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ON A BRYLINSKI CONJECTURE FOR COMPACT SYMPLECTIC MANIFOLDS *

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ABSTRACT

J. L. Brylinski in ([3], page 102) conjectured that for compact symplectic manifolds any de Rham cohomology class has a symplectically harmonic representative ν, that is, $d\nu = \delta\nu = 0$, for δ the Koszul differential. In this paper, a counterexample to this conjecture is constructed.

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1 Introduction

In this paper we shall consider symplectic manifolds as a particular type of Poisson manifolds; the Poisson bracket is given in a natural way by using the symplectic form as we explain bellow.

For a Poisson manifold M, J. L. Koszul in [12] defined a differential operator $\Delta : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)$. This operator is given by $\Delta = [i(G), d]$, where G is the Poisson tensor defined by A. Lichnerowicz [13], $i(G)$ denotes the contraction by G and d is the exterior derivative of M. We call it the Koszul differential; and using the notation of Brylinski [3] we shall write δ instead of Δ. Since $\delta^2 = 0$, it defines the so-called canonical homology of M.

For a symplectic manifold Brylinski [3] proved that δ is equal, up to a sign, to $\star d \star$, where $\star$ is the symplectic star operator defined in a similar way to the Hodge star operator for Riemannian manifolds.

As in the case of Riemannian manifolds, one can define a symplectic Laplacian $\Delta = d\delta + \delta d$, which is identically zero. A k-form ν such that $d\nu = \delta\nu = 0$ is called *symplectically harmonic*. Brylinski in ([3]; page 102), inspired by the Hodge's theorem for compact Riemannian manifolds, made the following

Conjecture: *If M is a symplectic manifold which is compact, any de Rham cohomology class in $H^*(M)$ has a representative ν such that $d\nu = \delta\nu = 0$, or, in other words, ν is a symplectically harmonic representative.*

For a compact manifold with (positive definite) Kähler metric Brylinski [3] proved that the conjecture is true. In this paper we shall give an example of a four dimensional compact symplectic manifold which *does not satisfy* the conjecture. Our example is the known compact symplectic manifold of Kodaira-Thurston (see [4, 6, 9, 11, 16]) which can be realized as the quotient of a nilpotent Lie group by a discrete subgroup.

The Kodaira-Thurston manifold KT is the simplest compact symplectic nilmanifold with no (positive definite) Kähler metric. KT cannot be Kählerian for topological reasons (see [4, 6, 7, 8, 10, 11, 16, 18]). However KT carries indefinite Kähler metrics ([4, 6, 7]), and hence symplectic structures.

2 The Koszul differential for symplectic manifolds

Let us briefly recall some results about the Koszul differential ([3, 12]). We also refer to [9] where a more detailed and more complete development may be found.

Let M be a symplectic manifold of dimension $2n$, with symplectic form ω. Denote by $\mathcal{X}(\mathcal{M})$ the Lie algebra of C^∞ vector fields on M and by $\mathcal{F}(\mathcal{M})$ the algebra of C^∞ functions on M. For every $f \in \mathcal{F}(\mathcal{M})$ we shall denote by X_f the Hamiltonian vector field of f, that is, the vector field on M such that

$$i_{X_f}\, \omega = df \ .$$

The Poisson bracket $\{\ ,\ \}$ on M is defined by

$$\{f, g\} = X_f(g) = -\omega(X_f, X_g), \tag{1}$$

for any $f, g \in \mathcal{F}(\mathcal{M})$. A. Lichnerowicz [13] remarked that the Poisson bracket gives rise to a non-singular covariant skewsymmetric tensor field G of order 2 such that $[G, G] = 0$, where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket. G is called Poisson tensor, and it satisfies:

$$i(G)(df \wedge dg) = \{f, g\}, \tag{2}$$

for $f, g \in \mathcal{F}(\mathcal{M})$, where $i(G)$ denotes the contraction by G. Conversely, such a tensor field G gives rise to a Poisson bracket and, hence, to a symplectic form.

Next, we shall denote by $\Lambda^k(M)$ the space of the differential k-forms on M. The *Koszul differential* $\delta : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)$ is defined as the commutator of $i(G)$ and the exterior derivative d, that is,

$$\delta\alpha = [i(G), d](\alpha) = i(G)(d\alpha) - d(i(G)\alpha)\ , \tag{3}$$

for all $\alpha \in \Lambda^k(M)$. Brylinski [3], inspired by the differential of the Chevalley-Eilenberg complex $C_*(\mathcal{F}(\mathcal{M}), \mathcal{F}(\mathcal{M}))$, has proved that δ can be alternatively defined by the formula:

$$\begin{aligned} \delta(f_0\ df_1 \wedge \cdots \wedge df_k) &= \sum_{1 \le i \le k} (-1)^{i+1}\ \{f_0, f_i\}\ df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k \\ &+ \sum_{1 \le i < j \le k} (-1)^{i+j}\ f_0\ d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k\ . \end{aligned} \tag{4}$$

From (3) or (4) one can check the following identity (see [3, 12]):

$$\delta^2 = 0\ . \tag{5}$$

Hence, we obtain the canonical complex of M:

$$\cdots \longrightarrow \Lambda^{k+1}(M) \xrightarrow{\ \delta\ } \Lambda^k(M) \xrightarrow{\ \delta\ } \Lambda^{k-1}(M) \longrightarrow \cdots$$

whose homology $H_*^{can}(M)$ is called the canonical homology of M.

Now, denote by $\Lambda^k(G)$, $(k \geq 0)$, the associated pairing $\Lambda^k(G) : \Lambda^k(T^*M) \times \Lambda^k(T^*M) \longrightarrow \mathcal{F}(\mathcal{M})$ which is $(-1)^k$-symmetric. We consider the volume form v_M on M given by $v_M = \dfrac{\omega^n}{n!}$. Then, imitating the Hodge star operator for Riemannian manifolds, Brylinski [3] defined the symplectic star operator $\star : \Lambda^k(M) \longrightarrow \Lambda^{2n-k}(M)$ by

$$\beta \wedge (\star\alpha) = \Lambda^k G(\beta, \alpha) \cdot v_M\ , \tag{6}$$

for all $\alpha, \beta \in \Lambda^k(M)$. Moreover, he proved that $\star^2$ is the identity, and

$$\delta\alpha = (-1)^{k+1} \star d \star (\alpha), \tag{7}$$

for all $\alpha \in \Lambda^k(M)$.

The $\star$ operator permits us to relate the canonical homology with the de Rham cohomology of M.

Theorem 2.1 *([3]) The operator $\star$ establishes an isomorphism of the canonical homology group $H_i^{can}(M)$ with the de Rham cohomology group $H^{2n-i}(M)$.*

In order to show, in the next section, that Brylinski's conjecture is not true we shall need some results.

Definition 2.2 For any $k \geq 0$, the map $L : \Lambda^k(M) \longrightarrow \Lambda^{k+2}(M)$ is given by

$$L(\alpha) = \alpha \wedge \omega \ , \tag{8}$$

for all $\alpha \in \Lambda^k(M)$.

Because ω is closed, the map L commutes with the exterior derivative:

$$L \circ d = d \circ L \ . \tag{9}$$

The following lemma gives the relations among L, $\star$ and G and it will be useful in the next section.

Lemma 2.3 *([9]) We have*

$$\begin{aligned} i(G) &= -\star L \star \ , & (10)\\ [L, \delta] &= -d \ . & (11) \end{aligned}$$

Moreover, any differential 1-form β on M satisfies

$$\star\,(\beta) = -(n-1)!\ L^{n-1}(\beta) \ . \tag{12}$$

3 A counterexample to Brylinski conjecture

Consider the Heisenberg group H that is, the connected, simply-connected and nilpotent Lie group of dimension 3 of the form

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} / \quad x, y, z \in \mathrm{R} \right\} .$$

A standar computation shows that a basis for the left invariant 1-forms on H is given by $\{dx, dy, dz - xdy\}$. Now, to get a compact quotient, we take the space of right cosets $\Gamma\backslash H$ where Γ is the uniform subgroup of H with integer coordinates. Thus, $\Gamma\backslash H$ is a 3-dimensional compact nilmanifold. Hence the left invariant 1-forms dx, dy, $dz - xdy$ all descend to 1-forms α, β, γ on $\Gamma\backslash H$, and so we have

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta \ . \tag{13}$$

Let us now consider the compact manifold $KT = \Gamma\backslash H \times S^1$. It is called the Kodaira-Thurston manifold.

Denote by η the canonical 1-form on S^1. Then, using the Nomizu's theorem [15], we compute the de Rham cohomology groups of KT:

$$\begin{aligned} H^0(KT) &= \{1\} \ , \\ H^1(KT) &= \{[\alpha], [\beta], [\eta]\} \ , \\ H^2(KT) &= \{[\alpha \wedge \gamma], [\alpha \wedge \eta], [\beta \wedge \gamma], [\beta \wedge \eta]\} \ , \\ H^3(KT) &= \{[\alpha \wedge \beta \wedge \gamma], [\alpha \wedge \gamma \wedge \eta], [\beta \wedge \gamma \wedge \eta]\} \ , \\ H^4(KT) &= \{[\alpha \wedge \beta \wedge \gamma \wedge \eta]\} \ . \end{aligned}$$

Furthermore, one can check that the 2-form ω given by

$$\omega = \alpha \wedge \gamma + \beta \wedge \eta \ , \tag{14}$$

is a symplectic form on KT.

Let $\{X, Y, Z, T\}$ be the dual basis to $\{\alpha, \beta, \gamma, \eta\}$, and let G be the Poisson tensor on KT associated with the symplectic form ω and defined by (14). Then we have that G is given by

$$G = Z \wedge X + T \wedge Y \ . \tag{15}$$

Therefore, using (4), we have that the Koszul differential δ on the generators of $\Lambda^k(KT)$, satisfies

$$\delta(\gamma \wedge \eta) = \alpha \ , \quad \delta(\beta \wedge \gamma \wedge \eta) = \alpha \wedge \beta \ , \tag{16}$$

and on the others is zero.

Theorem 3.1 *The cohomology class $[\beta\wedge\gamma\wedge\eta]$ in $H^3(KT)$ does not admit a representative ν such that $d\nu = \delta\nu = 0$. Therefore, Brylinski's conjecture is not true for compact symplectic no-Kähler manifolds.*

Proof : Let us suppose that the cohomology class $[\beta \wedge \gamma \wedge \eta]$ has a representative ν such that $\delta\nu = 0$. Then ν must be of the form

$$\nu = \beta \wedge \gamma \wedge \eta + d\theta \ , \tag{17}$$

for some differential 2-form $\theta \in \Lambda^2(KT)$. Since $\delta\nu = 0$, from (5), (13), (16) and (17) we have

$$\begin{aligned} 0 = \delta\nu &= \delta(\beta \wedge \gamma \wedge \eta + d\theta) \\ &= \alpha \wedge \beta - d(\delta\theta) \\ &= d(-\gamma - \delta\theta) \ , \end{aligned}$$

which implies that there is the cohomology class $[\gamma + \delta\theta] \in H^1(KT)$. Now, we distinguish the two cases: *i*) $[\gamma + \delta\theta]$ is zero in $H^1(KT)$, and *ii*) $[\gamma + \delta\theta]$ is non-zero.

In the case *i*), i.e. $[\gamma + \delta\theta] = 0$, we have

$$\gamma + \delta\theta = df \ , \tag{18}$$

for some function $f \in \mathcal{F}(\mathcal{KT})$. Applying L to both sides of (18), from (9) and (14), we get

$$-\beta \wedge \gamma \wedge \eta + L(\delta\theta) = d(Lf) \ ,$$

which implies

$$\beta \wedge \gamma \wedge \eta = L(\delta\theta) - d(Lf) \ . \tag{19}$$

Now, we shall compute $L(\delta\theta)$. Firstly, from (11), we have

$$L(\delta\theta) = \delta L(\theta) - d\theta \, . \tag{20}$$

Moreover, since θ is a differential 2-form on KT, we get $L(\theta) = g\alpha\wedge\beta\wedge\gamma\wedge\eta$, for some function $g \in \mathcal{F}(\mathcal{KT})$. Now, from (7) and (12), we obtain

$$\begin{aligned} \delta(L\theta) &= -\star d\star(g\alpha\wedge\beta\wedge\gamma\wedge\eta) \\ &= -\star(dg) \\ &= L(dg) \, . \end{aligned} \tag{21}$$

Substituting (20) and (21) in (19), it follows that

$$\begin{aligned} \beta\wedge\gamma\wedge\eta &= d(Lg) - d\theta - d(Lf) \\ &= d(Lg - \theta - Lf) \, , \end{aligned}$$

which is a contradiction because the cohomology class $[\beta\wedge\gamma\wedge\eta]$ is non-zero in $H^3(KT)$. Thus, the case $i)$ is not possible.

Let us now consider the case $ii)$, that is, $[\gamma+\delta\theta]$ is non-zero in $H^1(KT)$. Then

$$[\gamma+\delta\theta] = \lambda_1[\alpha] + \lambda_2[\beta] + \lambda_3[\eta] \, ,$$

for some $\lambda_i \in R$, $1 \le i \le 3$. This implies that

$$\gamma + \delta\theta = \lambda_1\alpha + \lambda_2\beta + \lambda_3\eta + dh \, , \tag{22}$$

for some function $h \in \mathcal{F}(\mathcal{KT})$. Applying L to both sides of (22) we obtain

$$\begin{aligned} -\beta\wedge\gamma\wedge\eta + L(\delta\theta) &= \lambda_1\alpha\wedge\beta\wedge\eta - \lambda_2\alpha\wedge\beta\wedge\gamma \\ &+ \lambda_3\alpha\wedge\gamma\wedge\eta + d(Lh) \, , \end{aligned}$$

from which we deduce

$$\begin{aligned} \beta\wedge\gamma\wedge\eta &- \lambda_2\alpha\wedge\beta\wedge\gamma + \lambda_3\alpha\wedge\gamma\wedge\eta \\ &= \lambda_1 d(\gamma\wedge\eta) - d(Lh) + L(\delta\theta) \, . \end{aligned} \tag{23}$$

Proceeding as in the case $i)$, we get $L(\delta\theta) = d(Lg) + d\theta$. Thus, from (23), it follows that the differential form $\beta\wedge\gamma\wedge\eta - \lambda_2\alpha\wedge\beta\wedge\gamma + \lambda_3\alpha\wedge\gamma\wedge\eta$ defines the zero cohomology class in $H^3(KT)$. But, it is not possible, because the cohomology classes $[\beta\wedge\gamma\wedge\eta]$, $[\alpha\wedge\beta\wedge\gamma]$ and $[\alpha\wedge\gamma\wedge\eta]$ are the generators of the cohomology group $H^3(KT)$. This completes the proof. $\boxed{QED}$

Furthermore, we consider the basis of vector fields $\{X, Y, Z, T\}$ on KT; and define an almost complex structure J on KT by

$$JX = Y,\ JZ = T\ . \tag{24}$$

It is easy to check that J is integrable (i.e. the Nijenhuis tensor of J vanishes). Let # denote the symmetric product and let

$$ds^2 = \beta \# \gamma - \alpha \# \eta\ . \tag{25}$$

Then ds^2 is an indefinite Kähler metric for J, and the corresponding Kähler form is the symplectic form ω.

Proposition 3.2 *A Brylinski conjecture for the compact nilmanifold KT with the indefinite Kähler metric given by (25) is not true.*

Proof : This follows from Theorem 3.1. $\boxed{QED}$

These results shows that is not possible to develop a symplectic harmonic Hodge theory.

Remark 3.3 O. Mathieu has proved in [14] that a symplectic harmonic Hodge theory can be developped on a symplectic manifold M if and only if it satisfies the hard Lefschetz Theorem.

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