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QUATERNIONIC DERIVABILITY II

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Abstract.

The article proposes a new type of complex and quaternionic derivative.

Since the hard quaternion derivatives:

$$f'_r(w_0) = \lim_{\substack{w\to w_0 \\ w\neq w_0}} \{[f(w) - f(w_0)]\,(w - w_0)^{-1}\} \tag{1}$$

$$f'_l(w_0) = \lim_{\substack{w\to w_0 \\ w\neq w_0}} \{(w - w_0)^{-1}\,[f(w) - f(w_0)]\}$$

give reduced sets of right or left quaternionic derivable functions (see [1], [2], [3], [7], [10]) the study of softer quaternionic derivatives appeared (see [1], [4], [5], [6], [8], [10]). The present article tries to relaunch (for the sake of physics applications) the study of the hard form of the quaternionic derivative in a generalized manner.

1. Preliminaries.

1.1. Definition. We define:

1.) $\mathrm{H}_1 = \{w \in \mathrm{H} \mid w = ix_2 + jx_3 + kx_4\}$, analogously for $\mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_4$.

2.) $\mathrm{H}_{1,2} = \{w \in \mathrm{H} \mid w = jx_3 + kx_4\}$, analogously for $\mathrm{H}_{1,3}, \dots, \mathrm{H}_{3,4}$.

3.) $\mathrm{H}_{1,2,3} = \{w \in \mathrm{H} \mid w = kx_4\}$, analogously for $\mathrm{H}_{1,2,4}, \mathrm{H}_{1,3,4}, \mathrm{H}_{2,3,4}$.

One can raise now the following question: are there functions, derivable in the sense of the formula (1), between sets belonging to the Definition 1.1.? The answer is given by the following:

1.2. Theorem 1.) The only functions $f : Q \subseteq H_i \longrightarrow H$ ($\forall i \in \{1, 2, 3, 4\}$ and Q an open set of the induced topology on H_i) which have the property that $f'(w_0) = \lim_{\substack{w\to w_0 \\ w\neq w_0}} \{[f(w) - f(w_0)]\,(w - w_0)^{-1}\}$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.

2.) The only functions $f : Q \subseteq H \longrightarrow H_i$ ($\forall i \in \{1, 2, 3, 4\}$ and Q an open set) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.

3.) The only functions $f : Q \subseteq H_i \longrightarrow H_j$ $(\forall i, j \in \{1, 2, 3, 4\}$ and Q an open set of the induced topology) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.
4.) The only functions $f : Q \subseteq H_{i,j} \longrightarrow H$ $(\forall i \neq j,\ i, j \in \{1, 2, 3, 4\}$ and Q an open set of the induced topology) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.
5.) The only functions $f : Q \subseteq H \longrightarrow H_{i,j}$ $(\forall i = j,\ i, j \in \{1, 2, 3, 4\}$ and Q an open set) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.
6.) For the functions $f : Q \subseteq H_{i,j} \longrightarrow H_k$ $(\forall i \neq j,\ i, j, k \in \{1, 2, 3, 4\}$ and Q an open set in the induced topology) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, one obtains the following results:

a. if $i = 3,\ j = 4,\ k = 4$, then f has the form:
$f(x_1 + ix_2) = h(x_1 + ix_2) + jC$, where $h : Q \subseteq C \longrightarrow C$ is an holomorph function,
b. if $i = 3,\ j = 4,\ k = 3$, then $f(x_1 + ix_2) = h(x_1 + ix_2) + kC$,
c. if $i = 3,\ j = 4,\ k = 2$, then $f(x_1 + ix_2) = c + jh(x_2 + ix_1)$.

Analogously for the rest of the cases.

7.) The only functions $f : Q \subseteq H_k \longrightarrow H_{i,j}$ $(\forall i \neq j,\ i, j, k \in \{1, 2, 3, 4\}$ and Q an open set of the induced topology) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.
8.) For the functions $f : Q \subseteq H_{i,j} \longrightarrow H_{k,l}$ $(\forall i, j, k, l \in \{1, 2, 3, 4\}$ $i \neq j, k \neq l$ and Q an open set of the induced topology) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, one can say that:

a. they are constant functions if $\{i, j\} \cap \{k, l\}$ has exact one element,
b. they are equivalent with the holomorph functions if $\{i, j\} = \{k, l\}$ or $\{i, j\} \cap \{k, l\} = \emptyset$.

9.) The functions $f : Q \subseteq H_{i,j,k} \longrightarrow H$ $(\forall i, j, k \in \{1, 2, 3, 4\}$ $i \neq j, i \neq k, j \neq k$ and Q an open set of the induced topology) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are equivalent with the derivable functions $f : Q \subseteq \mathbf{R} \longrightarrow \mathbf{R}^4$.
10.) The only functions $f : Q \subseteq H \longrightarrow H_{i,j,k}$ $(\forall i, j, k \in \{1, 2, 3, 4\}$ $i \neq j, i \neq k, j \neq k$ and Q an open set) which have the property that $f'(w_0)$ exists and is finite, $\forall w_0 \in Q$, are the constant functions.

Proof. The extended proof is analogous to that of the Theorem from [3] and has around 30 pages. As an example, we prove 6.) c.

Let Q be an open set from $\mathrm{H}_{3,4}$, $w_0 \in Q$ and $f : Q \longrightarrow \mathrm{H}_2$, hence $w = x_1 + ix_2$, $f = f(w)$ and $f = f_1 + jf_2 + kf_3$. According to the notations of [3] we have:

$$\bar{w} = x_1 - ix_2\,, \quad x_1 = (1/2)\,(w + \bar{w})\,, \quad x_2 = -(i/2)\,(w - \bar{w}).$$

Therefore

$$\begin{aligned} \mathrm{d}f &= (\partial f/\partial x_1)\,\mathrm{d}x_1 + (\partial f/\partial x_2)\,\mathrm{d}x_2 = \\ &= (\partial f/\partial x_1)\,(1/2)\,(\mathrm{d}w + \mathrm{d}\bar{w}) - (\partial f/\partial x_2)\,(i/2)(\mathrm{d}w - \mathrm{d}\bar{w}) = \\ &= (1/2)\,[(\partial f/\partial x_1) - (\partial f/\partial x_2)\,i\,]\,\mathrm{d}w + (1/2)\,[(\partial f/\partial x_1) + (\partial f/\partial x_2)\,i\,]\,\mathrm{d}\bar{w}. \end{aligned}$$

Following an analogous reasoning of that from the proof of the Theorem just mentioned above, it follows that the limit

$$\lim_{\substack{w \to w_0 \\ w \neq w_0}} \{[f(w) - f(w_0)]\,(w - w_0)^{-1}\}$$

exsists and is finite iff:

$$\partial f/\partial x_1 + (\partial f/\partial x_2)\, i = 0 \quad \text{and this is equivalent to}$$
$$\partial f_1/\partial x_1 + j\partial f_2/\partial x_1 + k\partial f_3/\partial x_1 + (\partial f_1/\partial x_2 + j\partial f_2/\partial x_2 + k\partial f_3/\partial x_2)\, i = 0$$

which is equivalent to

$$\begin{cases} \partial f_1/\partial x_1 = 0 \\ \partial f_1/\partial x_2 = 0 \\ \partial f_2/\partial x_2 = \partial f_3/\partial x_1 \\ \partial f_2/\partial x_2 = -\partial f_3/\partial x_2 \quad \text{q.e.d.} \end{cases}$$

One obtains analogous results if there are analyzed the following types of H-derivability definitions:

$$\lim_{\substack{w \to w_0 \\ w \neq w_0}} \{[f(w) - f(w_0)]\,(\bar{w} - \bar{w}_0)^{-1}\}$$

$$\lim_{\substack{w \to w_0 \\ w \neq w_0}} \{[f(w) - f(w_0)]\,(\tilde{w} - \tilde{w}_0)^{-1}\}$$

$$\lim_{\substack{w \to w_0 \\ w \neq w_0}} \{[f(w) - f(w_0)]\,(\hat{w} - \hat{w}_0)^{-1}\}.$$

(We recall some notations of the [3]:

$$w = x_1 + ix_2 + jx_3 + kx_4, \quad \bar{w} = x_1 - ix_2 - jx_3 - kx_4,$$
$$\tilde{w} = -x_1 + ix_2 - jx_3 - kx_4,\ \hat{w} = -x_1 - ix_2 + jx_3 - kx_4.)$$

So, it is obvious that if someone wants to use a hard H-derivative, it is necessary an extension of the notion of derivative. It is just what we are doing in the followings.

2. An extension of the C-derivative

First we present the idea of the H-derivative for the particular case of the complex numbers.

Throughout this paragraph Q is an open set of C, $z_0 \in Q$, $z = x + iy$ and $f : Q \subseteq C \longrightarrow C$ such that $f \in C^1\,(Q \subseteq I\!R^2, I\!R^2)$.
For f we have the decomposition:

$$\begin{aligned} \mathrm{d}f &= (\partial f/\partial x)\,\mathrm{d}x + (\partial f/\partial y)\,\mathrm{d}y = \\ &= (\partial f/\partial x)\,(1/2)(\mathrm{d}z + \mathrm{d}\bar{z}) - (\partial f/\partial y)\,(i/2)(\mathrm{d}z - \mathrm{d}\bar{z}) = \\ &= (1/2)\,(\partial f/\partial x - i\partial f/\partial y)\,\mathrm{d}z + (1/2)\,(\partial f/\partial x + i\partial f/\partial y)\,\mathrm{d}\bar{z}. \end{aligned}$$

According to this decomposition the limit $f'(z_0)$ exists iff:

$$\partial f/\partial x\,(z_0) + \partial f/\partial y\,(z_0)\, i = 0 \tag{2}$$

(due to the nonexistence of the limit $\lim_{\substack{z\to z_0\\ z\neq z_0}} (\bar{z} - \bar{z_0})/(z - z_0)$).

Let's consider:

1.) the set $\Gamma = \{\gamma : (-\epsilon, \epsilon) \longrightarrow C \,|\, \epsilon > 0,\ \gamma\,(0) = 0,\ \gamma = \gamma_1 + i\gamma_2$, γ is not constant, $\gamma \in C^1$, it exists and it is finite the limit:

$$\begin{aligned} L\,(\gamma) = \lim_{t\to 0}\, &[(\gamma_1^2(t) - \gamma_2^2(t))\,/(\gamma_1^2(t) + \gamma_2^2(t)) + \\ &- 2i\,(\gamma_1(t)\gamma_2(t))\,/(\gamma_1^2(t) + \gamma_2^2(t))\,]\,\} \end{aligned}$$

2.) the equivalence relation "$\sim$" on Γ:

$$\gamma \sim \delta \iff L\,(\gamma) = L\,(\delta),$$

3.) the quotient set $\Gamma\,/\sim\, = \tilde{\Gamma}$.

Definition 2.1. We call complex generalized derivative the following map:

$$D : C^1\,(Q \subseteq I\!R^2,\ I\!R^2) \times Q \times \tilde{\Gamma} \longrightarrow C$$

$$\begin{aligned} D\,(f, z_0, \hat{\gamma}) = &(1/2)\,[\partial f/\partial x\,(z_0) - (\partial f/\partial y\,(z_0))\, i\,] + \\ &+ (1/2)\,[\partial f/\partial x\,(z_0) + (\partial f/\partial y\,(z_0))\, i\,]\, L\,(\gamma). \end{aligned}$$

Remark 2.2 One uses the condition (2) to be sure that the limit $f'(z_0)$ exists for every sequence $z_n \longrightarrow z_0 \quad (z_n \neq z_0)$. If one accepts to have different limits of the function $[f(z) - f(z_0)] \cdot (z - z_0)^{-1}$, for different sequences $z_n \longrightarrow z_0$, one can extend the definition of the complex derivative as in Definition 1.1.. Of course, the price of this extension is an extra index for the generalized complex derivative.

Let's also remark that:
$L\,(\gamma) = \lim_{t\to 0} [\bar{\gamma}\,(t) - \bar{\gamma}\,(0)]\,/[\gamma\,(t) - \gamma\,(0)]$, and if f is holomorph, then $D\,(f, z_0, \hat{\gamma}) = f'(z_0)$, $\forall \gamma \in \Gamma$.

Proposition 2.3. Let $\epsilon > 0$ and $\gamma : (-\epsilon, \epsilon) \longrightarrow C$, $\gamma\,(t) = \mathrm{a} \cdot t$, $\mathrm{a} \in C^*$. Then $\gamma \in \Gamma$ and $L\,(\gamma) = \bar{\gamma}'(0)\,/\gamma'(0)$.

Proof. We know that $\gamma_1(t) = \alpha t$, $\gamma_2(t) = \beta t$ and $\mathrm{a} = \alpha + i\beta$. Hence:

$$\begin{aligned} L\,(\gamma) &= \lim_{t\to 0} [(\alpha^2 t^2 - \beta^2 t^2)\,/(\alpha^2 t^2 + \beta^2 t^2) - 2i(\alpha t \beta t)\,/(\alpha^2 t^2 + \beta^2 t^2)] = \\ &= (\alpha^2 - \beta^2)/(\alpha^2 + \beta^2) - 2i(\alpha\beta)/(\alpha^2 + \beta^2) = \\ &= \bar{\mathrm{a}}/\mathrm{a} = \bar{\gamma}'(0)/\gamma'(0). \end{aligned}$$

DEFINITION 2.4. We call rectilinear complex derivative, and we note it by $D(f, z_0, \mathrm{a})$, a particular case of generalized complex derivative for $\gamma(t) = \mathrm{a}\,t,\ \mathrm{a} \in C^*$ (see Proposition 2.3.), therefore:
$D : C^1(Q \subseteq I\!R^2, I\!R^2) \times Q \times C^* \longrightarrow C$
$D(f, z_0, \mathrm{a}) = D(f, z, \hat{\gamma})$, where $\gamma(t) = \mathrm{a}\,t,\ \mathrm{a} \in C^*$.

PROPOSITION 2.5. $D(f, z_0, \mathrm{a}) = \partial f/\partial z\,(z_0) + (\partial f/\partial \bar{z}\,(z_0)) \cdot \bar{\mathrm{a}}/\mathrm{a}$, $\forall \mathrm{a} \in C^*$, $\forall z_0 \in Q$, where we used Pompeiu's notation (see [9]):

$$\partial f/\partial z\,(z_0) = (1/2)\,[\partial f/\partial x\,(z_0) - \partial f/\partial y\,(z_0))\,i\,]$$
$$\partial f/\partial \bar{z}\,(z_0) = (1/2)\,[\partial f/\partial x\,(z_0) + \partial f/\partial y\,(z_0))\,i\,].$$

Proof.

$$\begin{aligned} D(f, z_0, \mathrm{a}) = D(f, z_0, \hat{\gamma}) &\overset{\substack{\text{def 1.1.}\\ \text{def 2.4.}}}{=} \partial f/\partial z\,(z_0) + \partial f/\partial \bar{z}\,(z_0))\,L(\gamma) = \\ &\overset{\text{prop 2.3.}}{=} \partial f/\partial z\,(z_0) + (\partial f/\partial \bar{z}\,(z_0))\,\bar{\gamma}'(0)\,/\gamma'(0) = \\ &= \partial f/\partial z\,(z_0) + \partial f/\partial \bar{z}\,(z_0))\,\bar{\mathrm{a}}/\mathrm{a}. \end{aligned}$$

EXAMPLE 2.6.
If $f : C \longrightarrow C$, $f(z) = \bar{z}$, then $D(f, z_0, \mathrm{a}) = \bar{\mathrm{a}}/\mathrm{a}$, $\forall z_0 \in C$, $\forall \mathrm{a} \in C^*$.
EXAMPLE 2.7.
If $f : C \longrightarrow C$, $f(z) = |z|$, then $D(f, z_0, \mathrm{a}) = 1/(2\,|z_0|) = (\bar{z}_0 + z_0 \cdot \bar{\mathrm{a}}/\mathrm{a})$.

3. Quaternionic derivative

Let's consider:
1.) the set $\Gamma = \{\gamma : (-\epsilon, \epsilon) \longrightarrow H \mid \epsilon > 0,\ \gamma(0) = 0,\ \gamma = \gamma_1 + i\gamma_2 + j\gamma_3 + k\gamma_4,\ \gamma \in C^1$, exists and are finite the following limits:

$$l_{11}(\gamma) = \lim_{t\to 0}\,[\gamma_1^2(t) - \gamma_2^2(t) - \gamma_3^2(t) - \gamma_4^2(t)]\,/\,|\gamma(t)|^2,$$
$$l_{22}(\gamma) = \lim_{t\to 0}\,[-\gamma_1^2(t) + \gamma_2^2(t) - \gamma_3^2(t) - \gamma_4^2(t)]\,/\,|\gamma(t)|^2,$$
$$l_{33}(\gamma) = \lim_{t\to 0}\,[-\gamma_1^2(t) - \gamma_2^2(t) + \gamma_3^2(t) - \gamma_4^2(t)]\,/\,|\gamma(t)|^2,$$
$$l_{ij}(\gamma) = \lim_{t\to 0}\,[\gamma_i(t) \cdot \gamma_j t\,]\,/|\gamma(t)|^2,\ \forall i, j \in \{1, 2, 3, 4\},\ i \neq j\}$$

2.) the equivalence relation "$\sim$" on Γ:

$$\gamma \sim \delta \iff l_{ij}(\gamma) = l_{ij}(\delta) \quad \forall i, j \in \{1, 2, 3, 4\},$$

3.) the quotient set $\Gamma\,/ \sim = \tilde{\Gamma}$.
In the followings Q is an open set in H (or in $I\!R^4$, since the topology is the same), $w = x_1 + ix_2 + jx_3 + kx_4$, $w_0 \in Q$, $f = f_1 + if_2 + jf_3 + kf_4$.

DEFINITION 2.8. We define the quaternionic derivative in the following manner:

$$D : C^1(Q \subseteq I\!R^4, I\!R^4) \times Q \times \tilde{\Gamma} \longrightarrow H$$

$$\begin{aligned} D\,(f, w_0, \hat{\gamma}) =& \partial f/\partial w\,(w_o) + \\ &+ (\partial f/\partial \bar{w}\,(w_o))\,[l_{11}(\gamma) - 2\,il_{12}(\gamma) - 2\,jl_{13}(\gamma) - 2\,kl_{14}(\gamma)] + \\ &+ (\partial f/\partial \tilde{w}\,(w_o))\,[l_{22}(\gamma) + 2\,il_{21}(\gamma) - 2\,jl_{24}(\gamma) - 2\,kl_{23}(\gamma)] + \\ &+ (\partial f/\partial \hat{w}\,(w_o))\,[l_{33}(\gamma) - 2\,il_{34}(\gamma) + 2\,jl_{31}(\gamma) + 2\,kl_{32}(\gamma)], \end{aligned}$$

where we used the definition from [3], i.e.:

$$\begin{aligned} \partial f/\partial w\,(w_0) &= (1/2)\,[\partial f/\partial x_1\,(w_0) - \partial f/\partial x_2\,(w_0))\,i + \\ &\quad - (\partial f/\partial x_3\,(w_0))\,j + \partial f/\partial x_4\,(w_0)\,k] \\ \partial f/\partial \bar{w}\,(w_0)) &= (1/2)\,[\partial f/\partial x_1\,(w_0) + (\partial f/\partial x_4\,(w_0))\,k\,] \\ \partial f/\partial \tilde{w}\,(w_0) &= (1/2)\,[-\partial f/\partial x_2\,(w_0))\,i + (\partial f/\partial x_4\,(w_0))\,k] \\ \partial f/\partial \hat{w}\,(w_0) &= (1/2)\,[-\partial f/\partial x_3\,(w_0))\,j + (\partial f/\partial x_4\,(w_0))\,k\,]. \end{aligned}$$

REMARK 2.9. One can write:

$$\begin{aligned} D\,(f, w_0, \hat{\gamma}) =& \partial f/\partial w\,(w_0) + (\partial f/\partial \bar{w}\,(w_0))\,\bar{l}\,(\gamma) + \\ &+ (\partial f/\partial \tilde{w}\,(w_0))\,\tilde{l}\,(\gamma) + (\partial f/\partial \hat{w}\,(w_0))\,\hat{l}\,(\gamma), \end{aligned}$$

using the notation:

$$\begin{aligned} \bar{l}\,(\gamma) &= l_{11}(\gamma) - 2\,il_{12}(\gamma) - 2\,jl_{13}(\gamma) - 2\,kl_{14}(\gamma) \\ \tilde{l}\,(\gamma) &= l_{22}(\gamma) + 2\,il_{21}(\gamma) - 2\,jl_{24}(\gamma) - 2\,kl_{23}(\gamma) \\ \hat{l}\,(\gamma) &= l_{33}(\gamma) - 2\,il_{34}(\gamma) + 2\,jl_{31}(\gamma) + 2\,kl_{32}(\gamma). \end{aligned}$$

Let's also remark that:

$$\begin{aligned} \bar{l}\,(\gamma) &= \lim_{t\to 0} \bar{\gamma}\,(t) \cdot \gamma^{-1}(t), \\ \tilde{l}\,(\gamma) &= \lim_{t\to 0} \tilde{\gamma}\,(t) \cdot \gamma^{-1}(t), \\ \hat{l}\,(\gamma) &= \lim_{t\to 0} \hat{\gamma}\,(t) \cdot \gamma^{-1}(t) \end{aligned}$$

and since:

$$\mathrm{d}f = (\partial f/\partial w)\,\mathrm{d}w + (\partial f/\partial \bar{w})\,\mathrm{d}\bar{w} + (\partial f/\partial \tilde{w})\,\mathrm{d}\tilde{w} + (\partial f/\partial \hat{w})\,\mathrm{d}\hat{w}$$

(see [3]), it results that D is a hard type of derivative, i.e.:

$$D\,(f, w_0, \hat{\gamma}) = \lim\,[f(w) - f(w_0)] \cdot (w - w_0)^{-1},$$

where the limit is calculated when w tends to w_o on the curve $t \longrightarrow w_0 + \gamma(t)$. More of this, if f is H-derivable (see [3]), then $D(f, w_0, \hat{\gamma}) = f'(w_0)$, $\forall \gamma \in \Gamma$, where f' is the quaternionic derivative defined in [3].

Proposition 2.10. If $\gamma \in \Gamma$, $\gamma(t) = \mathrm{a} \cdot t$, where $\mathrm{a} \in H^*$, $\mathrm{a} = \mathrm{a}_1 + i\mathrm{a}_2 + j\mathrm{a}_3 + k\mathrm{a}_4$, then:

1.) $l_{11}(\gamma) = (\mathrm{a}_1^2 - a_2^2 - \mathrm{a}_3^2 - \mathrm{a}_4^2)/|\mathrm{a}|^2$
$l_{22}(\gamma) = (-\mathrm{a}_1^2 + \mathrm{a}_2^2 - \mathrm{a}_3^2 - \mathrm{a}_4^2)/|\mathrm{a}|^2$
$l_{33}(\gamma) = (-\mathrm{a}_1^2 - a_2^2 + \mathrm{a}_3^2 - \mathrm{a}_4^2)/|\mathrm{a}|^2$
$l_{ij}(\gamma) = \mathrm{a}_i \cdot \mathrm{a}_j / |\mathrm{a}|^2$, $\forall i, j \in \{1, 2, 3, 4\}$ $i \neq j$,

2.) $\bar{l}(\gamma) = \bar{\mathrm{a}} \cdot \mathrm{a}^{-1} = \bar{\mathrm{a}} \cdot \bar{\mathrm{a}} \cdot |\mathrm{a}|^{-2}$,
$\tilde{l}(\gamma) = \tilde{\mathrm{a}} \cdot \mathrm{a}^{-1} = \tilde{\mathrm{a}} \cdot \bar{\mathrm{a}} \cdot |\mathrm{a}|^{-2}$,
$\hat{l}(\gamma) = \hat{\mathrm{a}} \cdot \mathrm{a}^{-1} = \hat{\mathrm{a}} \cdot \bar{\mathrm{a}} \cdot |\mathrm{a}|^{-2}$.

Proof. By straight calculation.

Definition 2.11. We call rectilinear quaternionic derivative the map:

$$D : C^1(Q \subseteq I\!R^4, I\!R^4) \times Q \times H^* \longrightarrow H$$

$$D(f, w_0, \mathrm{a}) = D(f, w_0, \hat{\gamma}), \quad \text{where } \gamma(t) = \mathrm{a} \cdot t \ \mathrm{a} \in H^*.$$

Proposition 2.12.

$$\begin{aligned} D(f, w_0, \mathrm{a}) =& \partial f/\partial w(w_0) + (\partial f/\partial \bar{w}(w_0))\, \bar{\mathrm{a}} \cdot \mathrm{a}^{-1} + \\ &+ (\partial f/\partial \tilde{w}(w_0))\, \tilde{\mathrm{a}} \cdot \mathrm{a}^{-1} + (\partial f/\partial \hat{w}(w_0))\, \hat{\mathrm{a}} \cdot \mathrm{a}^{-1}. \end{aligned}$$

Proof. From Remark 2.9. and Proposition 1.2..

Example 2.13.

1.) If $f : H \longrightarrow H$, $f(w) = \bar{w}$, then $D(f, w_0, \mathrm{a}) = \bar{\mathrm{a}} \cdot \mathrm{a}^{-1}$, $\forall w_0 \in H$.
2.) If $f : H \longrightarrow H$, $f(w) = \tilde{w}$, then $D(f, w_0, \mathrm{a}) = \tilde{\mathrm{a}} \cdot \mathrm{a}^{-1}$, $\forall w_0 \in H$.
3.) If $f : H \longrightarrow H$, $f(w) = \hat{w}$, then $D(f, w_0, \mathrm{a}) = \hat{\mathrm{a}} \cdot \mathrm{a}^{-1}$, $\forall w_0 \in H$.

Example 2.14.

If $f : H \longrightarrow H$, $\quad f(w) = w^2$, $\quad$ then

$$\begin{aligned} D(f; x_1 + ix_2 + jx_3 + kx_4; \mathrm{a}) =& 2\, x_1 + 2\, ix_2 + 2\, jx_3 + (ix_2 + jx_3)\, \bar{\mathrm{a}}\mathrm{a}^{-1} + \\ &+ (ix_2 - kx_4)\, \tilde{\mathrm{a}}\mathrm{a}^{-1} + (jx_3 - kx_4)\, \hat{\mathrm{a}}\mathrm{a}^{-1}. \end{aligned}$$

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