Meeting on
"QUATERNIONIC STRUCTURES IN MATHEMATICS AND PHYSICS"
Trieste, 5-9 September 1994

FOREWORD

The meeting 'Quaternionic Structures in Mathematics and Physics' took place at the International School for Advanced Studies (SISSA), Trieste, 5–9 September 1994, under the sponsorship of its Interdisciplinary Laboratory. The purpose of the meeting was to bring together scientists from different areas of mathematics and physics working in the field of quaternionic structures. New results were presented in the theory of quaternionic manifolds, together with progress in Clifford analysis and supersymmetry.

Interest for the scientific encounter was stronger than had been expected, and by popular demand the organizers decided to invite the speakers to contribute to what has become the present volume. The aim was to present recent advances to those working in the relevant sectors of geometry, analysis and theoretical physics.

On behalf of the Organizing Committee, we wish to thank SISSA and the Interdisciplinary Laboratory for their kind hospitality and financial support, the European Mathematical Society for having helped to cover the expenses of two speakers from Eastern Europe, and the Mathematics Committee of the Consiglio Nazionale delle Ricerche for support in the preparation of a preliminary version of this volume which appeared as a SISSA preprint, ILAS/FM–6/1996. We also owe a debt of gratitude to Rosanna Sain and Rosanna de Iurco for their invaluable help, assistance and patience, and to Angelo Bardelloni for the technical preparation of the electronic version of these Proceedings.

Finally, it gives us great pleasure to thank all the participants of the meeting for their interest and enthusiasm, and not least the contributors and the referees of the present volume.

Graziano Gentili
Stefano Marchiafava
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ORGANIZING COMMITTEE: D. Alekseevsky, E. Bonan, S. Cecotti, P. Frè, G. Gentili, S. Marchiafava, M. Pontecorvo, S. Salamon, G. Tomassini.

INTRODUCTION TO THE CONTRIBUTIONS

Given the importance of complex analysis and the concept of a complex manifold, it is not surprising that a great deal of effort has been made to develop analogous theories based upon the quaternions. Advances in the theory of four-dimensional geometry is one of the factors that has led to the successful exposition and development of various types of quaternionic manifolds, all of which attempt to generalize in some way structures in dimension 4 to dimensions $4n$ where $n \geq 1$. Independent progress in analysis related to the Dirac operator has led to a greater understanding of the definition and properties of functions possessing some form of quaternionic derivative, and one is beginning to see the analytical and geometrical aspects of the subject unite to serve each other.

From the point of view of homogeneous geometry, quaternion-Kähler manifolds form a natural class to investigate. Alekseevsky did a lot to get the subject underway by describing a class of Einstein homogeneous spaces with negative scalar curvature, corresponding to certain solvable Lie algebras. The more recent discovery that these spaces play a fundamental role in the classification of supergravity theories is explained in the paper by de Wit and Van Proeyen, who discuss them in parallel with the class of special Kähler manifolds that arise in connection with Calabi-Yau 3-folds. A detailed and complete classification of Alekseevsky spaces is carried out by Cortés from a different point of view, and he shows that the symmetric examples are precisely those with non-positive sectional curvature.

Nagano studies orbits of isotropy groups in Riemannian symmetric spaces, which automatically highlights the class of compact quaternion-Kähler manifolds. His approach unifies different types of geometrical structures and constructs their twistor spaces that parametrize families of almost complex structures. An algebro-geometric description of twistor spaces of hyper-Kähler manifolds is given by Fujiki, interpreting and extending the first known example of a non-projective compact algebraic manifold. The theory of hyper-Kähler metrics, whilst arguably under-represented in these proceedings, also underlies the paper of Swann, who investigates nilpotent coadjoint orbits, quotients and invariant theory. The resulting quaternionic structures are homogeneous for the action of a complex group, and incorporate the compact complex contact spaces characterized by Lichnerowicz in the context of Poisson manifolds and moment mappings. The theory of Poisson manifolds is also the setting for the paper of Fernández, Ibáñez and de León, who discuss symplectically harmonic forms on a nilmanifold.

Quaternionic manifolds, through their associated bundles discussed in papers already mentioned, give rise to a host of other geometries, some of which are of equal, if not greater, importance. Such is the case for the class of 3-Sasakian manifolds, described by Boyer, Galicki and Mann, whose theory to some extent mirrors that of quaternion-Kähler orbifolds. Whilst 3-Sasakian manifolds are Einstein, their holonomy does not reduce and they are intriguing for this very reason. They recur in a conformally-invariant setting in the paper by Ornea and Piccinni, who characterize certain Weyl structures and evaluate Betti numbers of associated hypercomplex manifolds.

The definition of anti-self-dual connections in higher dimensions pursues the analogy with 4-dimensional geometry and theoretical physics, and their moduli constitute a more radical type of auxiliary space that remains to be fully investigated. This topic is introduced by Nagatomo, who proves a number of vanishing theorems and then gives a monad description on the complex Grassmannian. The algebra of almost complex structures is exploited by Hangan to characterize Killing vector fields preserving a quaternionic structure, and exterior algebra associated to the fundamental 4-form is the subject of the paper by Bonan. Differential forms and generalizations of self-duality are also used in the paper by Fernández and Ugarte on the de Rham cohomology of 7-manifolds with a structure determined by the exceptional group G_2.

Clifford algebras and the Dirac operator form the bridge between the geometry and analysis represented by the contributions to the meeting's proceedings. Clifford algebras crop up in supergravity and other topics mentioned above, and the determination of the eigenvalues of the Dirac operator over compact manifolds with reduced holonomy is an active problem, addressed by Hijazi. The related concept of Killing spinor also occurs in the characterization of twistor spaces given by Moroianu and Semmelmann. On the other hand, various types of monogenic functions are defined as solutions of a suitable Dirac operator, and in dimension 4 correspond to the quaternionic holomorphic functions introduced by Fueter and described in Sudbery's well-known paper.

Conformal invariance of the Dirac operator is the starting point for Ryan, whose paper introduces Clifford analysis and the theory of monogenic functions. He proceeds to define a Fourier transform for L^2 functions over spheres with values in a complex Clifford algebra, and considers applications involving the resulting convolution and projection operators. Integral formulas for the Dirac operator form the basis of the paper by Cnops, who defines the Dirac operator in terms of Stokes' formula, and also considers spherical representations. Such analysis is an appropriate setting for quaternionic function theory, though the latter has a special flavour. Pernas adopts a direct approach to quaternionic holomorphic functions, defining them in the setting of hypercomplex manifolds, and then applying them to a study of hyperbolic space. An alternative definition of quaternionic derivatives is presented by Lugojan with useful references and examples.

Simon Salamon