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CANONICAL COHOMOLOGY OF COMPACT G_2-NILMANIFOLDS *

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ABSTRACT

In this paper we consider the *canonical complex* of a G_2-manifold [R1, R2]. An example of a compact cocalibrated G_2-nilmanifold whose de Rham cohomology groups are not isomorphic to the *canonical* cohomology groups is constructed.

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1. Introduction

G_2-manifolds are seven-dimensional Riemannian manifolds with a 2-fold vector cross product P, which can be considered as a natural generalization of an almost Hermitian structure. Corresponding to the Kähler form one has a *fundamental* 3-*form* φ.

Within the class of all G_2-manifolds one can find the G_2 analogues of the different types of almost Hermitian manifolds. For example, if φ is *coclosed* then M is a *cocalibrated* G_2-manifold [HL1, HL2], a G_2 analogue of a cosymplectic (or semi-Kähler) manifold; if $d * \varphi = \theta \wedge *\varphi$, where $*$ is the Hodge star operator of M and θ is the differential 1-form on M given by $\theta = \frac{1}{3} * (*d * \varphi \wedge *\varphi)$, then M is an *integrable* G_2-manifold [FU], a G_2 analogue of a complex manifold as we explain in Section 2. In particular, if φ is *closed* and *coclosed* then M has a subgroup of G_2 as holonomy group [G2], and M is a G_2 analogue of a Kähler manifold.

For a manifold with a G-structure, the theory of infinitesimal deformations of generalized instantons has been formulated by R. Reyes in [R1, R2] in terms of a *canonical complex*. Recently the authors of this paper have studied in [FU] the canonical cohomology groups for integrable G_2-manifolds and, in particular, for cocalibrated G_2-manifolds.

For a compact Kähler manifold L it is well-known the Hodge theorem: there is an isomorphism

$$H^q(L) \cong \bigoplus_{r+s=q} H^{r,s}(L), \tag{1}$$

where $H^q(L)$ denotes the q-th de Rham cohomology group of L, and $H^{r,s}(L)$ are the Dolbeault groups. In [FU] the corresponding isomorphism to (1) is proved for compact G_2-manifolds whose holonomy group is a subgroup of G_2 (see Section 2, Theorem 4). Our main purpose here is to show examples of compact integrable G_2-manifolds for which such isomorphism fails (see Section 3, Theorem 8 and Theorem 10). According to the classification of G_2-manifolds given in [FG], a manifold with Hol $\subseteq G_2$ can be defined as a nearly parallel G_2-manifold in the class $\mathcal{W}_3$ (i.e. the class of G_2-manifolds for which the fundamental 3-form satisfies $d * \varphi = 0$ and $d\varphi \wedge \varphi = 0$). In Theorem 8 we prove that the Hodge type theorem does not hold for arbitrary $\mathcal{W}_3$-manifolds, whereas Theorem 10 shows that such result can also fail for compact nearly parallel G_2-manifolds.

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2. G_2-manifolds and canonical cohomology

Let M be a C^∞ Riemannian manifold of dimension 7 with metric $\langle\ ,\ \rangle$. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M and by $\mathfrak{F}(M)$ the algebra of C^∞ functions on M.

A 2-*fold vector cross product* on M is a tensor field $P\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ satisfying the axioms:

i) $\langle P(X,Y), X\rangle = \langle P(X,Y), Y\rangle = 0,$

ii) $\|P(X,Y)\|^2 = \|X\|^2\|Y\|^2 - \langle X, Y\rangle^2,$

for $X, Y \in \mathfrak{X}(M)$.

A seven dimensional Riemannian manifold M with a 2-fold vector cross product P is called *G_2-manifold*. There is a representation of G_2 on each tangent space of M defined by means of the vector cross product P [BG, G2, G4, S].

To describe the geometry of a G_2-manifold M it is very useful to consider the differential 3-form φ on M given by

$$\varphi(X,Y,Z) = \langle P(X,Y), Z\rangle\,,$$

for $X, Y, Z \in \mathfrak{X}(M)$. This form φ, originally defined by Bonan [Bon], is called the *fundamental* 3*-form* of M.

Let $\Lambda^q(M)$ denote the space of differential q-forms on M, and let $\langle\ ,\ \rangle$ the inner product on $\Lambda^q(M)$ given by

$$\langle \alpha, \beta\rangle\, \Omega_M = \alpha \wedge *\beta, \tag{2}$$

for $\alpha, \beta \in \Lambda^q(M)$, where Ω_M denotes the volume form on M. In [FG] the first author of this paper and Gray proved that $\Lambda^q(M)$ splits orthogonally into components $\Lambda^q_l(M)$ of dimension l, which are irreducible under the action of G_2. The representation of G_2 on $\Lambda^1(M)$ is the irreducible 7-dimensional representation, and the representations of G_2 on $\Lambda^q(M)$ and $\Lambda^{7-q}(M)$ are the same because the Hodge star operator $*\colon \Lambda^q(M) \longrightarrow \Lambda^{7-q}(M)$ is an isometry. Therefore, it suffices to describe the representations of G_2 on $\Lambda^2(M)$ and $\Lambda^3(M)$. They are (see [Br, FG, S])

$$\begin{aligned}
\Lambda^2_7(M) &= \{*(\alpha \wedge *\varphi) \mid \alpha \in \Lambda^1(M)\},\\
\Lambda^2_{14}(M) &= \{\beta \in \Lambda^2(M) \mid \beta \wedge *\varphi = 0\},\\
\Lambda^3_1(M) &= \{f\varphi \mid f \in \mathfrak{F}(M)\},\\
\Lambda^3_7(M) &= \{*(\alpha \wedge \varphi) \mid \alpha \in \Lambda^1(M)\},\\
\Lambda^3_{27}(M) &= \{\gamma \in \Lambda^3(M) \mid \gamma \wedge \varphi = \gamma \wedge *\varphi = 0\}.
\end{aligned} \tag{3}$$

Now, from (2) and (3), it is easy to get

$$\Lambda^3_7(M) \oplus \Lambda^3_{27}(M) = \{\gamma \in \Lambda^3(M) \mid \gamma \wedge *\varphi = 0\}. \tag{4}$$

Recall [Br, C, FG, S] that the G_2-manifold M is said to be *parallel* if $\nabla\varphi = 0$ (or equivalently, $d\varphi = d*\varphi = 0$); *nearly parallel* if $d*\varphi = 0$ and $d\varphi = k*\varphi$, where $k \in \mathbb{R}$; *$\mathcal{W}_3$-manifold* if $d*\varphi = 0$ and $d\varphi \wedge \varphi = 0$; *cocalibrated* (*semi-parallel* or *cosymplectic*) if $d*\varphi = 0$; *integrable* if $d*\varphi = \theta \wedge *\varphi$, where θ is the differential 1-form on M given by $\theta = \frac{1}{3} * (*d*\varphi \wedge *\varphi)$.

Now, let us suppose that M is an *integrable* G_2-manifold. It is easy to see that $d\beta \wedge *\varphi = 0$, for $\beta \in \Lambda^2_{14}(M)$. Then, from (4) it follows the existence of the differential complex

$$\begin{aligned}
0 \longrightarrow\ \ &\Lambda^2_{14}(M) \xrightarrow{\hat{D}} \Lambda^3_7(M) \oplus \Lambda^3_{27}(M) \xrightarrow{\hat{D}}\\
&\Lambda^4(M) \xrightarrow{\hat{D}=d} \Lambda^5(M) \xrightarrow{\hat{D}=d} \Lambda^6(M) \xrightarrow{\hat{D}=d} \Lambda^7(M) \to 0,
\end{aligned} \tag{5}$$

where $\hat{D}$ is the restriction to $\Lambda^2_{14}(M)$ and $\Lambda^3_7(M) \oplus \Lambda^3_{27}(M)$ of the exterior derivative d of M, that is, (5) is a *differential subcomplex* of the de Rham complex. We shall denote by $\hat{H}{}^*(M)$ the cohomology of this complex; moreover, if α is a differential q-form on M such that $\hat{D}\alpha = 0$, we shall denote by $\{\alpha\}$ the cohomology class defined by α in $\hat{H}{}^q(M)$.

Next, we recall briefly some results about the canonical cohomology [R1, R2, S]. We also refer to [FU], where a more detailed and complete development may be found.

Let M be a G_2-manifold with fundamental 3-form φ, and denote by $\pi_7: \Lambda^2(M) \longrightarrow \Lambda^2_7(M)$ and $\pi_1: \Lambda^3(M) \longrightarrow \Lambda^3_1(M)$ the orthogonal projections of $\Lambda^2(M)$ and $\Lambda^3(M)$ onto $\Lambda^2_7(M)$ and $\Lambda^3_1(M)$, respectively. The maps $\overset{\vee}{D}_0: \Lambda^0(M) \longrightarrow \Lambda^1(M)$, $\overset{\vee}{D}_1: \Lambda^1(M) \longrightarrow \Lambda^2_7(M)$ and $\overset{\vee}{D}_2: \Lambda^2_7(M) \longrightarrow \Lambda^3_1(M)$ are defined by

$$\overset{\vee}{D}_0 = d, \tag{6}$$

$$\overset{\vee}{D}_1(\alpha) = \pi_7(d\alpha), \tag{7}$$

for $\alpha \in \Lambda^1(M)$, and

$$\overset{\vee}{D}_2(\beta) = \pi_1(d\beta), \tag{8}$$

for $\beta \in \Lambda^2_7(M)$.

It will be convenient to make no distinction to denote the maps $\overset{\vee}{D}_0$, $\overset{\vee}{D}_1$ and $\overset{\vee}{D}_2$. In fact, unless clarity is required, we shall write $\overset{\vee}{D}$ for these maps.

The following proposition is proved in [R1], [R2], [S] and [FU].

Proposition 1. *Let M be an integrable G_2-manifold. Then*

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{\overset{\vee}{D}=d} \Lambda^1(M) \xrightarrow{\overset{\vee}{D}} \Lambda^2_7(M) \xrightarrow{\overset{\vee}{D}} \Lambda^3_1(M) \longrightarrow 0 \tag{9}$$

is a differential complex.

Now, we stablish the notion of canonical cohomology.

Definition. Let M be an integrable G_2-manifold with fundamental 3-form φ. The complex (9) is called *canonical complex* of M. (It can be considered as the "Dolbeault complex" for G_2-manifolds.)

We denote by $\overset{\vee}{H}{}^*(M)$ the cohomology of this complex, and we call it *canonical cohomology* of M. For a differential q-form α such that $\overset{\vee}{D}\alpha = 0$ we shall denote by $[\alpha]$ the cohomology class in $\overset{\vee}{H}{}^q(M)$ defined by α.

From now on, M is an integrable G_2-manifold, and $H^*(M)$ denotes the de Rham cohomology of M. For $\alpha \in \Lambda^q(M)$ such that $d\alpha = 0$ we shall denote by $[\alpha]_{dR}$ the cohomology class in $H^q(M)$ defined by α.

Now, as a consequence of Proposition 1 we have that $\overset{\vee}{H}{}^0(M) = H^0(M)$ and the homomorphism $\mu: H^1(M) \longrightarrow \overset{\vee}{H}{}^1(M)$ given by

$$\mu([\alpha]_{dR}) = [\alpha], \tag{10}$$

for $[\alpha]_{dR} \in H^1(M)$, is injective.

Since the differential complexes defined by (5) and (9) are elliptic (see [R1, R2]), the cohomology groups $\overset{\vee}{H}{}^q(M)$ and $\hat{H}{}^q(M)$ have dimension $< \infty$ for every compact integrable G_2-manifold M. The following results [FU] show special properties of the canonical cohomology groups of particular classes of integrable G_2-manifolds.

Theorem 2. *Let M be a compact cocalibrated G_2-manifold. Then*

i) $\overset{\vee}{H}{}^1(M) \cong \overset{\vee}{H}{}^2(M)$,

ii) $\overset{\vee}{H}{}^3(M) \cong H^0(M)$.

Theorem 3. *Let M be a compact nearly parallel G_2-manifold.*

i) The homomorphism $\mu\colon H^1(M) \longrightarrow \overset{\vee}{H}{}^1(M)$ given by (10) *is an isomorphism. Therefore, $\overset{\vee}{H}{}^1(M) \cong H^1(M)$.*

ii) Furthermore, if M is nearly parallel non-parallel G_2-manifold, we have

$$\overset{\vee}{H}{}^1(M) \cong H^1(M) = \{0\} \qquad \text{and} \qquad H^2(M) \cong \hat{H}{}^2(M).$$

We finish this section with a Hodge type theorem for compact G_2-manifolds whose holonomy group is a subgroup of G_2. Its proof is given in [FU].

Theorem 4. *For a compact parallel G_2-manifold M we have*

$$\begin{aligned} H^1(M) &\cong \overset{\vee}{H}{}^1(M), \\ H^2(M) &\cong \overset{\vee}{H}{}^2(M) \oplus \hat{H}{}^2(M), \\ H^3(M) &\cong \overset{\vee}{H}{}^3(M) \oplus \hat{H}{}^3(M), \\ H^4(M) &\cong \hat{H}{}^4(M). \end{aligned} \tag{11}$$

Corollary 5. *For any compact parallel G_2-manifold M the Betti numbers $b_q(M)$ satisfy*

$$\begin{aligned} b_2(M) &\geq b_1(M), \\ b_3(M) &\geq b_1(M) + b_0(M). \end{aligned}$$

3. Compact integrable G_2-nilmanifolds

The main problem to construct counterexamples to Theorem 4 is the difficulty to compute the cohomology groups $\overset{\vee}{H}{}^q(M)$, $H^q(M)$ and $\hat{H}{}^q(M)$ of an integrable G_2-manifold M.

For a compact nilmanifold $M = \Gamma\backslash K$ it is well-known Nomizu's theorem [N], which asserts that the cohomology $H^*(\mathfrak{K}^*)$ of the complex $\Lambda^*(\mathfrak{K}^*)$ of left invariant differential forms on the nilpotent Lie group K ($\mathfrak{K}$ being the Lie algebra of K) is isomorphic to the de Rham cohomology $H^*(M)$.

For the authors is still unknown if there is a similar result for the canonical cohomology of an integrable G_2-nilmanifold. In [FU] they have proved some relations among the dimensions of the groups $H^q(M)$, $\overset{\vee}{H}{}^q(M)\oplus\hat{H}{}^q(M)$ and $\overset{\vee}{H}{}^q(\mathfrak{K}^*)\oplus\hat{H}{}^q(\mathfrak{K}^*)$, for any compact integrable G_2-nilmanifold $M=\Gamma\backslash K$. We explain below such relations, and we shall use them to show that (11) fails for compact integrable non-parallel G_2-nilmanifolds.

Let $M=\Gamma\backslash K$ be a compact nilmanifold of dimension 7, that is, K is a connected and simply-connected nilpotent Lie group of dimension 7, and Γ is a discrete subgroup of K such that the quotient $\Gamma\backslash K$ is a compact space.

Now, let us suppose that K is a G_2-manifold with a left invariant metric $\langle\ ,\ \rangle$, and a left invariant 2-fold vector cross product P. Denote by φ the left invariant fundamental 3-form on K. Then $\langle\ ,\ \rangle$, P and φ descend to a metric, a 2-fold vector cross product and a 3-form on M, respectively, for which we shall use the same notation, that is, $\langle\ ,\ \rangle$, P and φ, respectively. Therefore, M is a G_2-nilmanifold with vector cross product P and fundamental 3-form φ. Moreover, M is integrable, cocalibrated or, in particular, $\mathcal{W}_3$-manifold if and only if K is integrable, cocalibrated or $\mathcal{W}_3$-manifold, respectively.

Denote by $\mathfrak{K}$ the Lie algebra of K. Let

$$\cdots\longrightarrow\Lambda^{q-1}(\mathfrak{K}^*)\xrightarrow{d}\Lambda^q(\mathfrak{K}^*)\xrightarrow{d}\Lambda^{q+1}(\mathfrak{K}^*)\longrightarrow\cdots$$

be the differential complex, where $\Lambda^q(\mathfrak{K}^*)$ denotes the left invariant differential q-forms on K. The Chevalley-Eilenberg cohomology is defined by

$$H^q(\mathfrak{K}^*)=\frac{\operatorname{Ker}\{d\colon\Lambda^q(\mathfrak{K}^*)\longrightarrow\Lambda^{q+1}(\mathfrak{K}^*)\}}{\operatorname{Im}\{d\colon\Lambda^{q-1}(\mathfrak{K}^*)\longrightarrow\Lambda^q(\mathfrak{K}^*)\}}.$$

On the other hand, we introduce the differential complexes

$$0\longrightarrow\mathbb{R}\xrightarrow{\overset{\vee}{D}=0}\Lambda^1(\mathfrak{K}^*)\xrightarrow{\overset{\vee}{D}}\Lambda^2_7(\mathfrak{K}^*)\xrightarrow{\overset{\vee}{D}}\Lambda^3_1(\mathfrak{K}^*)\longrightarrow 0,\tag{12}$$

and

$$\begin{aligned}0\longrightarrow\ &\Lambda^2_{14}(\mathfrak{K}^*)\xrightarrow{\hat{D}}\Lambda^3_7(\mathfrak{K}^*)\oplus\Lambda^3_{27}(\mathfrak{K}^*)\xrightarrow{\hat{D}}\\ &\Lambda^4(\mathfrak{K}^*)\xrightarrow{\hat{D}=d}\Lambda^5(\mathfrak{K}^*)\xrightarrow{\hat{D}=d}\Lambda^6(\mathfrak{K}^*)\xrightarrow{\hat{D}=d}\Lambda^7(\mathfrak{K}^*)\to 0,\end{aligned}\tag{13}$$

where the spaces $\Lambda^q_l(\mathfrak{K}^*)$ are defined by similar relations to (3). (For example, $\Lambda^2_7(\mathfrak{K}^*)=\{*(\alpha\wedge*\varphi)\mid\alpha\in\Lambda^1(\mathfrak{K}^*)\}$.) The complexes (12) and (13) are differential subcomplexes of (9) and (5), respectively.

We denote by $\hat{H}{}^*(\mathfrak{K}^*)$ the cohomology of the complex (13), and by $\overset{\vee}{H}{}^*(\mathfrak{K}^*)$ the cohomology of the complex (12). Let $\hat{h}_q(\ )=\dim\hat{H}{}^q(\ )$ and $\overset{\vee}{h}_q(\ )=\dim\overset{\vee}{H}{}^q(\ )$; the following theorem is proved in [FU].

Theorem 6. *Let $M=\Gamma\backslash K$ be a compact integrable G_2-nilmanifold with fundamental 3-form φ, which arises from a left invariant fundamental 3-form φ on K. Then*

$$b_1(M)\le\overset{\vee}{h}_1(\mathfrak{K}^*)\le\overset{\vee}{h}_1(M).\tag{14}$$

Moreover, if M is cocalibrated, we have

$$b_2(M) \leq \overset{\vee}{h}_2(\mathfrak{K}^*) + \hat{h}_2(\mathfrak{K}^*) \leq \overset{\vee}{h}_2(M) + \hat{h}_2(M). \tag{15}$$

In order to construct a compact integrable G_2-nilmanifold for which (11) is not satisfied, we proceed as follows.

Consider the Lie group $K = H(2,1) \times \mathbb{R}^2$, where $H(2,1)$ is the generalized Heisenberg group [Ha] of dimension 5 consisting of the matrices of the form

$$\mathbf{a} = \begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $a_i, b_i, c \in \mathbb{R}$, $1 \leq i \leq 2$. A global system of coordinates $\{x_1, x_2, y_1, y_2, z\}$ on $H(2,1)$ is defined by

$$x_i(\mathbf{a}) = a_i, \qquad y_i(\mathbf{a}) = b_i, \qquad z(\mathbf{a}) = c,$$

for $1 \leq i \leq 2$. Denote by $\{t_1, t_2\}$ the natural coordinates on $\mathbb{R}^2$. Then it is easy to verify that $\{dx_1, dx_2, dy_1, dy_2, dz - x_1 dy_1 - x_2 dy_2, dt_1, dt_2\}$ is a basis for the left invariant 1-forms on $K = H(2,1) \times \mathbb{R}^2$.

Now, to get a compact quotient, we take $M = \Gamma\backslash K$, where Γ is the uniform subgroup of K given by $\Gamma = \Gamma(2,1) \times \mathbb{Z}^2$, $\Gamma(2,1)$ being the subgroup of $H(2,1)$ which consists of those matrices $\mathbf{a}$ whose entries a_1, a_2, b_1, b_2, c are integers.

The 1-forms dx_i, dy_i, $dz - x_1 dy_1 - x_2 dy_2$, dt_i all descend to M; denote the 1-forms induced on M by $\alpha_i, \beta_i, \gamma, \eta_i$ $(1 \leq i \leq 2)$, respectively. Therefore $\{\alpha_i, \beta_i, \gamma, \eta_i; 1 \leq i \leq 2\}$ is a basis for the 1-forms on M such that

$$\begin{cases} d\alpha_i = d\beta_i = d\eta_i = 0, \quad 1 \leq i \leq 2, \\ d\gamma = -\alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2. \end{cases} \tag{16}$$

Theorem 7. *There exists a (non-parallel) vector cross product on M such that the fundamental 3-form φ is coclosed and $d\varphi \wedge \varphi = 0$. Therefore, M is a cocalibrated G_2-nilmanifold and a $\mathcal{W}_3$-manifold.*

Proof. Let φ be the 3-form on M defined by

$$\begin{aligned} \varphi = \ & \beta_1 \wedge \alpha_2 \wedge \eta_1 + \alpha_2 \wedge \alpha_1 \wedge \eta_2 + \alpha_1 \wedge \eta_1 \wedge \beta_2 + \eta_1 \wedge \eta_2 \wedge \gamma \\ & + \eta_2 \wedge \beta_2 \wedge \beta_1 + \beta_2 \wedge \gamma \wedge \alpha_2 + \gamma \wedge \beta_1 \wedge \alpha_1. \end{aligned} \tag{17}$$

On the right hand side of (17) all of the terms are coclosed; and so $d * \varphi = 0$.

Define a metric on M by

$$\langle\ ,\ \rangle = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + \gamma^2 + \eta_1^2 + \eta_2^2.$$

Let $\{E_0, \ldots, E_6\}$ be the basis dual to $\{\beta_1, \alpha_2, \alpha_1, \eta_1, \eta_2, \beta_2, \gamma\}$. Then, a 2-fold vector cross product P on M is given by $P(E_i, E_j) = -P(E_j, E_i)$, and $P(E_i, E_{i+1}) = E_{i+3}$, $P(E_{i+3}, E_i) = E_{i+1}$, $P(E_{i+1}, E_{i+3}) = E_i$ $(i \in \mathbb{Z}_7)$. It is not hard to show that P satisfies the axioms for a 2-fold vector cross product, and moreover the form φ, given by (17), is the fundamental 3-form. To show that P is not parallel we prove that φ is not closed. Indeed, a calculation shows that $d\varphi = (-\alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2) \wedge \eta_1 \wedge \eta_2 \neq 0$.

Moreover, from (17) it follows that

$$\begin{aligned} d\varphi \wedge \varphi &= (-\alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2) \wedge (\beta_2 \wedge \gamma \wedge \alpha_2 + \gamma \wedge \beta_1 \wedge \alpha_1) \wedge \eta_1 \wedge \eta_2 \\ &= (-\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma + \alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma) \wedge \eta_1 \wedge \eta_2 \\ &= 0, \end{aligned}$$

which implies that M is a $\mathcal{W}_3$-manifold. ∎

Theorem 8. *For the compact $\mathcal{W}_3$-manifold M we have*

$$H^1(M) \ncong \overset{\vee}{H}{}^1(M), \tag{18}$$

$$H^2(M) \ncong \overset{\vee}{H}{}^2(M) \oplus \hat{H}{}^2(M). \tag{19}$$

Proof. It follows from (16) and Nomizu's theorem [N] that the de Rham cohomology groups $H^1(M)$ and $H^2(M)$ are

$$\begin{aligned} H^1(M) &= \{[\alpha_1]_{dR},\ [\alpha_2]_{dR},\ [\beta_1]_{dR},\ [\beta_2]_{dR},\ [\eta_1]_{dR},\ [\eta_2]_{dR}\}, \\ H^2(M) &= \{[\alpha_1 \wedge \alpha_2]_{dR},\ [\alpha_1 \wedge \beta_1]_{dR},\ [\alpha_1 \wedge \beta_2]_{dR},\ [\alpha_1 \wedge \eta_1]_{dR}, \\ &\quad [\alpha_1 \wedge \eta_2]_{dR},\ [\alpha_2 \wedge \beta_1]_{dR},\ [\alpha_2 \wedge \eta_1]_{dR},\ [\alpha_2 \wedge \eta_2]_{dR}, \\ &\quad [\beta_1 \wedge \beta_2]_{dR},\ [\beta_1 \wedge \eta_1]_{dR},\ [\beta_1 \wedge \eta_2]_{dR},\ [\beta_2 \wedge \eta_1]_{dR}, \\ &\quad [\beta_2 \wedge \eta_2]_{dR},\ [\eta_1 \wedge \eta_2]_{dR}\}.. \end{aligned}$$

Thus we have

$$b_1(M) = 6, \qquad b_2(M) = 14. \tag{20}$$

Next, because the left invariant differential forms on K all descend to differential forms on M, it will be convenient to make no distinction to denote the left invariant 1-forms $dx_i, dy_i, dz - x_1 dy_1 - x_2 dy_2, dt_i$ $(1 \leq i \leq 2)$ on K and the 1-forms $\alpha_i, \beta_i, \gamma, \eta_i$ $(1 \leq i \leq 2)$ on M, and as well as for the wedge products.

To compute $\overset{\vee}{H}{}^1(\mathfrak{K}^*)$ we see that α_i, β_i and η_i $(1 \leq i \leq 2)$ also define nonzero elements in $\overset{\vee}{H}{}^1(\mathfrak{K}^*)$. Moreover, we note that γ does not define a cohomology class in $H^1(M)$, because γ is not closed. However, by using that $*\varphi$ is given by

$$\begin{aligned} *\varphi = &-\alpha_1 \wedge \eta_2 \wedge \beta_2 \wedge \gamma + \beta_1 \wedge \eta_1 \wedge \beta_2 \wedge \gamma - \beta_1 \wedge \alpha_2 \wedge \eta_2 \wedge \gamma \\ &+\beta_1 \wedge \alpha_2 \wedge \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \alpha_1 \wedge \eta_1 \wedge \gamma - \beta_1 \wedge \alpha_1 \wedge \eta_1 \wedge \eta_2 \\ &-\alpha_2 \wedge \eta_1 \wedge \eta_2 \wedge \beta_2, \end{aligned} \tag{21}$$

one can check that $d\gamma \wedge *\varphi = 0$, and hence $d\gamma \in \Lambda^2_{14}(\mathfrak{K}^*)$, that is, $\overset{\vee}{D}\gamma = 0$ or, equivalently γ defines a nonzero cohomology class in $\overset{\vee}{H}{}^1(\mathfrak{K}^*)$. Thus

$$\overset{\vee}{H}{}^1(\mathfrak{K}^*) = \{[\alpha_1],\ [\alpha_2],\ [\beta_1],\ [\beta_2],\ [\gamma],\ [\eta_1],\ [\eta_2]\},$$

which implies

$$\overset{\vee}{h}_1(\mathfrak{K}^*) = 7. \tag{22}$$

Now, from (20), (22) and (14) it follows that

$$b_1(M) = 6 < \overset{\vee}{h}_1(\mathfrak{K}^*) = 7 \leq \overset{\vee}{h}_1(M),$$

which proves (18).

In order to prove (19) we compute a basis of the space $\Lambda^2_{14}(\mathfrak{K}^*) = \{\beta \in \Lambda^2(\mathfrak{K}^*) \mid \beta \wedge *\varphi = 0\}$. From (21) we get

$$\begin{aligned}\Lambda^2_{14}(\mathfrak{K}^*) = \{&\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1,\ \alpha_1 \wedge \beta_2 - \gamma \wedge \eta_2,\ \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2,\\ &\alpha_1 \wedge \beta_1 + \eta_1 \wedge \eta_2,\ \alpha_1 \wedge \eta_1 + \alpha_2 \wedge \gamma,\ \alpha_1 \wedge \eta_1 - \beta_1 \wedge \eta_2,\\ &\alpha_2 \wedge \eta_1 + \beta_2 \wedge \eta_2,\ \alpha_1 \wedge \gamma + \beta_2 \wedge \eta_2,\ \alpha_1 \wedge \alpha_2 - \beta_1 \wedge \beta_2,\\ &\alpha_1 \wedge \alpha_2 + \gamma \wedge \eta_1,\ \alpha_2 \wedge \eta_2 - \beta_1 \wedge \gamma,\ \alpha_2 \wedge \eta_2 - \beta_2 \wedge \eta_1,\\ &\alpha_1 \wedge \eta_2 + \beta_1 \wedge \eta_1,\ \alpha_1 \wedge \eta_2 - \beta_2 \wedge \gamma\}.\end{aligned}$$

Now, we compute $\hat{H}{}^2(\mathfrak{K}^*) = \{\beta \in \Lambda^2_{14}(\mathfrak{K}^*) \mid d\beta = 0\}$. It is given by

$$\begin{aligned}\hat{H}{}^2(\mathfrak{K}^*) = \{&\{\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1\},\ \{\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2\},\ \{\alpha_1 \wedge \beta_1 + \eta_1 \wedge \eta_2\},\\ &\{\alpha_1 \wedge \eta_1 - \beta_1 \wedge \eta_2\},\ \{\alpha_2 \wedge \eta_1 + \beta_2 \wedge \eta_2\},\ \{\alpha_1 \wedge \alpha_2 - \beta_1 \wedge \beta_2\},\\ &\{\alpha_2 \wedge \eta_2 - \beta_2 \wedge \eta_1\},\ \{\alpha_1 \wedge \eta_2 + \beta_1 \wedge \eta_1\}\}.\end{aligned}$$

Thus, we obtain

$$\hat{h}_2(\mathfrak{K}^*) = 8. \tag{23}$$

On the other hand, from Theorem 2 and Theorem 7 we have $\overset{\vee}{h}_2(\mathfrak{K}^*) = \overset{\vee}{h}_1(\mathfrak{K}^*)$. Then, from (22) we conclude

$$\overset{\vee}{h}_2(\mathfrak{K}^*) = 7. \tag{24}$$

Now, from (20), (23), (24) and (15) it follows that

$$b_2(M) = 14 < \overset{\vee}{h}_2(\mathfrak{K}^*) + \hat{h}_2(\mathfrak{K}^*) = 15 \leq \overset{\vee}{h}_2(M) + \hat{h}_2(M),$$

which completes the proof. ∎

Finally, we show an example of a compact nearly parallel G_2-manifold N for which $H^3(N) \ncong \overset{\vee}{H}{}^3(N) \oplus \hat{H}{}^3(N)$, and $H^4(N) \ncong \hat{H}{}^4(N)$. It is not a G_2-nilmanifold.

In general [G2], for an orientable hypersurface N of $\mathbb{R}^8$, with unit normal vector field U on N, there is a 2-fold vector cross product P on N given by

$$P(X, Y) = \overline{P}_{\pm}(U, X, Y), \tag{25}$$

for $X, Y \in \mathfrak{X}(N)$, where $\overline{P}_{\pm}$ denote the two non-isomorphic 3-fold vector cross products on $\mathbb{R}^8$ [Z].

Proposition 9. **[FG]** *Let N be an orientable hypersurface of $\mathbb{R}^8$ with 2-fold vector cross product P given by* (25). *Then*

i) N is cocalibrated G_2-manifold,

ii) N is parallel G_2-manifold if N is totally geodesic,

iii) N is nearly parallel G_2-manifold if N is totally umbilic (i.e. N is a part of a sphere).

Theorem 10. *Let S^7 be the 7-sphere with 2-fold vector cross product given by* (25). *Then S^7 is a nearly parallel (non-parallel) G_2-manifold with $H^3(S^7) \ncong \overset{\vee}{H}{}^3(S^7) \oplus \hat{H}{}^3(S^7)$ and $H^4(S^7) \ncong \hat{H}{}^4(S^7)$.*

Proof. From Proposition 9 it follows that S^7 is a nearly parallel (non-parallel) G_2-manifold. If $(\mathcal{B}^*(S^7), \hat{D})$ and $(\mathcal{A}^*(S^7), \overset{\vee}{D})$ denote the differential complexes (5) and (9), respectively, then the short exact sequence of complexes

$$0 \longrightarrow (\mathcal{B}^*(S^7), \hat{D}) \xrightarrow{\ i\ } (\Lambda^*(S^7), d) \xrightarrow{\ p\ } (\mathcal{A}^*(S^7), \overset{\vee}{D}) \longrightarrow 0,$$

where i and p are the natural inclusion and the orthogonal projection, respectively, induces the long exact sequence of cohomology groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(S^7) & \overset{H^1(p)}{\longrightarrow} & \overset{\vee}{H}{}^1(S^7) & \overset{H^1(d)}{\longrightarrow} \\
\hat{H}{}^2(S^7) & \overset{H^2(i)}{\longrightarrow} & H^2(S^7) & \overset{H^2(p)}{\longrightarrow} & \overset{\vee}{H}{}^2(S^7) & \overset{H^2(d)}{\longrightarrow} \\
\hat{H}{}^3(S^7) & \overset{H^3(i)}{\longrightarrow} & H^3(S^7) & \overset{H^3(p)}{\longrightarrow} & \overset{\vee}{H}{}^3(S^7) & \overset{H^3(d)}{\longrightarrow} \\
\hat{H}{}^4(S^7) & \overset{H^4(i)}{\longrightarrow} & H^4(S^7) & \longrightarrow & 0\,. &
\end{array} \tag{26}$$

Here $H^q(i)$ and $H^q(p)$ are the naturally induced maps, and $H^q(d)$ is the connecting homomorphism given by $H^q(d)([\alpha]) = \{d\alpha\} \in \hat{H}{}^{q+1}(S^7)$, for $[\alpha] \in \overset{\vee}{H}{}^q(S^7)$.

Since $H^q(S^7) = 0$ for $0 < q < 7$, from (26) it follows that $H^q(d)$ is an isomorphism for $q = 1, 2, 3$. Thus, from Theorem 2 and Theorem 3, we obtain

$$\overset{\vee}{H}{}^1(S^7) \cong \overset{\vee}{H}{}^2(S^7) \cong \hat{H}{}^2(S^7) \cong \hat{H}{}^3(S^7) \cong \{0\},$$

$$\overset{\vee}{H}{}^3(S^7) \cong \hat{H}{}^4(S^7) \cong \mathbb{R}.$$

Therefore, we have $\overset{\vee}{H}{}^3(S^7) \oplus \hat{H}{}^3(S^7) \cong \mathbb{R} \ncong H^3(S^7) = \{0\}$ and $\hat{H}{}^4(S^7) \cong \mathbb{R} \ncong H^4(S^7) = \{0\}$. ∎

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