Quaternionic Structures
in Mathematics and Physics
SISSA - Trieste, 5-9 September 1994

NAGATA THREEFOLD AND TWISTOR SPACE

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ABSTRACT

In this note we shall give a twistorial description of a certain threefold constucted by Nagata [6] in quite a different context. We also consider its higher dimensional analogue. This leads us to a formulation of a general result on the partial compactification of certain class of hyperkähler manifolds.

1. *Nagata's Example*

Let $C = \mathbf{P} := \mathbf{C}(t) \bigcup \{\infty\}$ be the complex projective line. Consider the product $Y = C \times C \times C$ as a fiber space over C with respect to the third projection $p = p_3 : Y \to C$. Then the fiber $Y_t = p^{-1}(t)$ over $t \in C$ is naturally identified with the product $C \times C = \Sigma_0$, where Σ_d denotes in general the Hirzebruch surface of degree d. Denote by Δ_t the diagonal of Y_t and take the blowing up $\hat{Y} \to Y$ of Y with center the disjoint union $\Delta_0 \bigcup \Delta_\infty$. Let Y_0' be the strict transform of Y_0 in $\hat{Y}$ which is still isomorphic to $Y_0 \cong \Sigma_0$. Then the normal bundle $N_{Y_0'/\hat{Y}}$ of Y_0' in $\hat{Y}$ is of type $(-1, -1)$ on $Y_0 = C \times C$ so that $\hat{Y}$ can be blown down to a manifold along either of the projections $p_i : Y_0' \to C, i = 1, 2$. The similar assertion also holds true for Y_∞', the strict transform of Y_∞ in $\hat{Y}$.

In view of this, we can find a blowing down map $q : \hat{Y} \to W$ of $\hat{Y}$ to a manifold W which is isomorphic outside $Y_0' \bigcup Y_\infty'$ and whose restriction to Y_0' (resp. Y_∞') is isomorphic to the *first* (resp. the *second*) projection to C. Set $E_0 = q(Y_0')$ and $E_\infty = q(Y_\infty')$. We call this manifold W the *Nagata threefold*, being constructed by Nagata [6] as the first example of a non-projective compact algebraic manifold. By construction there exists a natural holomorphic projection $\rho : W \to C$ whose fibers $W_t, t \in C$, are identified as

$$\begin{aligned} W_t &\cong \Sigma_0 \qquad t \neq 0, \infty \\ &\cong \Sigma_2 \qquad t = 0, \infty. \end{aligned} \tag{1}$$

In particular, W_t, ($t = 0$ or ∞) contains a unique curve of self-intersection number -2, which in fact is nothing but the curve E_t above. The fact that W is nonprojective is easily verified by the following implications;

$$E_0 \sim f_2 - f_1, E_\infty \sim f_1 - f_2 \implies E_0 + E_\infty \sim 0, \tag{2}$$

where f_i are general fibers of the i-th projections $p_i : W_t \to C, t \neq 0, \infty$ and $\sim$ denotes the homological equivalence.

We also call attenton to a distinguished divisor S in W which is by definition the strict transform in W of $\Delta \times C$ ($\subseteq Y$) and which by construction is still isomorphic to it. In particular, we have $S \cong C \times C$ with $\rho|S$ corresponding to the second projection. Moreover, the normal bundle $N_{S/W}$ of S in W is of type $(2, -2)$. So W can be contracted to a normal complex space $\bar{W}$ along the first projection $S \to C$. We have thus a diagram:

$$\begin{array}{ccc} W & \xrightarrow{u} & \bar{W} \\ \rho \downarrow & & \\ C & & \end{array} \tag{3}$$

where $\bar{W}$ is a compact orbifold with A_1-type singularities along $u(S) \cong C$.

2. *Twistorial Description*

Let $\mathbf{H}$ be the set of quaternions. Identify the curve C also with the subset

$$\{q \in \mathbf{H}; q^2 = -1\} = \{ai + bj + ck; a^2 + b^2 + c^2 = 1\} \cong S^2$$

in such a way that 0 and ∞ correspond to i and $-i$ respectively. Let $S_0 = S \bigcap W_0$. Then its self-intersection number in W_0 equals 2 and $S_0 \bigcap E_0 = \emptyset$. Thus the complement

$V := W_0 - S_0$ is identified with the total space of the cotangent bundle T^* of C, and hence, as a differentiable manifold it carries a natural hyperkähler metric (Eguchi-Hanson metric), which then gives rise to the associated twistor diagram

$$\begin{array}{ccc} & Z(V) & \xrightarrow{t} & V \\ f & \downarrow & & \\ & C & & \end{array} \tag{4}$$

where t is the (C^∞) twistor fibration and f is a smooth holomorphic map. Here, $t \times f : Z(V) \to V \times C$ gives a C^∞ trivialization of the holomorphic fibration f, and f thus gives a family $\{Z_q\}_{q\in C}$ of complex manifolds Z_q whose underlying C^∞ manifold is V. (We call this family, or f itself, the *Calabi family* of V.) Furthermore, after a suitable normalization we may assume that $Z_{\pm i} \cong T^*$ holomorphically. Now by Kronheimer [4] the above diagram is naturally compactified to the following diagram:

$$\begin{array}{ccccc} & W(\bar V) & \xrightarrow{v} & Z(\bar V) & \xrightarrow{\bar t} & \bar V \\ \bar f & \downarrow & & & & \\ & C & & & & \end{array} \tag{5}$$

where

1. $\bar V = V \bigcup \{*\}$ is a one-point conformal anti-self-dual orbifold compactification of V,
2. $Z(\bar V) = Z(V) \bigcup \ell_\infty$, where $\ell_\infty = \bar t^{-1}(*)$, is the twistor space of the anti-self dual orbifold V^* with $\bar t$ the twistor fibration. $Z(\bar V)$ is a complex orbifold with A_1-type singularities along $\ell_\infty \cong \mathbf{P}$,
3. v is the blowing up with center ℓ_∞ and
4. $W(\bar V)$ is a complex manifold, and f is a smooth holomorphic map.

Now our first theorem compares the two diagrams (3) and (5):

Theorem 1. *Let the notations and the assumptions be as above. There exists a natural isomorphism between the following two diagrams*

$$\begin{array}{ccccccccc} & W(\bar V) & \xrightarrow{v} & Z(\bar V) & & & W & \xrightarrow{u} & \bar W \\ \bar f & \downarrow & & & \cong & \rho & \downarrow & & \\ & C & & & & & C & & \end{array} \tag{6}$$

which induces an isomorphism $Z(V) \cong W - S$ of open subsets $Z(V)$ of $W(\bar V)$ and $W - S$ of W respectively. In particular, the Nagata threefold is a natural compactification of the twistor space of Eguchi-Hanson metric.

Remark. Let E be the zero section of $V \cong T^*$ (C^∞ isomorphism). Then under the identification $V = Z_{\pm i}(\cong T^*)$, E becomes a complex submanifold $E_{\pm i}$ of $Z_{\pm i}$ but with the opposite orientations since E is of odd complex dimension. Thus in the three manifold $Z \cong V \times C$ (C^∞ isomorphism) we have the homological equivalence $E_i + E_{-i} = E \times \{i\} + E \times \{-i\} \sim 0$. This is the twistor version of the argument (2).

The proof of the theorem is accomplished by specifying a number of properties of a fiber space $h : X \to C$ in general, where X is a compact complex manifold of dimension three, and then showing that these properties actually characterize both the fiber spaces $\bar{f} : W(\bar{V}) \to C$ and $\rho : W \to C$. The properties we consider are as follows.

1. The fibers of h are given by the formula (1).

2. For $t = 0$ and ∞, h gives a Kuranishi family of X_t.

3. There exists a smooth divisor S in X which is isomorphic to $C \times C$ such that $h|S : S \to C$ is identified with the first projection. Moreover, for $t = 0$ and ∞, S_t is a section of the natural projection $X_t \to C$ and its self-intersection number in X_t equals two.

4. X is not projectively embeddable.

Note that these conditions are not irredundant. For instance, if in the Nagata's construction we blow up Y more generally with center the union $\Delta_0 \bigcup g(\Delta_\infty)$ for any $g \in SL(2, \mathbf{C})$, we end up with a holomorphic family of "generalized Nagata threefolds" $\{Y_g\}_{g \in SL(2,\mathbf{C})}$ parametrized by $SL(2, \mathbf{C})$, (which turns out to give a Kuranishi family of $Y = Y_e$ at $g = e$ (unit element)), where $SL(2, \mathbf{C})$ operates on $Y_\infty = C \times C$ ($\supseteq \Delta$) by the trivial action on the first factor and by the natural action on the second factor. We may distinguish Y_e and $Y_g, g \neq e$, only through the condition 3).

3. *Higher Dimensional Generalization*

The construction of Nagata above can be generalized straightforwardly to higher dimensions as follows. Let Y be the product $Y = \mathbf{P}^n \times \mathbf{P}^n \times C$, where $\mathbf{P}^n$ is the complex projective space of dimension n. Let $p : Y = \mathbf{P}^n \times \mathbf{P}^n \times C \to C$ be the natural projection. We fix a smooth divisor Δ of bidegree $(1, 1)$ in $Y_t = \mathbf{P}^n \times \mathbf{P}^n$ independently of t. Then as in the three dimensional case, we blow up Y with center $\Delta_0 \bigcup \Delta_\infty$ and then blow down suitably, ending up with a fiber space $\rho : W \to C$ with

$$\begin{array}{rcll} W_t & \cong & \mathbf{P}^n \times \mathbf{P}^n & t \neq 0, \infty \\ & \cong & \mathbf{P}(T_n^* \oplus 1) & t = 0, \infty, \end{array} \tag{7}$$

where T_n^* is the cotangent space of $\mathbf{P}^n$ (In general for a vector bundle E, $\mathbf{P}(E)$ denotes the associated bundle of projective spaces.) Moreover, the strict transform S in W of $\Delta \times C$ ($\subseteq Y$), which is still isomorphic to $\Delta \times C$, can be contracted in W along the projection $S \to \Delta$. Let $u : W \to \bar{W}$ be the blowing down map:

$$\begin{array}{ccc} & W & \overset{u}{\to} & \bar{W} \\ \rho & \downarrow & & \\ & C & & \end{array} \tag{8}$$

where $\bar{W}$ is an orbifold with A_1-type singularities along $u(S) \cong \Delta$. We have $S_0 \cong \mathbf{P}(T_n^*)$ and as before $V := W_0 - S_0$ is diffeomorphic to the total space of the cotangent bundle

T_n^* of $\mathbf{P}^n$. On the other hand, we have a natural hyperkähler structure on T_n^* as a C^∞ manifold (cf. [2][1]), and hence, get the associated twistor diagram

$$\begin{array}{cccc} & Z(V) & \stackrel{t}{\to} & V \\ f & \downarrow & & \\ & C & & \end{array} \tag{9}$$

similar to (4) with $Z(V)_i \cong T_n^*$ holomorphically. Then, analogous to Theorem 1 we get the following:

Theorem 2. *The diagram* (9) *is naturally compactified to a diagram:*

$$\begin{array}{cccccc} & W(\bar{V}) & \stackrel{v}{\to} & Z(\bar{V}) & \stackrel{\bar{t}}{\to} & \bar{V} \\ \bar{f} & \downarrow & & & & \\ & C & & & & \end{array} \tag{10}$$

where

1. *$\bar{V} = V \bigcup R$ is a compactification of V as a quaternionic orbifold (in the sense of Salamon* [8]*),*
2. *R is the Grassmann manifold $Gr(n, 2)$ of complex 2-planes in $\mathbf{C}^n$, considered naturally as a quaternionic Kähler manifold,*
3. *$Z(\bar{V}) = Z(V) \bigcup D$ is the twistor space of the quaternionic orbifold $\bar{V}$ with the associated twistor fibration $\bar{t} : Z(\bar{V}) \to \bar{V}$,*
4. *D is isomorphic to $\mathbf{P}(T_n^*)$, and $Z(\bar{V})$ is a compactification of $Z(V)$ as a complex orbifold which admits singularities of type A_1 along D,*
5. *$\bar{t}|D : D \to R$ is identified with the twistor fibration $\mathbf{P}(T_n^*) \to Gr(n, 2)$ of the quaternionic Kähler manifold $Gr(n, 2)$,*
6. *v is the blowing up of $Z(\bar{V})$ with center D, and*
7. *$W(\bar{V})$ is a smooth compactification of $Z(V)$ and $\bar{f}$ is a smooth holomorphic map.*

Moreover, there exists an isomorphism between the following two diagrams:

$$\begin{array}{cccc} & W(\bar{V}) & \stackrel{v}{\to} & Z(\bar{V}) \\ \bar{f} & \downarrow & & \\ & C & & \end{array} \quad \cong \quad \begin{array}{cccc} & W & \stackrel{u}{\to} & \bar{W} \\ \rho & \downarrow & & \\ & C & & \end{array} \tag{11}$$

which induces an isomorphism $Z(V) \cong W - S$ of open subsets $Z(V)$ of $W(\bar{V})$ and $W - S$ of W respectively.

Remark. If, the other way round, one starts from the hyperkähler viewpoint, then we may start with any rational homogeneous space X instead of $\mathbf{P}^n$; e.g. the Grassmann manifold $Gr(n, r), 0 < r < n$. In fact, then on the total space V of the cotangent bundle T_X^* there exists a natural hyperkähler structure (e.g. [1]) and we have the associated

Calabi family $f : Z(V) \to C$ with $Z(V)_i \cong T_X^*$ as above. But there is no corresponding Nagata's construction in general. More precisely, although each $Z(V)_q$ admits a natural compactification $\bar{Z}(V)_q := Z(V)_q \bigcup \mathbf{P}(T_X^*)$, we can show that they cannot form a global family over C unless $X = \mathbf{P}^n$. So the phenomenon in the above theorem is a very special one.

Compared with Theorem 1, a new ingredient of Theorem 2 is that here the method of Kronheimer mentioned before Theorem 1 does not apply. In fact, in obtaining the compactification $\bar{V}$ in Theorem 1 the fact that V is an ALE (asymptotically locally Euclidian) manifold was essential, while in the case $n > 1$ V is definitely not ALE (even topologically).

Although for the part which does not involve $\bar{V}$, we can use a complex analytic method to obtain a compactified diagram on the left hand side of (11), and may then proceed as in the proof of Theorem 1, in order to get the whole diagram (10) involving the compactification $\bar{V}$, we have to rely on the construction of V as the hyperkähler quotient for the action of the circle group S^1 on the right $\mathbf{H}$-vector space $\mathbf{H}^n$, where $t \in S^1$ acts on $\mathbf{H}^n$ by the left multiplications; $(q_1, \dots, q_n) \to (tq_1, \dots, tq_n)$. This action clearly extends to the quaternionic projective space $\mathbf{PH}^n$ with respect to the natural inclusion $\mathbf{H}^n \subseteq \mathbf{PH}^n$ (quaternionic compactification), and the induced S^1-action on the complement $\mathbf{PH}^{n-1}$ gives rise to $Gr(n, 2)$ via quaternionic Kähler quotient due to Galicki and Lawson [3].

Our contention is then that the union of these two quotients V and $Gr(n, 2)$ fit nicely together to yield a global compactification of V as a quaternionic manifold as above. In fact, this can be done by the well-known explicit description of moment maps, and the argument indeed applies also to arbitrary hyperkähler quotients of a compact Lie group acting on $\mathbf{H}^n$. The latter generalization is of some interesting because the moduli spaces of certain framed anti-selfdual connections on ALE spaces are contstucted in this way [5] and the construction of natural (partial) compactification of the moduli spaces is in general an important problem.

So we formulate part of the above result in a more general framework which deals with partial compactifications of hyperkähler quotients of $\mathbf{H}^n$. For this purpose, however, it seems conceptually more tranparent to work in a bit more general setting obtained by replacing $\mathbf{H}^n$ by the class of hyperkähler manifolds considered by Swan [9].

4. *Partial Compactification of certain Hyperkähler Manifolds*

Let M be a quaternionic Kähler manifold of dimension $4n$. The holonomy group induces a natural reduction of the structure group of the frame bundle of M to the subgroup $Sp(n)Sp(1)$. Then associated to the projection $Sp(n)Sp(1) \to Sp(1)/\{\pm 1\}$ composed with the inclusion $Sp(1)/\{\pm 1\} \hookrightarrow \bar{\mathbf{H}}^* := \mathbf{H}^*/\{\pm 1\}$ we get a natural principal $\bar{\mathbf{H}}^*$-bundle $U = U(M) \to M$, where $\mathbf{H}^* = \mathbf{H} - \{0\}$. Suppose that the scalar curvature t of M is positive. Then by Swan [9] U admits a natural hyperkähler metric induced by the given quaternionic Kähler metric on M. Let

$$\begin{array}{ccc} Z(U) & \xrightarrow{t} & U \\ h \downarrow & & \\ C & & \end{array} \tag{12}$$

be the associated twistor diagram. Suppose that a compact Lie group K acts on the bundle $U \to M$ preserving the relevant structures. Then by Swan [9] there exists a natural hyperkähler moment map

$$\mu : M \to \underline{k}^* \otimes \underline{s},$$

where $\underline{k}$ is the Lie algebra of K, $\underline{k}^*$ the dual vector space and $\underline{s} = sp(1)$.

Let $\underline{c}^*$ be the subspace of $\underline{k}^*$ consisting of the fixed vectors of the coadjoint action. Then for any element b of $\underline{c}^* \otimes \underline{s}$ such that the action of K on $\mu^{-1}(b)$ is locally free, i.e. with finite stabilizer at each point, the quotient orbifold $V = V_b = \mu^{-1}(b)/K$ has a naturally induced hyperkähler structure. Thus we can speak of the associated twistor fibration

$$\begin{array}{ccc} & Z(V) & \xrightarrow{t} & V \\ f & \downarrow & & \\ & C & & \end{array} \tag{13}$$

We want to compactify a hyperkähler manifold V obtained in this way as a quaternionic orbifold $\bar{V}$, such that the associated twistor diagram gives a partial compactification of the diagram (13) above. More precisely we can show the following :

Theorem 3. *Let the notations and the assumptions be as above. then the diagram (13) is naturally partially compactified into a diagram:*

$$\begin{array}{ccccc} & W(\bar{V}) & \xrightarrow{v} & Z(\bar{V}) & \xrightarrow{\bar{t}} & \bar{V} \\ \bar{f} & \downarrow & & & & \\ & C & & & & \end{array} \tag{14}$$

where

1. *$\bar{V} = V \bigcup R$ is a partial compactification of V as a quaternionic orbifold with R a quaternionic Kähler orbifold of (quaternionic) codimension 1,*
2. *$\bar{t} : Z(\bar{V}) \to \bar{V}$ is a twistor fibration associated to the quaternionic orbifold $\bar{V}$,*
3. *$Z(\bar{V}) = Z(V) \bigcup D$ is a partial compactification of $Z(V)$ as a complex orbifold, where D is of complex codimension two in $Z(\bar{V})$ and $Z(\bar{V})$ has A_1-type singularities along D,*
4. *$\bar{t}|D : D \to R$ is a twistor fibration of the quaternionic Kähler manifold R,*
5. *v is the blowing up of $Z(\bar{V})$ with center D and*
6. *$W(\bar{V})$ is a complex orbifold which is a union of the form $W(\bar{V}) = Z(V) \bigcup (D \times C)$. Each fiber $W(\bar{V})_q$ of $\bar{f}$ for $q \in C$ is naturally identified with the union $V_q \bigcup D$; thus $Z(\bar{V})_q \to \bar{V}$ is considered as a (quaternionic) blowing down of $Z(\bar{V})_q$ to $\bar{V}$ along the twistor fibration $\bar{t}|D : D \to R$.*

Remark. Strictly speaking, the above construction does not actually cover all the compactifications by Kronheimer of ALE hyperkähler manifolds mentioned above. If K

contains the involution $(q_i) \to -(q_i)$, then the corresponding hyperkähler quotient is in the above class with $M = \mathbf{PH}^n$; otherwise, it is a double covering of the manifolds in the above class.

We end this note with the following

Question. Let V be as in Theorem 3. Then is the twistor space $Z(V)$ of V always an algebraic variety ?

If this is the case, $Z(V)$ can always be compactifed to a complete algebraic variety by a general theorem of Nagata [7], which would then be one distinguished property of the hyperkähler manifolds in the above class.

References

[1] Burns, D., Some example of twistor construction, In: *Contributions to several complex variables, in honor of W. Stoll* (eds. A. Howard and P.M. Wong), 51–67, Vieweg, Braunschweig 1986.

[2] Calabi,E., Isometric families of Kähler structures, In: The Chern Symposium 1979, Berlin-Heidelberg-New York: Springer, 1980, 23–40.

[3] Galicki, K., and Lawson, H.B., Quaternionic reduction and quaternionic orbifolds, *Math. Ann.*, **282** (1988), 1–21.

[4] Kronheimer, P.B., A Torelli-type theorem for graviational instantons, *J. Diff. Geom.*, **29** (1989), 685–697.

[5] Kronheimer, P.B. and Nakajima, H., Yang-Mills instantons on ALE gravitational instantons, *Math. Ann.* 288, (1990), 263–307.

[6] Nagata, M., Existence theorems for nonprojective complete algebraic varieties, *Ill. J. Math.*, **2** (1958) 490–498.

[7] Nagata, M., Imbedding of an abstract variety in a complete variety, J. of Math. Kyoto Univ., **2** (1962), 1–10.

[8] Salamon, S., Differential geometry of quaternionic manifolds, *Ann. Sci. Ec. Norm. Sup.*, **19** (1986), 31–55.

[9] Swan, A., Hyperkähler and quaternionic Kähler geometry, *Math. Ann.*, **289** (1991), 421–450.

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