}^q)$ $p+q=n,\ r\le p\le q$	$G_p(\mathbb{H}^n)$
7	$\mathrm{DIII}(2r)$	$G_{4r}(\mathbb{R}^{2n})$	$\mathrm{SO}(2n)$	$\mathrm{UII}(r)$	$G_{2r}(\mathbb{C}^n)$	$\mathrm{DIII}(n)$
8	$\mathrm{DIII}(n-1)$	$G_2(\mathbb{R}^{2n})\tilde{}$	$\mathrm{SO}(2n)\tilde{}$	$\mathrm{SO}(n-1)$	$S^{n-1}\times S^{n-1}$	$G_2(\mathbb{R}^{2n})\tilde{}$
9	CP^{n-1}	$\mathrm{DIII}(n)$	$\mathrm{SO}(2n)$	$\mathbb{R}P^{n-1}$	$\mathrm{SO}(n)$	$G_n(\mathbb{R}^{2n})$
10	CP^5	EII	E_6	$\mathbb{R}P^5$	$\mathrm{CI}(4)^{\%}$	EI
11	S^2	EII	E_6	S^1	$\mathrm{CI}(4)^{\%}$	EI
12	$G_2^o(\mathbb{R}^{10})$	EIII	E_6	$S^4\cdot S^4$	$G_2(\mathbb{H}^4)^{\%}$	EI
13	S^2	EII	E_6	S^1	$S^2\cdot G_3(\mathbb{C}^6)$	EII
14	$G_2^o(\mathbb{R}^{10})$	EIII	E_6	$S^3\cdot S^5$	$G_2(\mathbb{C}^6)$	EII
15	S^2	EII	E_6	S^1	$\mathrm{DIII}(5)$	EIII
16	$G_2^o(\mathbb{R}^{10})$	EIII	E_6	$S^1\cdot S^7$	$G_2^o(\mathbb{R}^{10})$	EIII
17	$G_2^o(\mathbb{R}^{10})$	EIII	E_6	S^8	FII	EIV
18	$G_2^o(\mathbb{R}^{12})$	EVI	E_7	$S^5\cdot S^5$	$G_4(\mathbb{C}^8)$	EV
19	S^2	EVI	E_7	S^1	$G_4(\mathbb{C}^8)^{\%}$	EV
20	CP^7	$\mathrm{EV}^{\%}$	E_7	$\mathbb{R}P^7$	$\mathrm{AI}(8)/\mathbb{Z}_4$	$\mathrm{EV}^{\%}$

No.	$\mathbf{G_{ev}/G_0}$	$\mathbf{G/G_{ev}}$	$\mathbf{G}$	$\mathbf{K_{ev}/K_0}$	$\mathbf{K/K_{ev}}$	$\mathbf{M}$
21	S^2	EVI	E_7	S^1	$S^2 \cdot \mathrm{DIII}(6)$	EVI
22	$G_2^o(\mathbb{R}^{12})$	EVI	E_7	$S^3 \cdot S^7$	$G_4^o(\mathbb{R}^{12})$	EVI
23	S^2	EVI	E_7	S^1	EIII	EVII
24	$G_2^o(\mathbb{R}^{12})$	EVI	E_7	$S^1 \cdot S^9$	EIII	EVII
25	S^2	EIX	E_8	S^1	$\mathrm{DIII}(8)^{\%}$	EVIII
26	$G_2^o(\mathbb{R}^{16})$	EVIII	E_8	$S^7 \cdot S^7$	$G_8(\mathbb{R}^{16})^{\#}$	EVIII
27	S^2	EIX	E_8	S^1	$S^2 \cdot \mathrm{EVII}$	EIX
28	$G_2^o(\mathbb{R}^{16})$	EVIII	E_8	$S^3 \cdot S^{11}$	EVI	EIX
29	S^2	FI	F_4	S^1	$S^2 \cdot \mathrm{CI}(3)$	FI
30	$G_2^o(\mathbb{R}^9)$	FII	F_4	$S^3 \cdot S^4$	$\mathrm{G}_1(\mathbb{H}^3)$	FI
31	$G_2^o(\mathbb{R}^9)$	FII	F_4	S^7	S^8	FII
32	S^2	GI	G_2	S^1	$S^2 \cdot S^2$	GI
A1	$G_3(\mathbb{C}^6)$	EII	E_6	$\mathrm{U}(3)$	$S^2 \cdot G_3(\mathbb{C}^6)$	EII
A2	$\mathrm{DIII}(6)$	EVI	E_7	$\mathrm{UII}(3)$	$S^2 \cdot \mathrm{DIII}(6)$	EVI
A3	EVII	EIX	E_8	$\mathrm{AII}(4)/\mathbb{Z}_2$	$\mathrm{DIII}(8)^{\%}$	EVIII
A4	EVII	EIX	E_8	$\mathrm{T} \cdot \mathrm{EIV}$	$S^2 \cdot \mathrm{EVII}$	EIX
A5	$\mathrm{CI}(3)$	FI	F_4	$\mathrm{UI}(3)$	$S^2 \cdot \mathrm{CI}(3)$	FI
A3a	EVII	$\{p\}$	E_7	$\mathrm{AII}(4)/\mathbb{Z}_2$	$\{p\}$	EV
A4a	EVII	$\{p\}$	E_7	$\mathrm{T} \cdot \mathrm{EIV}$	$\{p\}$	EVII

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