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ON INFINITESIMAL AUTOMORPHISMS OF QUATERNIONIC MANIFOLDS

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Abstract. On a quaternionic manifold an infinitesimal homographic transformation (i.h.t.) is a vector field in the direction of which the Lie derivative of the tensor B introduced by E.Bonan vanishes; the resulting equation is of the first order and of tensorial type $(1,1)$. One shows that, if moreover, the manifold is Kählerian and compact an i.h.t. can be defined by a divergence type equation, which is of the second order but of tensorial type $(0,1)$.

1. Introduction. On a Kähler manifold $(M^{2n}, \boldsymbol{\mathcal{I}}, g)$ an infinitesimal transformation ξ (vector field) is analytic if the Lie derivative $\boldsymbol{\mathcal{L}}_\xi \boldsymbol{\mathcal{I}}$ of the complex structure $\boldsymbol{\mathcal{I}}$ in direction ξ vanishes. Denote $\mathrm{a}(\xi) := \frac{1}{2}(\boldsymbol{\mathcal{L}}_\xi \boldsymbol{\mathcal{I}}) \circ \boldsymbol{\mathcal{I}}$. If M^{2n} is compact, A. Lichnerowicz in [1] has shown that a vector field ξ is analytic iff $\delta \mathrm{a}\,(\xi) = 0$ where

$$\delta \mathrm{a}\,(\xi)_x := \sum_{r=1}^{2n} (\nabla_{e_r} \mathrm{a}\,(\xi))\,(e_r)\ ;$$

in this formula $\{e_1, \ldots, e_{2n}\} \subset (TM^{2n})_x$ is an orthonormal basis of the tangent space at x to M^{2n}. The proof results from the equation

$$\mathrm{div}\, \xi^* \circ \mathrm{a}\,(\xi) = < \mathrm{a}\,(\xi), \mathrm{a}\,(\xi) >_g + g\,(\xi, \delta \mathrm{a}\,(\xi))$$

through integration over M^{2n}. Here $\xi^* \circ \mathrm{a}\,(\xi)$ is the 1-form defined by

$$\xi^* \circ \mathrm{a}\,(\xi)\,(v) := g\,(\xi, \mathrm{a}\,(\xi)\,(v))_x\ ,\ v \in (TM^{2n})_x\ ,$$

and $<\,,\,>_g$ denotes the scalar product the tensor g induces in $(TM^{2n}) \otimes (TM^{2n})^*$. As it will be shown, analogous results are valid on quaternionic Kähler manifolds and on quaternionic grassmannians for homographic infinitesimal transformations in place of analytic ones. The central role in the quaternionic case is played by a tensor field $\boldsymbol{\mathcal{B}}$ of type $(2,2)$ introduced by E. Bonan in [2].

2. A quaternionic Kähler manifold $(M^{4n}, \boldsymbol{\mathcal{S}}, g)$ is a riemannian $4n$-dimensional manifold, endowed with a 2-dimensional sphere bundle $\boldsymbol{\mathcal{S}}$ of hermitian almost-complex stuctures of quaternionic type which is invariant under parallel transport, see [3] for a more detailed definition. This has as consequence the fact that M^{4n} carries a field $\boldsymbol{\mathcal{B}}$ of involutoric automorphisms of the tensorial square $TM^{4n} \otimes TM^{4n}$ of the tangent bundle TM^{4n}, expressible locally in the form

$$\boldsymbol{\mathcal{B}}|_U = \frac{1}{2}\left[\mathrm{Id} \otimes \mathrm{Id} - \sum_{\alpha=1}^{3} \underset{\alpha}{\boldsymbol{\mathcal{I}}} \otimes \underset{\alpha}{\boldsymbol{\mathcal{I}}}\right].$$

In this formula, U is a trivialisation neighbourhood for $\boldsymbol{\mathcal{S}}$, and $\underset{\alpha}{\boldsymbol{\mathcal{I}}}$, $\alpha = 1, 2, 3$ are three anticommuting sections of $\boldsymbol{\mathcal{S}}|_U$ such that $\boldsymbol{\mathcal{I}}_1 \circ \boldsymbol{\mathcal{I}}_2 = \boldsymbol{\mathcal{I}}_3, \ldots;$

they are determined modulo an orthogonal transformation. Moreover, equation

$$\nabla^g \boldsymbol{\mathcal{B}} = 0$$

where ∇^g is the Levi-Civita connection of the riemannian structure g, expresses the invariance of the bundle $\boldsymbol{\mathcal{S}}$ under parallel transport. Locally, this equation becomes

$$\sum_\alpha \left(\nabla^g_X \underset{\alpha}{\boldsymbol{\mathcal{I}}}\right) \otimes \underset{\alpha}{\boldsymbol{\mathcal{I}}} + \sum_\alpha \underset{\alpha}{\boldsymbol{\mathcal{I}}} \otimes \nabla^g_X \underset{\alpha}{\boldsymbol{\mathcal{I}}} = 0$$

for any vector field X tangent to M^{4n}.

3. On $(M^{4n}, \boldsymbol{\mathcal{S}}, g)$ a vector field ξ is an infinitesimal homography if the Lie derivative of field $\boldsymbol{\mathcal{B}}$ in direction ξ vanishes. Therefore, locally, equation $\boldsymbol{\mathcal{L}}_\xi \boldsymbol{\mathcal{B}} = 0$ becomes

$$\sum_\alpha \boldsymbol{\mathcal{L}}_\xi \underset{\alpha}{\boldsymbol{\mathcal{I}}} \otimes \underset{\alpha}{\boldsymbol{\mathcal{I}}} + \sum_\alpha \underset{\alpha}{\boldsymbol{\mathcal{I}}} \otimes \boldsymbol{\mathcal{L}}_\xi \underset{\alpha}{\boldsymbol{\mathcal{I}}} = 0\,.$$

From the formula

$$\boldsymbol{\mathcal{L}}_\xi \boldsymbol{\mathcal{I}} = \nabla^g_\xi \boldsymbol{\mathcal{I}} - [\nabla^g \xi, \boldsymbol{\mathcal{I}}]$$

which expresses the Lie derivative in terms of covariant derivation, one gets, for an infinitesimal homography ξ,

$$\sum_\alpha \left[\nabla^g \xi, \underset{\alpha}{\boldsymbol{\mathcal{I}}}\right] \otimes \underset{\alpha}{\boldsymbol{\mathcal{I}}} + \sum_\alpha \underset{\alpha}{\boldsymbol{\mathcal{I}}} \otimes \left[\nabla^g \xi, \underset{\alpha}{\boldsymbol{\mathcal{I}}}\right] = 0\,.$$

This last equation is equivalent with

$$P\left(\nabla^g \xi\right) = 0$$

where $P : \mathrm{End}\, TM^{4n} \to \mathrm{End}\, TM^{4n}$ is the projector defined locally, over U, by

$$P\,(\mathrm{l}) := \frac{3}{4}\mathrm{l} + \frac{1}{4}\sum_\alpha \underset{\alpha}{\boldsymbol{\mathcal{I}}} \circ \mathrm{l} \circ \underset{\alpha}{\boldsymbol{\mathcal{I}}} + \frac{1}{4n}\sum_\alpha (\mathrm{tr}\ \mathrm{l} \circ \underset{\alpha}{\boldsymbol{\mathcal{I}}}) \underset{\alpha}{\boldsymbol{\mathcal{I}}}\,.$$

Denoting Q the complementary projector

$$Q := \mathrm{id}_{\mathrm{End}\, TM^{4n}} - P$$

the hypothesis on $\boldsymbol{\mathcal{S}}$, with respect to g, to be settled of hermitian almost-complex structures, implies that

$$\mathrm{End}\, TM^{4n} = P\,(\mathrm{End}\, TM^{4n}) \oplus Q\,(\mathrm{End}\, TM^{4n})$$

is an orthogonal direct sum with respect to the scalar product $<,>_g$ that g induces in $TM^{4n} \otimes (TM^{4n})^*$.

Therefore

$$< P(\mathrm{l}), \mathrm{l} >_g = < P(\mathrm{l}),\, P(\mathrm{l}) >_g \ , \ \forall \mathrm{l} \in \ \mathrm{End}\, TM^{4n} \, .$$

4. Consider now a vector field ξ on M^{4n} ; it represents an infinitesimal homography iff

$$\mathrm{h}\,(\xi) := P\,(\nabla^g \xi)$$

vanishes. Define the 1-form $\xi^* \circ \mathrm{h}\,(\xi)$ through

$$\xi^* \circ \mathrm{h}\,(\xi)\,(v) := g\,(\xi, \mathrm{h}\,(\xi)\,(v))_x \, , \quad v \in (TM^{4n})_x \, .$$

One has

$$\mathrm{div}\, \xi^* \circ \mathrm{h}\,(\xi) = < \mathrm{h}\,(\xi), \mathrm{h}\,(\xi) >_g + g\,(\xi, \delta \mathrm{h}\,(\xi))$$

where

$$\delta \mathrm{h}\,(\xi)_x := \sum_{r=1}^{4n} (\nabla_{e_r} \mathrm{h}\,(\xi))\,(e_r)$$

and $\{e_1, \ldots, e_{4n}\} \subset (TM^{4n})_x$ is an orthonormal basis of $((TM^{4n})_x, g_x)$. If M^{4n} is compact one gets therefore through integration the result:

on a compact quaternionic Kähler manifold the equation $\delta h\,(\xi) = 0$ is a necessary and sufficient condition for the vector field ξ to be an infinitesimal homography, i.e. $\boldsymbol{\mathcal{L}}_\xi \boldsymbol{\mathcal{B}} = 0$.

The standard manifold which has inspired this result is the quaternionic projective space endowed with the Fubini-Study metric.

For quaternionic grassmannians analogous results are valid, see [4]. The role of the field $\boldsymbol{\mathcal{B}}$ is played now by a tensor field of the same type $(2, 2)$, denoted $S_{\mathbf{H}}$, which combines $\boldsymbol{\mathcal{B}}$ and the half symmetry

$$(u \otimes v) \otimes (u' \otimes v') \longrightarrow (u \otimes v') \otimes (u' \otimes v)$$

which characterizes the vector spaces which are real tensor-products, see [5] for a precise definition.

Infinitesimal homographies ξ are defined through the equation

$$\boldsymbol{\mathcal{L}}_\xi S_{\mathbf{H}} = 0$$

and the projector $P : \operatorname{End} TM \to \operatorname{End} TM$ which defines

$$\mathrm{h}\,(\xi) = P\,(\nabla^g \xi)$$

is gotten from the formula

$$(\dim M) \times \mathrm{h}\,(\xi) = \; C_2^2\,(S_{\mathbf{H}} \circ \boldsymbol{\mathcal{L}}_\xi S_{\mathbf{H}})$$

where C_2^2 denotes the contraction in the last indices of covariance and contravariance of the field included in parenthesis.

References

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