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Kählerian Killing Spinors, Complex Contact Structures and Twistor Spaces[1]

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ABSTRACT

We collect our recent results ([5] and [8]) and we get the equivalence of the three notions of the title under some conditions. We then use this equivalence in order to prove some consequences about Sasakian manifolds, complex almost contact structures and complex k-contact structures.

Keywords - Contact structure, Sasakian manifold, spinor, twistor space.

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0. Introduction. The notion of a *complex contact structure* was introduced in the late 50's by S. Kobayashi (cf. [6]), in analogy to real contact structures.

In 1982 in [9], S. Salamon investigated quaternionic Kähler manifolds. In particular, he defined the *twistor space* over such a manifold as a generalization of the classical notion of twistor space over a self-dual 4-manifold.

In 1986, K.D. Kirchberg was led to define *Kählerian Killing Spinors*, in order to characterize Kähler spin manifolds of odd complex dimension admitting the smallest possible eigenvalue of the Dirac operator (cf. [4]). Some important contributions to this problem are also due to O. Hijazi (cf. [1]).

The aim of this paper is to collect our recent results (cf. [5] and [8]), in order to explain the close connection between these three notions and to derive some corollaries.

1. Previous results. In this section we describe the three notions introduced above, and recall relevant results obtained in each of these directions.

Let M be a compact spin Kähler manifold of odd complex dimension $m = n/2$ and positive scalar curvature R. Then, each eigenvalue λ of the Dirac operator D satisfies the inequality (cf. [4])

$$\lambda^2 \geq \frac{m+1}{4m} \inf_M R.$$

In the limiting case of this inequality, M is Einstein and any eigenspinor Ψ of D corresponding to the eigenvalues $\pm\sqrt{(m+1)R/4m}$ is a *Kählerian Killing spinor*, i.e., satisfies the following first-order differential equation (cf. [1], [4]):

$$\nabla_X \Psi + \frac{1}{n+2} X \cdot D\Psi + \frac{1}{n+2} J(X) \cdot \tilde{D}\Psi = 0.$$

We call such M a *limiting manifold.* Conversely, any compact Kähler manifold admitting Kählerian Killing spinors is a limiting manifold. The first known examples of such manifolds were the complex projective spaces CP^{2k+1}.

Using complex contact structures it is possible to construct other manifolds admitting Kählerian Killing spinors. We will shortly describe the construction of [5].

Definition 1 (cf.[6]) *Let M^{2m} be a complex manifold of complex dimension $m = 2k+1$. A complex contact structure is a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:*

(i) $\{U_i\}$ *is an open covering of M.*

(ii) ω_i *is a holomorphic 1-form on U_i.*

(iii) $\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0} M)$ *is different from zero at every point of U_i.*

(iv) $\omega_i = f_{ij}\omega_j$ *in* $U_i \cap U_j$, *where* f_{ij} *is a holomorphic function on* $U_i \cap U_j$.

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex contact structure. Then there exists an associated holomorphic line subbundle $L_{\mathcal{C}} \subset \Lambda^{1,0}(M)$ with transition functions $\{f_{ij}^{-1}\}$ and local sections ω_i. From condition (iii) immediately follows the isomorphism $L_{\mathcal{C}}^{k+1} \cong K$, where $K = \Lambda^{m,0}(M)$ denotes the canonical bundle of M. If we assume k to be an odd integer then M admits a canonical spin structure. It is given by the isomorphism

$$L_{\mathcal{C}}^{\frac{k+1}{2}} \quad \cong \quad K^{\frac{1}{2}} \quad \cong \quad S_0. \tag{1}$$

Here S_0 is the subbundle of the spinor bundle S which is defined as the eigenspace of Ω for the eigenvalue $-i\,m$, where the Kähler form Ω is considered as endomorphism of S. We construct now a section $\Psi_{\mathcal{C}}$ of the spinor bundle which is associated to the contact structure $\mathcal{C}$. For doing so we fix $(U, \omega) \in \mathcal{C}$ and define $\Psi_{\mathcal{C}}$ over the open set U by

$$\Psi_{\mathcal{C}} \mid_U := |\Psi_\omega|^{-2}\, \bar{\eta}_\omega \ \cdot \ \Psi_\omega\ , \tag{2}$$

where $\Psi_\omega \in \Gamma(S_0 \mid_U)$ is the local section in S_0 corresponding to $\omega^{\otimes \frac{k+1}{2}}$ under the identification (1) and $\eta_\omega := \omega \wedge (\partial\omega)^{\frac{k-1}{2}}$. From the condition (iv) it follows that the spinor $\Psi_{\mathcal{C}}$ is globally defined. We have the following

Proposition 1 (cf. [5]) *Let* (M, g, J) *be a compact Kähler-Einstein manifold of complex dimension* $m = 2k+1$ *with* k *odd, and let* $\mathcal{C}$ *be a complex contact structure on* M. *Then the spinor* $\Psi_{\mathcal{C}}$ *associated with* $\mathcal{C}$ *satisfies the equation*

$$D^2\, \Psi_{\mathcal{C}} \;=\; \frac{m+1}{4m}\, R\, \Psi_{\mathcal{C}},$$

where R *is the scalar curvature of* (M, g). *In particular, the spinors* $\Psi_{\mathcal{C}}^{\pm} := \lambda_1 \Psi_{\mathcal{C}} \pm D\Psi_{\mathcal{C}}$ *are Kählerian Killing spinors, where* $\lambda_1 = \sqrt{\frac{m+1}{4m}R}$.

A class of manifolds satisfying the assumptions of Proposition 1 are the twistor spaces of quaternionic Kähler manifolds introduced by S. Salamon (cf. [9]).
A *quaternionic Kähler manifold* is defined to be an oriented 4n-dimensional Riemannian manifold whose restricted holonomy group is contained in the subgroup $Sp(n)Sp(1) \subset SO(4n)$ $(n \geq 2)$. Salamon's idea is to construct over each such manifold M a natural CP^1– bundle Z, admitting a Kähler metric such that the bundle projection is a Riemannian submersion. He called this bundle the *twistor space* of M.

Proposition 2 (cf.[9]) *Let* M^{4k} *be a quaternionic Kähler manifold with positive scalar curvature. Then its twistor space* Z *admits a Kähler Einstein metric of positive scalar curvature and a complex contact structure. Moreover,* Z *is spin for odd* k *and* Z *is spin for even* k *iff* $Z = CP^{2k+1}$.

From Propositions 1 and 2 we obtain that all the twistor spaces of quaternionic Kähler manifolds M^{4k} ($k \equiv 1(2)$) with positive scalar curvature admits Kählerian Killing spinors, i.e. they are limiting manifolds.

The only explicitly known manifolds of this kind are the following three families:

- $Sp(k+1)/Sp(k) \times U(1) \quad \cong \quad CP^{2k+1}$,
- $SU(k+2)/S(U(k) \times U(1) \times U(1))$,
- $SO(k+4)/S(O(k) \times O(3) \times O(2))$.

and the 15–dimensional exceptional space $F_4/Sp(3)U(1)$.

It is now interesting to see that each such limiting manifold (i.e. each spin Kähler manifold of odd complex dimension and positive scalar curvature admitting Kählerian Killing spinors) has to be a twistor space. This is due to the following classification result:

Proposition 3 (cf. [8]) *The limiting manifolds of complex dimension $4l+3$ are exactly the twistor spaces associated to quaternionic Kähler manifolds of positive scalar curvature. The only limiting manifold of complex dimension $4l+1$ is CP^{4l+1}.*

The idea of the proof is the following. Take a limiting manifold M and consider a maximal root of the canonical line bundle with some hermitian metric. The associated principal $U(1)$-bundle over M, say P_M, with a carefully chosen metric, is spin, and any spinor on M induces a *projectable* spinor on P_M. Moreover, a Kählerian Killing spinor induces a projectable real Killing spinor on P_M. This forces P_M to admits a regular Sasakian 3-structure and M to be the twistor space over the quotient of P_M by the Sasakian 3-structure.

The last part of the proposition follows from the fact that the only spin twistor space of complex dimension 4l+1 is CP^{4l+1}.

2. The results.

Combining the above propositions we have

Theorem 4 *Let M be a compact spin Kähler manifold of positive scalar curvature and complex dimension $4l+3$. Then the following statements are equivalent:*

(i) *M admits Kählerian Killing spinors;*

(ii) *M is Kähler-Einstein and admits a complex contact structure;*

(iii) *M is the twistor space of some quaternionic Kähler manifold of positive scalar curvature.*

As an immediate corollary we have the following result:

Corollary 4.1 *If M is a Kähler-Einstein manifold of complex dimension $4l+3$ which admits a complex contact structure, then M is the twistor space of some quaternionic Kähler manifold of positive scalar curvature.*

This corollary is in fact part of a very recent theorem of C. LeBrun (cf. [7]). He proves the same statement but without the restriction on the dimension and also using a different method. The interest of our proof lies in the unexpected appearance of the Dirac operator. As a less obvious corollary we have the following

Corollary 4.2 *Let M be a Riemannian manifold of real dimension $n = 8l + 7$, admitting a Sasakian 3-structure which is regular in one direction. Then it is regular in all directions.*

Proof. Let V be the Killing vector field in the regular direction. We denote by N the quotient of M by the S^1-action in the direction of V. Regularity just means that N is a manifold. Now a simple calculation (cf. [2]) shows that N is a Kähler–Einstein manifold admitting a complex contact structure.

Corollary 4.1 yields that N is the twistor space of some quaternionic Kähler manifold Q, of positive scalar curvature. Using [2] once again, we see that the 2-distribution given by the two other Killing vector fields of the Sasakian 3-structure, projects on the 2-distribution Θ which gives the complex contact structure on N. So the quotient of M by the Sasakian 3-structure is diffeomorphic to the space of leaves of Θ, which is exactly the manifold Q. Thus our Sasakian 3-structure is regular.

Remark 1 *Corollary* 4.2 *is also true for $n = 8l + 3$. We just have to use the result of C. LeBrun* ([7]) *instead of Corollary* 4.1 *in the above proof.*

In [3] S. Ishihara and M. Konishi introduced the concept of *complex almost contact structures.* These are the hermitian manifolds of odd complex dimension $2n+1$ whose structure group can be reduced to $U(1) \times (Sp(n) \otimes U(1))$. They proved that each such manifold under an additional normality condition admits a Kähler–Einstein metric and also a complex contact structure. In [2] they also showed the existence of a normal complex almost contact structure on the S^1–quotient of a 3–Sasakian space which is regular in one direction. From Theorem 4 we then have

Corollary 4.3 *Let M be a complete Hermitian manifold with a complex almost contact structure. Then the structure is normal iff M is the twistor space of some quaternionic Kähler manifold of positive scalar curvature.*

To give a last application of Theorem 4 we consider a generalization of complex contact structures. For this let $\mathcal{C} = \{U_i, \omega_i\}$ be a family of (local) r–forms which again satisfies conditions (i) – (iv) of Definition 1, where (iii) has to be changed into: *(iii)* '

$$\omega_i \wedge (\partial\omega_i)^s \in \Gamma(\Lambda^{m,0}M\,|_{U_i}) \quad \text{is different from zero at each point of} \quad U_i.$$

Here $s = \frac{m-r}{r+1}$ must be an integer. Such a family was called a complex r–contact structure in [5]. If s is an odd integer then M again admits a canonical spin structure. In this situation it is once more possible to construct a Kählerian Killing spinor $\psi_{\mathcal{C}}$ (similar to (2)). Theorem 4 then implies

Proposition 5 *Let (M^{2m}, g, J) be a compact Kähler–Einstein manifold with positive scalar curvature which admits a complex r–contact structure such that $s = (m-r)/(r+1)$ is an odd integer. Then M is a complex contact manifold.*

Remark 2 *As it was recently pointed out to us by K. Galicki, Corollary 4.2 is an old result of S. Tanno [10].*

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