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TWISTOR OPERATORS AND EIGENVALUES OF THE DIRAC OPERATOR

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Abstract. Twistor operators are introduced on Riemannian, Kählerian and Quaternionic-Kähler spin manifolds. Using these operators with the Lichnerowicz formula, one gets sharp lower bounds for the square of the eigenvalues of the Dirac operator in terms of the dimension and the geometrical structure of the compact manifold.

0. Introduction

The Lichnerowicz theorem could be stated as a gap phenomenon in the spectrum of the Dirac operator:

Theorem 0. [Li 1] *On a compact Riemannian spin manifold with positive scalar curvature S, the Dirac operator has no eigenvalues in the closed interval*

$$\left[\, -\frac{1}{2}\sqrt{S_o}\,,\, +\frac{1}{2}\sqrt{S_o}\,\right],$$

where S_0 denotes the infimum of the scalar curvature.

A fundamental consequence of this result is that it gives topological obstructions to the existence of Riemannian metrics with positive scalar curvature. In fact, since the Dirac operator D is self-adjoint, its analytical index is zero. For an even dimensional manifold, the spinor bundle splits under the action of the volume element into the direct sum of positive and negatif spinors. The corresponding splitting of the Dirac operator is given by $D = D_+ + D_-$, where D_+ (resp. D_-) interchanges positive and negative spinors (resp. negative and positive spinors). The analytical index of the elliptic operator D_+ is given by

$$\operatorname{Ind} D_+ = \dim\ker D_+ - \dim\ker D_- \,.$$

By the Lichnerowicz theorem, if the scalar curvature is positive then $\ker D_+ = \ker D_- = \{0\}$, hence $\operatorname{Ind} D_+ = 0$. By the Atiyah-Singer Index theorem, the analytical index is equal to the topological index, therefore on a compact Riemannian spin manifold with

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a non-vanishing topological index, there exists no metric with positive scalar curvature. Important extentions of positive scalar curvature obstructions were obtained by Hitchin, Gromov-Lawson and Stolz [Hn, GL 1, 2, 3, St 1, 2].

In this paper we present optimal refinements of Theorem 0 based on the use of twistor operators. Under the curvature assumption, three families of compact spin manifolds are of interest: Riemannian, Kählerian and Quaternionic-Kähler manifolds. It turns out that on the spectral level, via the twistor construction, the three structures are related. The round sphere, the complex projectif space (of odd complex dimension) and the Quaternionic projectif space are the model spaces of these structures. The whole spectrum of the Dirac operator of these model spaces has been computed [Su, CGLS, Mt]. In case the holonomy group is the special orthogonal group or the unitary group, if λ_1 *denotes* the first eigenvalue of the Dirac operator, then the ratio $\frac{\lambda_1^2}{S_0}$ of the given manifold is at least equal to the corresponding ratio of the model space [Ki 1, Ki 2, Hi 4]. In the Quaternionic case, the above is also true in small dimensions [HM 2, 3].

It is worth noting that the size of the gap in the spectrum of the Dirac operator increases with the rigidity of the geometry of the Riemannian compact spin manifold. In this presentation, we shall emphasize on the computation of the minimal eigenvalues for the above mentioned families of manifolds, without getting into the classification problem of manifolds with lowest eigenvalues to which references for known results are given.

The paper is organized as follows : In Section 1 we state and sketch the proof of basic results on the first eigenvalues of the Dirac operator on compact spin manifolds, with positive scalar curvature. The second part concerns the corresponding problem when, in addition, the manifold is Kähler. The results are mainly due to Kirchberg, but our presentation has the advantage to be self-contained and aims to illustrate the next section. In the last section, we review some results of a joint work with J.-L. Milhorat on Quaternionic-Kähler spin manifolds. The main tools are presented without proofs. Our goal in this exposition is to show how to determine the twistor operators of manifolds with special geometrie, then to make use of these operators to study the lowest eigenvalues of the Dirac operator.

1. Spin Compact Manifolds

In this section, we review different refinements of the Lichnerowicz theorem on Riemannian compact spin manifold (M^n, g, γ) with positive scalar curvature. Here, the Riemannian metric is denoted by g and the spin structure by γ. For basic facts concerning spin geometry, we refer to [LM, Bo, CGLS]. First we present Friedrich's Inequality [Fr 1] based on the use of a modified spinor connection. We give a simple and natural proof of this result by means of the twistor operator [Li 7, 8, 9]. The second step is obtained by the author [Hi 1] where the conformal covariance of the Dirac operator plays a central role (see also [Li 2-7, HL]).

Denote by ∇ the Levi-Civita connection on the tangent bundle as well as its extensions on the bundle of exterior forms $\wedge T^*M$ and the bundle of complex spinors ΣM. The Dirac operator D acting on the space of sections of the spinor bundle is the elliptic self-adjoint operator defined as the composition of the Levi-Civita connection with Clifford multiplication. Locally, with respect to a g-orthonormal tangent frame $\{e_1, \ldots, e_n\}$ one has:

$$\begin{aligned} D: \quad \Gamma(\Sigma M) &\xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma M) \xrightarrow{\mathcal{M}} \Gamma(\Sigma M) \\ \psi &\longmapsto \sum_i e_i \otimes \nabla_{e_i}\psi \quad \longmapsto \sum_i e_i \cdot \nabla_{e_i}\psi \,. \end{aligned}$$

Note that in case one considers the Clifford bundle (which is isomorphic as a vector bundle to the bundle of exterior forms) instead of the spinor bundle ΣM, then the corresponding Dirac operator is the Hodge-deRham operator $d + \delta$. A fundamental result on the Dirac operator is the Lichnerowicz formula:

$$D^2 = \nabla^*\nabla + \frac{1}{4}\,S\,, \tag{1.1}$$

where ∇^* is the adjoint of the Levi-Civita connection with respect to the natural Hermitian scalar product $(\ ,\ )$ on the spin bundle. For completeness, we show how (1.1) yields to the Lichnerowicz theorem.

Proof of Theorem 0. Applying (1.1) to a spinor field ψ, taking its scalar product with ψ and integrating on the compact manifold with respect to the volume element v_g, give

$$\int_M |\nabla\psi|^2\, v_g = \int_M |D\psi|^2\, v_g - \frac{1}{4}\int_M S\,|\psi|^2\, v_g\,. \tag{1.2}$$

Since the l.h.s of (1.2) is non-negative, it follows that if λ is an eigenvalue of D, then

$$\lambda^2 \geq \frac{1}{4}\,S_0\,.$$

If equality holds, then the scalar curvature is constant and $\nabla\psi \equiv 0$ where ψ is a corresponding eigenspinor. Hence,

$$\lambda^2 = \frac{1}{4}\,S_0 = \frac{1}{4}\,S = 0\,,$$

which contradicts the curvature assumption. $\square$

The next step is to improve this result so that equality could be achieved. We have

Theorem 1. [Fr 1] *On a compact Riemannian spin manifold with positive scalar curvature the Dirac operator has no eigenvalue in the open interval*

$$\left]\, -\frac{1}{2}\sqrt{\frac{n}{n-1}\,S_o}\,,\, +\frac{1}{2}\sqrt{\frac{n}{n-1}\,S_o}\,\right[\,.$$

Moreover, manifolds with minimal eigenvalues are Einstein.

Remark 1. *Denote by* $\lambda_1^2 := \lambda_1^2(M^n, g, \gamma)$ *the first eigenvalue of the Dirac operator on a Riemannian compact spin manifold* (M^n, g, γ) *and let* ν_1 *be the ratio* $\frac{\lambda_1^2}{S_0}$. *Since* $\lambda_1^2(S^n, g_0, \gamma_0) = \frac{n^2}{4}$ *(see* [Su]*), then the first part of Theorem 1 could be stated as*

$$\nu_1(M^n, g, \gamma) \geq \nu_1(S^n, g_0, \gamma_0)\,, \tag{1.3}$$

where (S^n, g_0, γ_0) *denotes the standard* n*-dimensional sphere with its canonical structure.*

Before proving Theorem 1, we recall that on a spin manifold there is another basic "operator" introduced by R. Penrose, called the twistor operator which will be denoted by P [Li 7–9]. The twistor operator is defined as the complement of the Dirac operator, more precisely, it is the composition of the covariant derivative with the projection on the kernel of Clifford multiplication. Locally, one has

$$\begin{aligned} P: \quad \Gamma(\Sigma M) &\xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma M) \xrightarrow{\pi} \ker \mathcal{M} \\ \psi \longmapsto \sum_i e_i \otimes \nabla_{e_i}\psi \quad &\longmapsto \sum_i e_i \otimes (\nabla_{e_i}\psi + \frac{1}{n} e_i \cdot D\psi)\,, \end{aligned}$$

hence,

$$P_X\,\psi := \nabla_X\,\psi + \frac{1}{n}\,X \cdot D\psi\,, \qquad \forall X \in \Gamma(TM), \quad \forall \psi \in \Gamma(\Sigma M)\,. \tag{1.4}$$

Note that by definition, the twistor operator satisfies

$$\sum_{i=1}^{n} e_i \cdot P_{e_i}\psi = 0, \qquad \forall \psi \in \Gamma(\Sigma M)\,. \tag{1.5}$$

A crucial relation between the Dirac operator and the twistor operator is given by:

Lemma 1. [Li 3] *For any spinor field* ψ, *the following Identity is true*

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n}\,|D\psi|^2\,. \tag{1.6}$$

Proof. By (1.5) and since Clifford multiplication by vectors is skew-symmetric, it follows:

$$\begin{aligned} |P\psi|^2 &= \sum_{i=1}^{n} (P_{e_i}\psi, P_{e_i}\psi)\,, \\ &= \sum_{i=1}^{n} (P_{e_i}\psi, \nabla_{e_i}\psi + \frac{1}{n} e_i \cdot D\psi)\,, \\ &= \sum_{i=1}^{n} (P_{e_i}\psi, \nabla_{e_i}\psi)\,, \\ &= \sum_{i=1}^{n} (\nabla_{e_i}\psi + \frac{1}{n} e_i \cdot D\psi, \nabla_{e_i}\psi)\,, \\ &= |\nabla\psi|^2 - \frac{1}{n}\,|D\psi|^2\,. \quad \square \end{aligned}$$

Proof of Theorem 1. Combining (1.5) with (1.2), gives

$$\int_M |P\psi|^2 v_g = \frac{n-1}{n} \int_M |D\psi|^2 \, v_g - \frac{1}{4} \int_M S\, |\psi|^2 \, v_g \,, \tag{1.7}$$

which implies the desired result. If equality holds in (1.3), i.e., if there exists a spinor field ψ such that $D\psi = \lambda_0 \psi$, with $\lambda_0^2 = \frac{n}{4(n-1)} S_0$, then

$$\nabla_X \psi = -\frac{\lambda_0}{n} \, X \cdot \psi \,, \qquad \forall X \in \Gamma(TM) \,. \tag{1.8}$$

The fact that in this case the manifold is Einstein, is an immediate consequence of (1.8) and the Ricci Identity [Fr 1]. □

Again, with the help of the Ricci Identities, one has the following

Lemma 2. [Hi 1] *For a real function f and a non-trivial spinor field ψ on a spin manifold (M^n, g, γ), define the metric connection ∇^f by*

$$\nabla^f_X \psi := \nabla_X \psi + \frac{f}{n} X \cdot \psi, \qquad \forall X \in \Gamma(TM) \,.$$

If there exists a non-trivial spinor field ψ satisfying $\nabla^f \psi \equiv 0$, then the manifold is compact and

$$f^2 = \lambda_0^2 = \frac{n}{4(n-1)} \, S_0.$$

In particular, the manifold is Einstein.

A non-trivial spinor field ψ such that $P\psi \equiv 0$ (resp. $\nabla^f \psi \equiv 0$) is *called* twistor-spinor (resp. real Killing spinor). It should be mentioned that real Killing spinors first appeared in mathematical physics in the context of supergravity and recently in superstring theories. In our context, they appear as being eigenspinors associated with the minimal eigenvalues of the Dirac operator. All manifolds with real Killing spinors have been classified. In low dimensions, the approach was a case by case proof [BGKF] and in any dimension the approach is based on the Berger-Simons holonomy theorem [Bä 2].

On the other hand, if one considers a complex valued function f, then the condition $\nabla^f \psi \equiv 0$ implies that (see [Li 3–7]) the function is either real and the manifold is compact or purely imaginary and the manifold is non-compact. In the second case, a ∇^f-parallel spinor is called an imaginary Killing spinor. All manifolds with imaginary Killing spinors are also classified. The first step in this direction was obtained by A. Lichnerowicz [Li 5]. In the case where the imaginary function is constant this is due to H. Baum [BFGK] and the general case is due to H.B. Rademacher [Ra].

In the remaining part of this section we give a qualitative improvement of Theorem 1:

Theorem 2. [Hi 2] *On a compact Riemannian spin manifold* (M^n, g, γ) *with positive scalar curvature any eigenvalue* λ *of the Dirac operator satisfies*

$$\lambda^2(M^n, g, \gamma) \geq \frac{n}{4(n-1)} \sup_u \inf_M (\overline{S}\, e^{2u}) , \tag{1.9}$$

where $\overline{S}$ *is the scalar curvature of the metric* $\overline{g} = e^{2u} g$.

A well-known result of N. Hitchin says that the dimension of the space of harmonic spinors is a conformal invariant [Hn]. The proof relies on the conformal covariance of the Dirac operator which could be seen as follows: If the conformal class of g is *denoted* by $[g]$, an element in this class is a metric $\overline{g}$ such that $\overline{g} = e^{2u} g$, for a real function u on M. At a given point x of M, we consider a g-orthonormal basis $e = \{e_1, \ldots, e_n\}$ of $T_x M$. The corresponding $\overline{g}$-orthonormal basis is *denoted* by $\overline{e} = \{e^{-u} e_1, \ldots, e^{-u} e_n\}$. This correspondence extends to the spin level to give an isometry between the corresponding spinor bundles. We agree to put a (–) above every object which is naturally associated with the metric $\overline{g}$. Then, for any spinorfields ψ and φ, one has

$$(\overline{\psi}, \overline{\varphi}) = (\psi, \varphi) ,$$

where (,) *denotes* the natural Hermitian scalar products on $\Gamma(\Sigma M)$, and on $\Gamma(\Sigma \overline{M})$ which make an isometry from the isomorphism between these spaces. The Levi-Civita connection $\overline{\nabla}$ is related to ∇ by

$$\overline{\nabla}_X \overline{\psi} = \overline{\nabla_X \psi} - \frac{1}{2} \overline{X \cdot du \cdot \psi} - \frac{1}{2} X(u) \overline{\psi} .$$

With the help of this Identity, one shows that ([Hn], [Ba]) the Dirac operators satisfy

$$\overline{D} (e^{-\frac{(n-1)}{2} u} \overline{\psi}) = e^{-\frac{n+1}{2} u} \overline{D\psi} . \tag{1.10}$$

Note that by (1.10)

$$D\psi = \lambda\psi \quad \Longrightarrow \quad \overline{D}\, \overline{\varphi} = \lambda e^{-u}\, \overline{\varphi} , \qquad \text{with} \quad \varphi = e^{-\frac{(n-1)}{2} u}\, \psi . \tag{1.11}$$

Proof of Theorem 2. With respect to the metric $\overline{g} = e^{2u} g$, apply Identity (1.7) to the spinor field $\overline{\varphi}$ satisfying (1.10) to get

$$\begin{aligned} \frac{n}{n-1} \int_M |\overline{P}\, \overline{\varphi}|^2 v_{\overline{g}} &= \int_M |\overline{D}\, \overline{\varphi}|^2 v_{\overline{g}} - \frac{1}{4} \int_M \overline{S} |\overline{\varphi}|^2 v_{\overline{g}} , \\ &= \int_M \left(\lambda^2 - \frac{n}{4(n-1)} \overline{S} e^{2u} \right) e^{-2u} v_{\overline{g}} \geq 0 . \quad \square \end{aligned}$$

If equality holds in (1.9), then

$$\overline{P}\, \overline{\varphi} \equiv 0 , \qquad \text{with} \qquad \overline{D}\, \overline{\varphi} = \lambda e^{-u}\, \overline{\varphi} ,$$

which, by Lemma 2 implies that the function u is constant, hence the spinor field $\psi = e^{\frac{(n-1)}{2}u}\varphi$ is a real Killing spinor.

We next show that the expression $\sup_u \inf_M (\overline{S}\, e^{2u})$ in (1.9) could be interpreted as a scalar spectral invariant. For this we recall some definitions. For $n \geq 3$, let L be the Yamabe operator, i.e., the scalar conformal Laplacian given by

$$L := 4\frac{n-1}{n-2}\Delta + S\,.$$

For $\overline{g} \underset{n\geq 2}{=} e^{2u}g \underset{n\geq 3}{=} h^{\frac{4}{n-2}}g$, the scalar curvatures S and $\overline{S}$ are related by

$$\begin{aligned} h^{-1}L(h) &\underset{n\geq 3}{=} 4\frac{n-1}{n-2}h^{-1}\Delta\, h + S \\ &\underset{n\geq 3}{=} \overline{S}\, h^{\frac{4}{n-2}} = \overline{S}\, e^{2u} \underset{n\geq 2}{=} 2(n-1)\Delta u - (n-1)(n-2)|du|^2 + S\,. \end{aligned} \tag{1.12}$$

Let h_1 be an eigenfunction of L associated with its first eigenvalue μ_1. Let u_1 be the function corresponding to h_1 by (1.12). By defintion of the Yamabe operator L, one gets $\mu_1 = \overline{S}\, e^{2u_1}$. It is easy to see that ([Hi 5])

$$\mu_1(M^n, g) = \sup_u \inf_M (\overline{S}\, e^{2u})\,,$$

and the supremum is achieved by a function u if and only if $u = u_1$. As a corollary, one has

Theorem 3. [Hi 3] *On a compact Riemannian spin manifold (M^n, g, γ) with positive scalar curvature and dimension $n \geq 3$, any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2(M^n, g, \gamma) \geq \frac{n}{4(n-1)}\mu_1(M^n, g)\,. \tag{1.13}$$

It is clear that $\mu_1 \geq \inf_M S$, and equality is achieved if and only if the scalar curvature is constant. A second corollary of Theorem 2 could be obtained by comparing $\mu_1(M^n, g)$ to the Yamabe number, *denoted* by μ. This is the conformal invariant scalar defined by

$$\mu(M^n, [g]) \;:\underset{n\geq 2}{=} \inf_{\overline{g}\in[g]} \frac{\int_M \overline{S}\, v_{\overline{g}}}{(\mathrm{vol}\,(M, \overline{g}))^{(n-2)/n}} \;:\underset{n\geq 3}{=} \inf_{h>0} \frac{\int_M hL(h)\, v_g}{\|h\|^2_{2n/(n-2)}}\,. \tag{1.14}$$

By a direct application of the Hölder Inequality, it follows

Theorem 4. [Hi 3] *On a compact Riemannian spin manifold (M^n, g, γ) with positive scalar curvature and dimension $n \geq 2$, any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2(M^n, g, \gamma)\; (\mathrm{vol}(M^n, g))^{\frac{2}{n}} \geq \frac{n}{4(n-1)}\; \mu(M^n, [g])\,. \tag{1.15}$$

In case $n = 2$, by the Gauss-Bonnet formula, one gets (see also [Bä 1]) $\mu = 4\pi\, \chi(M)$, where $\chi(M)$ is the Euler-Poincaré characteristic class.

We also mention that Inequality (1.13) is also true for second order differential operators introduced in [HL] and there is a similar Inequality for the twistor operator [Li 8].

2. Kählerian Spin Manifolds

This section is devoted to a self-contained exposition of known results on the first eigenvalues of the Dirac operator on Kählerian spin compact manifolds. We recall the basic properties of these manifolds. Let (M^n, g, J, γ) be a Kähler spin manifold of dimension $n = 2m$, with a Riemannian metric g, a parallel complex structure J and a spin structure γ. For any tangent vector fields X and Y, the Kähler form Ω, defined by the complex structure J, satisfies the following relations

$$\Omega(X, Y) = g(J(X), Y) = -g(X, J(Y))\,,$$

$$\nabla\Omega = 0\,,$$

$$\omega = \frac{1}{m!}\underbrace{\Omega\wedge\ldots\wedge\Omega}_{m\ \text{times}},$$

where ω is the volume element defining the orientation of (M^{2m}, g, J, γ). In what follows we consider a local orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent bundle TM, and at any point x in M, we choose normal coordinates at this point so that $(\nabla e_i)(x) = 0$, for all $i \in \{1, \ldots, n\}$. All the computations will be made in such charts. We recall the following result concerning the spectrum of the Dirac operator on compact spin Kähler manifolds.

Theorem 5. [Hi 1] *On a compact Kähler spin manifold (M^{2m}, g, J, γ) with positive scalar curvature S, any eigenvalue λ satisfies*

$$\lambda^2(M^{2m}, g, J, \gamma) > \frac{n}{4(n-1)}\,\mu_1(M^{2m}, g, J)\,, \tag{2.1}$$

where $n = 2m$ is the real dimension and μ_1 is the first eigenvalue of the Yamabe operator.

Proof. It is sufficient to prove that a compact spin manifold with a real Killing spinor could not carry a Kähler structure. Assume that ψ is such a spinor. Then by Lemma 2, the spinor ψ satisfies

$$\nabla_X\psi = -\frac{\lambda_0}{n}\,X\cdot\psi, \qquad \forall X \in \Gamma(TM)\,, \tag{2.2}$$

where $\lambda_0^2 := \frac{n}{4(n-1)}\,S_0 = \frac{n}{4(n-1)}\,\mu_1$. By contradiction assume that the manifold has a Kähler structure and denote by Ω the Kähler form. Since Ω is parallel, one gets

$$\begin{aligned}
D(\Omega\cdot\psi) &= \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}(\Omega\cdot\psi)\,,\\
&= \sum_{i=1}^{n} e_i\cdot\Omega\cdot\nabla_{e_i}\psi\,,\\
&= -\frac{\lambda_0}{n}\sum_{i=1}^{n} e_i\cdot\Omega\cdot e_i\cdot\psi\,,\\
&= -\frac{n-4}{n}\,\lambda_0\,\Omega\cdot\psi\,,
\end{aligned} \tag{2.3}$$

where we have used the fact that for any k-form α, one has

$$\sum_{i=1}^{n} e_i \cdot \alpha \cdot e_i = (-1)^k (n-2k)\, \alpha \,.$$

Since λ_0 is minimal, for $n \neq 4$, Equation (2.3) implies that the spinor $\Omega\cdot\psi$ is an eigenspinor for D associated with an eigenvalue strictly smaller than the smallest eigenvalue, hence $\Omega \cdot \psi = 0$. For $n = 4$, the same conclusion holds by the Lichnerowicz theorem. Since Clifford multiplication of Ω by itself m-times gives the volume element, it follows that $\psi \equiv 0$, a contradiction. $\square$

Theorem 6. [Li 3] *A Riemannian compact spin manifold with a non-trivial Killing spinor has no non-trivial parallel forms.*

Proof. The same argument used in the proof of Theorem 5 could be applied to any parallel k-form α to deduce that $\alpha \cdot \psi = 0$. Since by defintion of Clifford multiplication, for any k-form α and vector field X, one has

$$\begin{aligned} X \cdot \alpha &= X \wedge \alpha - \imath_X \alpha \,, \\ \alpha \cdot X &= (-1)^k \left(X \wedge \alpha + \imath_X \alpha \right) \,, \end{aligned}$$

by differentiation and for any vector field X, one gets

$$\begin{aligned} 0 &= \alpha \cdot X \cdot \psi \,, \\ &= (-1)^k \left((X \wedge \alpha) \cdot \psi + (\imath_X \alpha) \cdot \psi \right) \,, \\ &= (-1)^k \left(X \cdot \alpha \cdot \psi + 2(\imath_X \alpha) \cdot \psi \right) \,, \\ &= 2(-1)^k (\imath_X \alpha) \cdot \psi \,, \end{aligned}$$

where $X\wedge$ (resp. $\imath_X$) *denotes* the exterior (resp. interior) multiplication by the 1-form associated with X. One then gets the result by induction. $\square$

The following result is due to Kirchberg.

Theorem 7. [Ki 1, 3] *On a compact Kähler spin manifold (M^{2m}, g, J, γ) with positive scalar curvature S, any eigenvalue λ of the Dirac operator satisfies*

$$(*) \qquad \lambda^2 \geq \frac{m+1}{4m} \inf_M S \,, \qquad \text{if } m \text{ is odd,}$$

and

$$(**) \qquad \lambda^2 \geq \frac{m}{4(m-1)} \inf_M S \,, \qquad \text{if } m \text{ is even} \,.$$

The remaining part of this section is devoted to the proof of Theorem 7. For this, we need some Lemmas which are essential for the determination of Kählerian twistor operators. Most of these Lemmas are due to Kirchberg [Ki 1, 2, 3] but our presentation is different.

Lemma 3. *As an endomorphism of the spinor bundle, the Kähler form is given locally by*

$$\Omega = \frac{1}{2}\sum_{k=1}^{n} e_k \cdot J(e_k) = -\frac{1}{2}\sum_{k=1}^{n} J(e_k)\cdot e_k\,. \tag{2.4}$$

Proof. By definition of the 2-form Ω, one has

$$\begin{aligned}\Omega &= \frac{1}{2}\sum_{k,l=1}^{n} \Omega_{kl}\; e_k \wedge e_l\,,\\ &= \frac{1}{2}\sum_{k,l=1}^{n} g(J(e_k),e_l)\; e_k\cdot e_l\,, \quad \square\\ &= \frac{1}{2}\sum_{k}^{n} e_k\cdot J(e_k)\,.\end{aligned}$$

Lemma 4. *For any vector field X, one has*

$$[\Omega, X] = 2J(X), \qquad \text{and} \qquad [\Omega, J(X)] = -2X\,. \tag{2.5}$$

Proof. Since $J^2 = -\operatorname{Id}$, the second Identity follows from the first. By linearity, it is sufficient to take $X = e_k$, for $1 \le k \le n$, fixed. We have

$$\begin{aligned}\Omega\cdot e_k &= \frac{1}{2}\sum_{l}^{n} e_l\cdot J(e_l)\cdot e_k\,,\\ &= -\frac{1}{2}\sum_{l}^{n} e_l\cdot\left(e_k\cdot J(e_l) + 2g(J(e_l),e_k)\right)\,,\\ &= J(e_k) + \frac{1}{2}\sum_{l}^{n}\left(e_k\cdot e_l + 2\delta_{kl}\right)J(e_l)\,,\\ &= 2J(e_k) + e_k\cdot\Omega\,. \quad \square\end{aligned}$$

Lemma 5. *Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum*

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M\,, \tag{2.6}$$

where each sub-bundle $\Sigma_r M$ is an eigenbundle for Ω with eigenvalue $i\,\mu_r := i\,(2r-m)$ and whose rank is $\binom{r}{m}$.

An orthonormal local frame $\{e_1,\dots,e_m,e_{m+1},\dots,e_{2m}\}$ is said to be *adapted* if $J(e_j) = e_{m+j}$ and $J(e_{m+j}) = -e_j$. Let

$$\begin{cases} p_+(e_j) := z_j := \frac{e_j - ie_{m+j}}{2}\,,\\ p_-(e_j) := \overline{z}_j := \frac{e_j + ie_{m+j}}{2}\,.\end{cases} \tag{2.7}$$

It is a straightforward verification that $\{z_1, \dots, z_m, \overline{z}_1, \dots, \overline{z}_m\}$ forms a de Witt basis with respect to the complex extension of the Riemannian metric g. In fact, one has

$$\begin{cases} g(z_j, z_k) = g(\overline{z}_j, \overline{z}_k) = 0\,, \\ g(z_j, \overline{z}_k) = g(\overline{z}_j, z_k) = \frac{1}{2}\delta_{jk}\,. \end{cases} \tag{2.8}$$

The multiplication rule in the complex Clifford algebra $\mathbb{C}l_x M$ could be expressed by

$$\begin{cases} z_j \cdot z_k + z_k \cdot z_j = \overline{z}_j \cdot \overline{z}_k + \overline{z}_k \cdot \overline{z}_j = 0 \\ z_j \cdot \overline{z}_k + \overline{z}_j \cdot z_k = -\delta_{jk}\,. \end{cases} \tag{2.9}$$

Let $\overline{\boldsymbol{\omega}} := \overline{z}_1, \cdot \dots \cdot \overline{z}_m$. Then, at a point $x \in M$, the set

$$\{z_{j_1} \cdot \dots \cdot z_{j_r} \cdot \overline{\boldsymbol{\omega}} : 1 \le j_1 < \dots < j_r \le m\,, 0 \le r \le m\}\,,$$

is a basis of $\Sigma_x M$. An explicit isomorphism between the Clifford algebra and the algebra of endomorphisms of the spin bundle is given by

$$\begin{aligned} \rho : \quad \mathbb{C}l_x M &\longrightarrow \operatorname{End} \Sigma_x M \\ v := z_{j_1} \cdot \dots \cdot z_{j_p} \cdot \overline{z}_{k_1} \cdot \dots \cdot \overline{z}_{k_q} &\longmapsto \rho(v)\,, \end{aligned}$$

where

$$\rho(v)\,(\,z_{l_1} \cdot \dots \cdot z_{l_r} \cdot \overline{\boldsymbol{\omega}}\,) = z_{j_1} \cdot \dots \cdot z_{j_p} \cdot \overline{z}_{k_1} \cdot \dots \cdot \overline{z}_{k_q} \cdot z_{l_1} \cdot \dots \cdot z_{l_r} \cdot \overline{\boldsymbol{\omega}}\,.$$

The action of Ω could be expressed as

$$\Omega = i\,m + 2\,i \sum_{j=1}^{m} \overline{z}_j \cdot z_j\,. \tag{2.10}$$

Proof of Lemma 5. For each $r \in \{0, \dots, m\}$, define the space $\Sigma_r M$ by

$$\Sigma_r M := \{\,z_{j_1} \cdot \dots \cdot z_{j_r} \cdot \overline{\boldsymbol{\omega}} : 1 \le j_1 < \dots < j_r \le m\,\}\,.$$

Since the action by Clifford multiplication is fiberwise, all the computations are pointwise. Consider a spinor $\psi_r := z_{j_1} \cdot \dots \cdot z_{j_r} \cdot \overline{\boldsymbol{\omega}} \in \Sigma_r M$. By (2.9), it follows

$$\begin{aligned} \Omega \cdot \psi_r &= i \left[m + 2 \sum_{j=1}^{m} \overline{z}_j \cdot z_j \right] \cdot z_{j_1} \cdot \dots \cdot z_{j_r} \cdot \overline{\boldsymbol{\omega}}\,, \\ &= i\,m\,\psi_r - 2\,i\,(m-r)\,\psi_r\,, \\ &= i\mu_r\,\psi_r\,. \quad \square \end{aligned}$$

Lemma 6. *For any tangent vector field X and $r \in \{0, \dots, m\}$, one has*

$$p_+(X) \cdot \Sigma_r M \subset \Sigma_{r+1} M \quad \text{and} \quad p_-(X) \cdot \Sigma_r M \subset \Sigma_{r-1} M \,. \tag{2.11}$$

with the convention that $\Sigma_{-1} M = \Sigma_{m+1} M = M \times \{0\}$.

Proof. The proof is a consequence of Lemma 4. It is also possible to use the explicit decomposition given in Lemma 5 to get the result. To see this, consider a nontrivial spinor $\psi_r := z_{j_1} \cdot \ldots \cdot z_{j_r} \cdot \overline{\boldsymbol{\omega}} \in \Sigma_r M$. By linearity it is sufficient to show (2.11) for $X = e_k$, for any $k \in \{1, \dots, m\}$. Lemma 5 and (2.9) imply that $z_k \cdot \psi = 0$ if $k \in \{j_1, \dots, j_r\}$ and $z_k \cdot \psi \in \Sigma_{r+1} M$ if $k \notin \{j_1, \dots, j_r\}$. The second inclusion could be seen similarly. □

We now show that Lemma 6 implies that the image of the restriction of the Dirac operator on sections of the subbundle $\Sigma_r M$ lies in the space of sections of $\Sigma_{r-1} M \oplus \Sigma_{r+1} M$.

Associated with J, there is an elliptic self-adjoint operator $\widetilde{D}$ (see [Hn, Mi, Ki 1]) defined locally by

$$\widetilde{D} = \sum_{i=1}^{n} J(e_i) \cdot \nabla_{e_i} = - \sum_{i=1}^{n} e_i \cdot \nabla_{J(e_i)} \,.$$

Since the almost complex structure J is parallel, Lemma 4 yields to the following relations

Lemma 7. *The operator $\widetilde{D}$ satisfies*

$$[\Omega, D] = 2\widetilde{D} \,, \qquad \text{and} \qquad [\Omega, \widetilde{D}] = -2D \,, \tag{2.12}$$

$$\widetilde{D} D + D \widetilde{D} = 0 \,, \tag{2.13}$$

$$\widetilde{D}^2 = D^2 \,. \tag{2.14}$$

Proof. The Identities (2.12) follow from (2.5). Equation (2.13) is a direct consequence of the identity

$$X \cdot J(X) + J(X) \cdot X = -2g(X, J(X)) = 0 \,, \qquad \forall X \in \Gamma(TM) \,.$$

By (2.12) one gets

$$2D^2 = D\,[\widetilde{D}, \Omega] = D\,\widetilde{D}\,\Omega - D\,\Omega\,\widetilde{D} \,, \tag{2.15}$$

and

$$2\widetilde{D}^2 = \widetilde{D}\,[\Omega, D] = \widetilde{D}\,\Omega\,D - \widetilde{D}\,D\,\Omega \,. \tag{2.16}$$

Equations (2.15), (2.16) and (2.13) imply

$$2\widetilde{D}^2 - 2D^2 = \widetilde{D}\,\Omega\,D + D\,\Omega\,\widetilde{D} \,. \tag{2.17}$$

On the other hand, by (2.12)and (2.13) it follows

$$\begin{aligned} \widetilde{D}\,\Omega\,D + D\,\Omega\,\widetilde{D} &= (\Omega\,\widetilde{D} + 2D)\,D + (\Omega\,D - 2\widetilde{D})\,\widetilde{D} \,, \\ &= 2D^2 - 2\widetilde{D}^2 \,. \end{aligned} \tag{2.18}$$

Equations (2.17) and (2.18) imply (2.14). □

Lemma 6 and Lemma 7 immediately imply:

Lemma 8. *Define the two operators D_+ and D_- by*

$$p_-(D) := D_- := \frac{1}{2}\,(D + i\,\widetilde{D})\,, \qquad \text{and} \qquad p_+(D) := D_+ := \frac{1}{2}\,(D - i\,\widetilde{D})\,.$$

We have the following differential complexes

$$\begin{aligned} &\{0\} \longrightarrow \Gamma(\Sigma_m M) \ldots \Gamma(\Sigma_r M) \xrightarrow{D_-} \Gamma(\Sigma_{r-1} M) \xrightarrow{D_-} \ldots \Gamma(\Sigma_0 M) \longrightarrow \{0\}\,, \\ &\{0\} \longrightarrow \Gamma(\Sigma_0 M) \ldots \Gamma(\Sigma_r M) \xrightarrow{D_+} \Gamma(\Sigma_{r+1} M) \xrightarrow{D_+} \ldots \Gamma(\Sigma_m M) \longrightarrow \{0\}\,. \end{aligned} \tag{2.19}$$

Now we are ready to define Kählerian twistor operators. On a Kähler spin manifold $(M^{n=2m}, g, J, \gamma)$, for each $r \in \{0, \ldots, m\}$, the Kählerian twistor operator $P^{(r)}$ is the composition of the restriction of the covariant derivative to the subbundle $\Gamma(\Sigma_r M)$ with the projection on $\ker \mathcal{M}_r$:

$$P^{(r)} \;:\quad \Gamma(\Sigma_r M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma_r M) \xrightarrow{\pi_r} \ker \mathcal{M}_r\ ,$$

where $\mathcal{M}_r$ is the restricted Clifford multiplication defined by

$$\mathcal{M}_r \;:\quad \Gamma(T^*M \otimes \Sigma_r M) \longrightarrow \Gamma(\Sigma_{r-1} M) + \Gamma(\Sigma_{r+1} M)\,,$$

and π_r is the projection on $\ker \mathcal{M}_r \subset \Gamma(T^*M \otimes \Sigma_r M)$. We have

Proposition 1. *For each $r \in \{0, \ldots, m\}$ and for any spinor field $\psi_r \in \Gamma(\Sigma_r M)$, the local expression of the Kählerian twistor operator $P^{(r)}$ is given by*

$$P^{(r)}\psi_r = \sum_{i=1}^{2m} e_i \otimes \left[\nabla_{e_i}\psi_r + a_r\, p_-(e_i)\cdot D_+\psi_r + b_r\, p_+(e_i)\cdot D_-\psi_r\right]\,, \tag{2.20}$$

where

$$a_r = \frac{1}{2(r+1)} \qquad \text{and} \qquad b_r = \frac{1}{2(m-r+1)}\,.$$

Proof. By Lemmas 6 and 8, we first notice that the image by $P^{(r)}$ of any ψ_r lies in $\Gamma(T^*M \otimes \Sigma_r M)$. We now compute the constants a_r and b_r so that the image of $P^{(r)}$ lies in $\ker \mathcal{M}_r$. For this, it is sufficient to compute the following two traces:

$$\sum_{i=1}^{n} e_i \cdot p_-(e_i) \qquad \text{and} \qquad \sum_{i=1}^{n} e_i \cdot p_+(e_i)\ ,$$

viewed respectively as acting on the space $\Gamma(\Sigma_{r+1} M)$ and $\Gamma(\Sigma_{r-1} M)$. By definition of $p_-(e_i)$, Equation (2.4) and Lemma 5, one gets

$$\begin{aligned} \sum_{i=1}^{n} e_i \cdot p_-(e_i) &= \frac{1}{2}\sum_{i=1}^{n} e_i \cdot (e_i + i\,J(e_i))\ , \\ &= \frac{1}{2}\,(-n + 2\,i\,\Omega)\ , \\ &= \frac{1}{2}\,(-n - 2\,\mu_{r+1})\ , \\ &= -2(r+1)\ . \end{aligned}$$

Similarly, for the second trace, it follows

$$\begin{aligned}\sum_{i=1}^{n} e_i \cdot p_+(e_i) = & \ \frac{1}{2} \sum_{i=1}^{n} e_i \cdot (e_i - i\,J(e_i)) \ , \\ = & \ -\frac{1}{2}\,(n + 2\,i\,\Omega) \ , \\ = & \ -\frac{1}{2}\,(n - 2\,\mu_{r-1}) \ , \\ = & \ -2\,(m - r + 1) \ .\end{aligned}$$

By definition of $P^{(r)}$, the coefficients a_r and b_r should satisfy

$$\frac{1}{a_r} = -\sum_{i=1}^{n} e_i \cdot p_+(e_i)|_{\Gamma(\Sigma_{r-1}M)} \ = \ 2\,(m - r + 1) \ ,$$

and

$$\frac{1}{b_r} = -\sum_{i=1}^{n} e_i \cdot p_-(e_i)|_{\Gamma(\Sigma_{r+1}M)} \ = 2(r + 1) \ . \quad \square$$

By definition, for each integer r the Kählerian twistor operator $P^{(r)}$ satisfies

$$P^{(r)}_X \ : \quad \Gamma(\Sigma_r M) \longrightarrow \Gamma(\Sigma_r M)\,, \qquad \forall X \in \Gamma(TM)\,, \tag{2.21}$$

and, for any spinor field $\psi_r \in \Gamma(\Sigma_r M)$,

$$\sum_{i=1}^{n} e_i \cdot P^{(r)}_{e_i} \psi_r = 0\,. \tag{2.22}$$

Moreover, we have:

Lemma 9. *For each integer r and any spinor field $\psi_r \in \Gamma(\Sigma_r M)$, the Kählerian twistor operator $P^{(r)}$ satisfies*

$$\sum_{i=1}^{n} J(e_i) \cdot P^{(r)}_{e_i} \psi_r = 0\,. \tag{2.23}$$

Proof. The Identities (2.5), (2.21) and (2.22), imply

$$\begin{aligned}0 &= \Omega \cdot \big(\sum_{i=1}^{n} e_i \cdot P^{(r)}_{e_i} \psi_r\big) \ , \\ &= \sum_{i=1}^{n} \big(e_i \cdot \Omega + 2J(e_i)\big) \cdot P^{(r)}_{e_i} \psi_r \ , \\ &= i\,\mu_r \sum_{i=1}^{n} e_i \cdot P^{(r)}_{e_i} \psi_r + 2 \sum_{i=1}^{n} J(e_i) \cdot P^{(r)}_{e_i} \psi_r \ , \\ &= 2 \sum_{i=1}^{n} J(e_i) \cdot P^{(r)}_{e_i} \psi_r \ . \quad \square\end{aligned}$$

Proof of Theorem 6. From the classification of Clifford algebras [ABS], it follows that the spin bundle of a Kähler spin manifold carries an anti-linear map j satisfying the relations (see also [Ki 1]):

$$\nabla j = 0\,, \tag{2.24}$$

$$j^2 = (-1)^{\frac{m(m+1)}{2}}\, I_{\Sigma M}\,, \tag{2.25}$$

$$[X, j] = 0\,, \qquad \forall X \in \Gamma(TM)\,, \tag{2.26}$$

$$(j\psi, j\varphi) = (\varphi, \psi)\,, \qquad \forall \psi, \varphi \in \Gamma(\Sigma M)\,. \tag{2.27}$$

From these identities one derives the following relations

$$[D^2, j] = 0\,, \tag{2.28}$$

$$[\Omega, j] = 0\,. \tag{2.29}$$

Moreover, for $\psi_r \in \Sigma_r M$, by (2.29) and the anti-linearity of j, it comes

$$\Omega \cdot (j\psi_r) = j\,\Omega \cdot \psi_r = -i\;\mu_r\, j\psi_r = i\;\mu_{m-r}\, j\psi_r\,,$$

hence , j defines an isomorphism between $\Sigma_r M$ and $\Sigma_{m-r} M$. Consider the space $\Sigma^\lambda M := \{\psi \in \Gamma(\Sigma M) : D^2\psi = \lambda^2\,\psi\}$. Since $[\Omega, D^2] = 0$, it follows that there exists an orthogonal decomposition of the space $\Sigma^\lambda M$ into

$$\Sigma^\lambda M = \mathop{\oplus}_{r=0}^{m} \Sigma_r^\lambda M\,. \tag{2.30}$$

In the decomposition (2.30), take r to be the minimum integer for which $\Sigma_r^\lambda M \neq \{0\}$. With the help of the isomorphism j, it follows that the integer r satifies $0 \le r < \frac{m}{2}$. Since $[D^2, D_\pm] = 0$, then for $0 \neq \psi_r \in \Sigma_r^\lambda M$, one has $D_-\psi_r = 0$, i.e., $D\psi_r = D_+\psi_r$. The Identities (2.22) and (2.23) imply

$$\begin{aligned}
\int_M |P^{(r)}\psi_r|^2\, v_g &= \int_M \sum_{i=1}^{n} |P^{(r)}_{e_i}\psi_r|^2\, v_g\,,\\
&= \int_M \sum_{i=1}^{n} \big(P^{(r)}_{e_i}\psi_r, \nabla_{e_i}\psi_r\big)\, v_g\,,\\
&= \int_M \sum_{i=1}^{n} (\nabla_{e_i}\psi_r + \frac{1}{2(r+1)}\, p_-(e_i)\cdot D_+\psi_r\; v_g\,, \nabla_{e_i}\psi_r)\,.
\end{aligned}$$

Since Clifford multiplication by X or $J(X)$ is skew-symmetric, it follows

$$\int_M |P^{(r)}\psi_r|^2\, v_g = \int_M |\nabla\psi_r|^2\, v_g - \frac{1}{2(r+1)} \int_M |D\psi_r|^2\, v_g\,. \tag{2.31}$$

Equation (2.31) combined with the Lichnerowicz formula, yields

$$\lambda^2 \geq \frac{2r+2}{2r+1}\,\frac{S_0}{4}\,, \qquad 0 \leq r < \frac{m}{2}\,. \tag{2.32}$$

Since $\frac{2r+2}{2r+1}$ is a decreasing function of r, one gets $(*)$ and $(**)$ respectively for $2r = m-1$ and $2r = m-2$. $\square$

Equality in $(*)$ is characterized by the existence of a non-trivial solution, *called* Käkhlerian Killing spinor, of an over-determined differential equation similar to the equations of real Killing spinors [Hi 5, Ki 4]. These manifolds has been classified recently by A. Moroianu [Mo] (see also [KS]). Manifolds with eigenvalues satisfying equality in $(**)$ are classified, for $m = 2$, by Th. Friedrich [Fr 3] and for $m \geq 4$, by A. Lichnerowicz [Li 12].

3. Quaternionic-Kähler Spin Manifolds

In this section, we review the main results in [HM 1, 2, 3]. The key point is to define the Quaternionic twistor operators. Since formally this is done along the same lines as in our presentation for the Kähler case, we only give the main ingredients needed to prove the following:

Theorem 8. [HM 3] *On a compact quaternionic spin manifold of dimension $4m \geq 8$ with positive scalar curvature S, let λ be any eigenvalue of the Dirac operator. Then, the following two inequalities hold*

$$(*_r) \qquad \exists r \in \mathbb{N}\,,\ 0 \leq r < m\,, \qquad \lambda^2 \geq \frac{2(r+1)(m+r+3)}{2(r+1)(m+r+3)-(r+2)}\,\frac{S}{4}\,,$$

and

$$(**_r) \qquad \exists r \in \mathbb{N}\,,\ 0 < r \leq m\,, \qquad \lambda^2 \geq \frac{2(r+1)(m-r+1)}{2(r+1)(m-r+1)-r}\,\frac{S}{4}\,,$$

where r is an integer given by a generic eigenvalue $\boldsymbol{\mu}_r = 6\,m - 4r(r+2)$ of the fundamental 4-form $\boldsymbol{\Omega}$, acting on the spin bundle by Clifford multiplication.

The eigenvalues $\boldsymbol{\mu}_r$ of $\boldsymbol{\Omega}$ and the associated eigenspaces are determined in [HM 1]. Let λ_1 be the first eigenvalue of the Dirac operator and *denote* by $\boldsymbol{\nu}_1$ the ratio $\frac{\lambda_1^2}{S}$. Then, Inequalities $(*_r)$ and $(**_r)$ imply (see Corollary 2):

Corollary 1. *Under the conditions of Theorem 8, for $m = 2$ or for $m = 3$, one has*

$$\boldsymbol{\nu}_1 \geq \boldsymbol{\nu}_0 \,, \tag{3.1}$$

with $\boldsymbol{\nu}_0$ the corresponding ratio for the Quaternionic projective space $(I\!HP^m, can)$.

In [Mt], it is shown that

$$\boldsymbol{\nu}_0 = \frac{m+3}{4(m+2)} \;.$$

It is natural to *expect* that Inequality (3.1) is true in all dimensions. Independently, Alekseevsky-Marchiafava [AM] and Lebrun [Le] showed recently that for compact Quaternionic manifolds with positive scalar curvature, Inequality (3.1) holds for the eigenvalues of the scalar Laplacian with equality if and only if the manifold is isometric to $(I\!HP^m, can)$.

It should be also pointed out that for S positive and m odd, the manifold is isometric to the projective Quaternionic space [Sa], and by results of LeBrun-Salamon [LS], Quaternionic compact manifolds with positive scalar curvature are in some sense "rare".

We start by reviewing some basic relations on Quaternionic spin manifolds [HM 1, 2] which will be used in this paper. In fact, all what is needed to define the Quaternionic twistor operators, is the computation of the iterated commutators of $\boldsymbol{\Omega}$ with Clifford multiplication by vector fields. For this, let (M^{4m}, g, F, γ) be such a manifold of dimension $4m \geq 8$. The Quaternionic structure is given by a rank 3 subbundle F of $\mathrm{End}(TM)$ such that: there exists a trivialization for which the associated local sections J_α, $\alpha = 1, 2, 3$, satisfy [Sa, Be]

$$g(J_\alpha(X), J_\alpha(Y)) = g(X, Y) \;, \quad \forall X, Y \in TM, \; \alpha = 1, 2, 3, \tag{3.2}$$

$$J_\alpha \circ J_\beta = -\delta_{\alpha\beta} \; Id \; + \varepsilon^{123}_{\alpha\beta\gamma} \; J_\gamma \;, \tag{3.3}$$

where $\varepsilon^{123}_{\alpha\beta\gamma} = \pm 1$ depending on whether (α, β, γ) is an even or odd permutation of $(1, 2, 3)$, $\varepsilon^{123}_{\alpha\beta\gamma} = 0$ otherwise, and

$$\nabla \; J_\alpha = \sum_{\beta=1}^{3} \omega^\beta_\alpha \; J_\beta \;. \tag{3.4}$$

Here ∇ *denotes* the natural extension of the Levi-Civita connection on M and ω^β_α are local 1-forms on M.

The above conditions imply:

$$g(J_\alpha X, X) = -g(X, J_\alpha X) = 0 \;, \quad \alpha = 1, 2, 3, \quad \forall X \in TM \,, \tag{3.5}$$

$$g(J_\alpha X, J_\beta X) = \delta_{\alpha\beta} \; g(X, X) \;, \quad \alpha, \beta = 1, 2, 3, \quad \forall X \in TM \,, \tag{3.6}$$

Let Ω_α, $\alpha = 1, 2, 3$, be the local "Kähler" 2-forms defined by

$$\Omega_\alpha(X, Y) = g(J_\alpha X, Y)\ , \quad \alpha = 1, 2, 3 \quad , \forall X, Y \in TM\ . \tag{3.7}$$

The fundamental global 4-form $\mathbf{\Omega}$ is then defined by [Bn, Kr]

$$\mathbf{\Omega} = \sum_{\alpha=1}^{3} \Omega_\alpha \wedge \Omega_\alpha\ , \tag{3.8}$$

which satisfies

$$\nabla \mathbf{\Omega} = 0\ . \tag{3.9}$$

As an endomorphism of the spin bundle, the global form $\mathbf{\Omega}$ is related to the local 2-forms Ω_α by

$$\mathbf{\Omega} = \sum_{\alpha=1}^{3} \Omega_\alpha \cdot \Omega_\alpha \ + \ 6m\ Id \tag{3.10}$$

For any vector field X, one has:

$$[\Omega_\alpha, X] = 2\ J_\alpha(X)\ , \quad , \quad \forall\, \alpha = 1, 2, 3\ , \tag{3.11}$$

$$\frac{1}{4}\,[\mathbf{\Omega}, X] = \sum_{\alpha=1}^{3}\ J_\alpha(X) \cdot \Omega_\alpha - 3\,X\ . \tag{3.12}$$

Denote the r.h.s of Equation (3.12) by $\mathcal{J}(X)$. With the help of (3.11) and (3.6), it is straightforward to see that $\mathcal{J}(X)$ could also be written as

$$\mathcal{J}(X) = \sum_{\alpha=1}^{3}\ \Omega_\alpha \cdot J_\alpha(X) + 3\,X\ . \tag{3.13}$$

We now make use of the decomposition of the spin bundle under the action of the fundamental 4-form to define the Quaternionic twistor operators. For this, we need the following:

Proposition 2. [HM 1] *Under the action of $\mathbf{\Omega}$, the spin bundle ΣM splits into*

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M\ ,$$

each fiber of $\Sigma_r M$ is an eigenspace of $\mathbf{\Omega}$ for the eigenvalue

$$\boldsymbol{\mu}_r = 6m - 4r\,(r + 2)\ .$$

The multiplicity of $\boldsymbol{\mu}_r$ is given by

$$\operatorname{rank}(\Sigma_r M) = (r+1)\left(\binom{2m}{m-r} - \binom{2m}{m-r-2}\right)\ ,$$

with $\binom{p}{q} = 0$ if $q < 0$.

In the following proposition gives some interesting properties of the operator $\mathcal{J}(X)$.

Proposition 3. *As endomorphisms of* $\Gamma(\Sigma M)$*, one has*

$$X \cdot \mathcal{J}(X) + \mathcal{J}(X) \cdot X = 0 \,, \tag{3.14}$$

$$\begin{aligned} [\mathbf{\Omega}, \mathcal{J}(X)] &= 4 \left(3X + 2\, \mathcal{J}(X) - (\mathbf{\Omega} - 6m) \cdot X\right), \\ &= 4 \left(3X - 2\, \mathcal{J}(X) - X \cdot (\mathbf{\Omega} - 6m)\right). \end{aligned} \tag{3.15}$$

Moreover, if $\{e_1, \ldots, e_{4m}\}$ *denotes an orthonormal basis of the tangent space at some point* x *of* M*, then*

$$\mathbf{\Omega} = \frac{1}{2} \sum_{i=1}^{4m} e_i \cdot \mathcal{J}(e_i) \,. \tag{3.16}$$

As in the Kähler case, with the help of the commutation relations it is easy to see that Clifford multiplication by vector fields will send any subbundle $\Sigma_r M$ into $\Sigma_{r-1}M \oplus \Sigma_{r+1}M$, for $r \in \{0, \ldots, m\}$ with the convention that $\Sigma_{-1}M = \Sigma_{m+1}M = M \times \{0\}$. Since the 4-form $\mathbf{\Omega}$ is parallel, the Dirac operator will also send $\Sigma_r M$ into $\Sigma_{r-1}M \oplus \Sigma_{r+1}M$. More precisely,

Definition 1. *For any tangent vector field* X *and* $r \in \{0, \ldots, m\}$*, define*

$$\begin{aligned} p_-^{(r)}(X) &= \frac{1}{4(r+1)} \left[(2r+3)X + \mathcal{J}(X)\right], \\ p_+^{(r)}(X) &= \frac{1}{4(r+1)} \left[(2r+1)X - \mathcal{J}(X)\right]. \end{aligned} \tag{3.17}$$

Notice that these "projectors" are defined so that for all $r \in \{0, \ldots, m\}$, and any vector field X, decomposes as $X = p_-^{(r)}(X) + p_+^{(r)}(X)$, where $p_-^{(r)}(X)$ and $p_+^{(r)}(X)$ are defined by the sum of forms of degree 1 and 3.

Lemma 10. *For any tangent vector field* X *and* $r \in \{0, \ldots, m\}$*, one has*

$$p_-^{(r)}(X) \cdot \Sigma_r M \subset \Sigma_{r-1} M \quad \text{and} \quad p_+^{(r)}(X) \cdot \Sigma_r M \subset \Sigma_{r+1} M \,. \tag{3.18}$$

Definition 2. *Define the operators* $\widetilde{D}$ *and* $D_\pm^{(r)}$ *by*

$$\begin{aligned} \widetilde{D} &= \frac{1}{4} [\mathbf{\Omega}, D] \,, \\ D_-^{(r)} &:= \frac{1}{4(r+1)} \left[(2r+3)D + \widetilde{D}\right], \\ D_+^{(r)} &:= \frac{1}{4(r+1)} \left[(2r+1)D - \widetilde{D}\right]. \end{aligned}$$

Then by (3.12), it is straightforward to see that the operator $\widetilde{D}$ could be written locally as

$$\widetilde{D} = \sum_{i=1}^{4m} \mathcal{J}(e_i) \cdot \nabla_{e_i} \,.$$

Moreover,

Lemma 11. *For any $r \in \{0, \dots, m\}$, the operators $D_{\pm}^{(r)}$ satisfy*

$$D_{-}^{(r)}(\Sigma_r M) \subset \Sigma_{r-1} M \quad \text{and} \quad D_{+}^{(r)}(\Sigma_r M) \subset \Sigma_{r+1} M \,. \tag{3.19}$$

On a Quaternionic spin manifold (M^{4m}, g, F, γ), for each $r \in \{0, \dots, m\}$, the Quaternionic twistor operator $\boldsymbol{P}^{(r)}$ is the composition of the restriction of the covariant derivative to the subbundle $\Gamma(\Sigma_r M)$ with the projection on $\ker \mathcal{M}_r$:

$$\boldsymbol{P}^{(r)} \ : \quad \Gamma(\Sigma_r M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma_r M) \xrightarrow{\pi_r} \ker \mathcal{M}_r \ ,$$

where $\mathcal{M}_r$ is the restricted Clifford multiplication defined by

$$\mathcal{M}_r \ : \quad \Gamma(T^*M \otimes \Sigma_r M) \longrightarrow \Gamma(\Sigma_{r-1} M) + \Gamma(\Sigma_{r+1} M) \,,$$

and π_r is the projection on $\ker \mathcal{M}_r \subset \Gamma(T^*M \otimes \Sigma_r M)$. We have

Proposition 4. *For every $r \in \{0, \dots, m\}$ and any spinor field $\psi_r \in \Gamma(\Sigma_r M)$, the local expression of the Quaternionic twistor operator $\boldsymbol{P}^{(r)}$ is given by*

$$\boldsymbol{P}^{(r)} \psi_r = \sum_{i=1}^{4m} e_i \otimes \left[\nabla_{e_i} \psi_r + a^{(r)} \, p_{-}^{(r+1)}(e_i) \cdot D_{+}^{(r)} \psi_r + b^{(r)} \, p_{+}^{(r-1)}(e_i) \cdot D_{-}^{(r)} \psi_r \right] \,,$$

where

$$a^{(r)} = \frac{r+2}{2(r+1)(m+r+3)} \qquad \text{and} \qquad b^{(r)} = \frac{r}{2(r+1)(m-r+1)} \,.$$

By Lemmas 10 and 11, we first notice that the image by $\boldsymbol{P}^{(r)}$ of any ψ_r lies in $\Gamma(T^*M \otimes \Sigma_r M)$. The constants $a^{(r)}$ and $b^{(r)}$ are determined by requiring that the image of $P^{(r)}$ lies in $\ker \mathcal{M}_r$.

Moreover, we have:

Lemma 12. *For each integer r and any spinor field $\psi_r \in \Gamma(\Sigma_r M)$, the quaternionic twistor operator $\boldsymbol{P}^{(r)}$ satisfies*

$$\begin{aligned} \sum_{i=1}^{4m} e_i \cdot \boldsymbol{P}_{e_i}^{(r)} \psi_r &= 0 \,, \\ \sum_{i=1}^{4m} \mathcal{J}(e_i) \cdot \boldsymbol{P}_{e_i}^{(r)} \psi_r &= 0 \,. \end{aligned} \tag{3.20}$$

Sketch of the proof of Theorem 8. Consider the space $\Sigma^\lambda M = \{\psi \in \Gamma(\Sigma M) : D^2 \psi = \lambda^2 \, \psi\}$. Since $[\boldsymbol{\Omega}, D^2] = 0$, it follows that there exists an orthogonal decomposition of the space $\Sigma^\lambda M$ into

$$\Sigma^\lambda M = \overset{m}{\underset{r=0}{\oplus}} \Sigma_r^\lambda M \,. \tag{3.21}$$

We choose r to be minimum (resp. maximum) so that $D_-^{(r)}\psi_r = 0$ (resp. $D_+^{(r)}\psi_r = 0$). If r is minimum, then by Lemma 12 and the Lichnerowicz formula, one easily gets

$$\int_M |P^{(r)}\psi_r|^2 = (1 - a^{(r)}) \int_M |D\psi_r|^2 - \frac{1}{4}\int_M S\,|\psi_r|^2\,,$$

which gives $(*_r)$ since the l.h.s is non-negative. For r maximum, the same argument yields

$$\int_M |P^{(r)}\psi_r|^2 = (1 - b^{(r)}) \int_M |D\psi_r|^2 - \frac{1}{4}\int_M S\,|\psi_r|^2\,,$$

which gives $(**_r)$. $\square$

We now conclude by showing that, in low dimensions, the above result imply Inequality (3.1).

Corollary 2. *Denote by $\kappa_1(r,m)$ (resp. $\kappa_2(r,m)$) the coefficient of S in Inequality $(*_r)$ (resp. $(**_r)$). Then,*

$$\kappa_1(r,m) \geq \nu_1 \qquad \Longleftrightarrow \qquad r = 0\,, \tag{3.22}$$

$$\kappa_2(r,m) \geq \nu_1 \qquad \Longleftrightarrow \qquad [\frac{m}{2}] + 1 \leq r \leq m\,, \tag{3.23}$$

where [ ] stands for the integer part. In particular, Inequality (3.1) *holds for $m = 2$ and for $m = 3$.*

Proof. The proof of (3.22) and (3.23) is given by elementary calculus. For the last statement, we consider the decomposition (3.21). For $m = 2$ or $m = 3$, there exists at least one r, with $r = 0$ or $[\frac{m}{2}] + 1 \leq r \leq m$, for which $\Sigma_r^\lambda M \neq \{0\}$. Hence by (3.22) and (3.23), Inequalities $(*_r)$ and $(**_r)$ imply (3.1). $\square$

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