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Stokes' Formula and the Dirac Operator on Embedded Manifolds

J. Cnops*

ABSTRACT

In this paper we give an alternative approach to the definition of the Dirac operator on manifolds, taking as a starting point Stokes' formula. This definition can be used on a quite general class of manifolds, and has the advantage of being very simple when the structure of the manifold is. In order to be able to use Stokes' formula on manifolds with a non-definite metric the formula is slightly changed so as to avoid the use of the notion of outward normal vector. Conformal transformation properties of the operator are given, as well as some examples.

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* Post-doc researcher NFWO, Belgium. Address: RUG, Galglaan 2, B-9000 Gent, Belgium.

1 Introduction

The definition of the Dirac operator as we construct it here is inspired both by the idea of the areolar derivative as introduced by Theodoresco in [7] and the derivative of a differential form, which can be found e.g. in [4].

Theodoresco defined for Euclidean space the areolar derivative as follows. Given a function f defined and, say, C^1 in a neighbourhood of some point $\vec{x}$. Then the areolar derivative of f in $\vec{x}$ is defined as the limit for $r \to 0$ of the integral over the sphere, centred in $\vec{x}$ with radius r, of f multiplied with the outer normal of the sphere. In order to perform this multiplication with a vector, Theodoresco used the Clifford algebra $\mathbb{R}_{0n}$.

We shall only define the Dirac operator for (pseudo-)Riemannian, orientable manifolds which can be conformally embedded in a (pseudo-)Euclidean space.

This is not a restriction, really, since (up to a smoothness condition) any metric manifold can be isometrically (and hence conformally) embedded in some (pseudo-)Euclidean space of sufficiently big dimension. This was proved for the Riemannian case by Nash [6], where the embedding is in Euclidean space, and in general by Clarke, [2].

Quite often however a manifold is embedded in a natural way in a pseudo-Riemannian space, and the structure of the embedding space is readily at hand. The general notion of embedding provides then a link between several possible embeddings. Due to the embedding in a pseudo-Euclidean space we could do without the notion of tangent space in the abstract sense, and only use the definition of tangent space in its classical geometrical meaning. Moreover, since the Clifford algebra $\mathbb{R}_{pq}$ is available, the notion of forms could be disposed with (even if the notion of course is present in the background), and we use Clifford algebra valued functions instead. We have defined integrals in terms of classical integrals on manifolds, where also the notion of k-forms is not explicitly introduced, even if, again, this notion is implicitly present in the background.

Regarding terms and notations, recall that a pseudo-Riemannian manifold M is a manifold such that, at any point $\vec{x}$ of the manifold there is a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\vec{x}}$ on the tangent space $T_{\vec{x}}$, where the mapping $\vec{x} \to \langle\cdot,\cdot\rangle_{\vec{x}}$ is C^∞. The manifold is called Riemannian if this bilinear form is positive definite.

Clifford algebras

Definitions. Basic properties. Let $\mathbb{R}^{pq}$ be the real n-dimensional space, where $n = p+q$, endowed with the nondegenerate inner product $\mathcal{B}(\cdot,\cdot)$ of signature (p,q). We assume that an orthonormal basis $e_1, \ldots, e_n$ is given, that is a basis such that

$$\begin{aligned} \mathcal{B}(e_i,e_i) &= -1, \quad i = 1,\ldots,p \\ \mathcal{B}(e_i,e_i) &= 1, \quad i = p+1,\ldots,n \\ \mathcal{B}(e_i,e_j) &= 0, \quad i \neq j. \end{aligned}$$

It should be noticed that the notation is taken such that Euclidean space with positive definite inner product is written as $\mathbb{R}^{0n}$ (which will sometimes be abbreviated to $\mathbb{R}^n$).

The Clifford algebra over $\mathbb{R}^{pq}$ is denoted by $\mathbb{R}_{pq}$. It is the free algebra generated by the e_i modulo the relations

$$e_ie_j + e_je_i = -2\mathcal{B}(e_i,e_j). \tag{1}$$

Elements of $\mathbb{R}_{pq}$ are called Clifford numbers or Clifford scalars. As a vector space the Clifford algebra has a basis of elements of the form $e_{t_1}\ldots e_{t_k}$ where $t_1 < \ldots < t_k$. We shall use two shorthand notations for such a product:

$$e_{t_1}\dots e_{t_k} = e_{t_1\dots t_k} = e_{\{t_1,\dots,t_k\}}, \qquad t_1 < \dots < t_k.$$

The dimension of $\mathbb{R}_{pq}$ over $\mathbb{R}$ is equal to 2^n.

If A has k elements, then e_A is called a k-vector. Likewise any linear combination of k-vectors is called a k-vector, and the vector space of k-vectors is written as $\mathbb{R}_{pq}^k$. Obviously $\mathbb{R}_{pq}$ is the direct sum of all $\mathbb{R}_{pq}^k$ for $k \leq n$, and the projection of a Clifford scalar a on $\mathbb{R}_{pq}^k$ will be written as $[a]_k$. Instead of 1-vectors the term vectors is used, and vectors are identified with elements of $\mathbb{R}^{pq}$. They will be denoted by $\vec{x}$, $\vec{y}$ etc. Also the term bivectors is used for 2-vectors. The n-vector $e_{\{1,\dots,n\}}$ is called the *pseudoscalar*. It is often written as e_N.

The following (anti)automorphisms will be used:

- The *main antiautomorphism* defined by

$$\bar{\vec{x}} = -\vec{x} \qquad \overline{(ab)} = \bar{b}\bar{a}.$$

- The *main automorphism* is defined by

$$\vec{x}' = -\vec{x} \qquad (ab)' = a'b'.$$

These (anti)morphisms will mainly be applied to products of vectors. Explicitly one obtains for a product of k vectors:

$$\begin{aligned} \overline{(\vec{x}_1\dots\vec{x}_k)} &= (-\vec{x}_k)\dots(-\vec{x}_1) \\ (\vec{x}_1\dots\vec{x}_k)' &= (-\vec{x}_1)\dots(-\vec{x}_k). \end{aligned}$$

It is not uncommon to define a so-called *dot product* on the Clifford algebra. We shall only need it when both arguments are vector valued, and in this case the dot product is given by

$$\vec{x}\cdot\vec{y} = \frac{1}{2}(\vec{x}\vec{y}+\vec{y}\vec{x}) = -\mathcal{B}(\vec{x},\vec{y}).$$

The *wedge product* between two vectors is given by $\vec{x}\wedge\vec{y} = \frac{1}{2}(\vec{x}\vec{y}-\vec{y}\vec{x})$. If b is a k-vector, then

$$\vec{x}\wedge b = \frac{1}{2}(\vec{x}b + (-1)^k b\vec{x}),$$

giving a $k+1$-vector. This can be extended in a unique way to the general case $a\wedge b$, using linearity and demanding associativity, to the wedge product of the so-called Grassman algebra, but it will be needed here only when the first argument is a vector.

Notice that, if $\vec{x}$ is parallel (orthogonal) to $\vec{y}$, then $\vec{x}\cdot\vec{y}$ ($\vec{x}\wedge\vec{y}$) is equal to $\vec{x}\vec{y}$: parallel vectors commute, while orthogonal vectors anticommute. As a consequence $\vec{x}^2$ is real for any vector $\vec{x}$, and $\vec{x}$ is invertible as an element of the Clifford algebra if and only if $\vec{x}^2 \neq 0$. Indeed, if $\vec{x}^2 = 0$, then $\vec{x}$ is a zero divisor and hence not invertible, while in the opposite case we can put $\vec{x}^{-1} = \vec{x}(\vec{x}^2)^{-1}$.

Let V be a right module over $\mathbb{R}_{pq}$. An *inner product* on V is a real-bilinear mapping $V\times V \to \mathbb{R}_{pq}$ such that it is $\mathbb{R}_{pq}$-linear in the second argument and moreover either

$(f,g) = \overline{(g,f)}$ (symmetric) or $(f,g) = -\overline{(g,f)}$ (antisymmetric inner product), for all f and g. From this a real valued inner product can be derived putting

$$(f,g)_{\mathbb{R}} = [(f,g)]_0.$$

Analoguous definitions can be given for left modules. Notice that here the inner product is linear in the first argument.

Subalgebra and subgroups. The *even subalgebra* $\mathbb{R}_{pq}^+$ is the set of Clifford numbers a such that $a = a'$. Alternatively it can be defined as the subalgebra generated by products of even numbers of vectors. Several groups in the Clifford algebra will be of interest. We have

- The *Clifford group* $\Gamma(p,q)$, sometimes called the Lipschitz group. It is generated, as a group, by the invertible vectors. Notice that for an invertible vector $\vec{x}$, $\vec{x}\bar{\vec{x}} = \mathcal{B}(\vec{x},\vec{x})$ is real and non-zero. By induction, for any a in $\Gamma(p,q)$, $a\bar{a}$ is real and non-zero.

- The *Pin group* $\mathrm{Pin}(p,q)$ is the subgroup of $\Gamma(p,q)$ of elements a for which $a\bar{a} = \pm 1$. It can be considered as the group generated by the unit vectors, i.e. vectors with $\vec{x}\bar{\vec{x}} = \pm 1$.

- The *Pin plus group* $\mathrm{Pin}^+(p,q)$ is the group of those elements a of $\mathrm{Pin}(p,q)$ such that $a\bar{a} = +1$. For $p = 0$, $\mathrm{Pin}^+(0,n) = \mathrm{Pin}(0,n)$. Notice that $\mathrm{Pin}^+(p,q)$ usually is *not* the group generated by the vectors such that $\vec{x}\bar{\vec{x}} = +1$. So, e.g., $\mathrm{Pin}^+(n,0)$ is not trivial for $n > 1$, while there is no vector in $\mathbb{R}^{n0}$ with $\vec{x}\bar{\vec{x}} = 1$.

- The *Spin group* $\mathrm{Spin}(p,q)$ is the intersection of $\mathrm{Pin}(p,q)$ with $\mathbb{R}_{pq}^+$. It is the group consisting of the products of even numbers of unit vectors.

If $p = 0$ often the notations $\Gamma(n)$ etc. instead of $\Gamma(0,n)$ etc. will be used, in analogy with the short notation $\mathbb{R}^n$ for $\mathbb{R}^{0n}$ It should be noticed that for any element of the Clifford group the equality

$$a' = \varepsilon a$$

holds, where $\varepsilon = +1$ when a is in the even subalgebra and $\varepsilon = -1$ if this is not the case. For an element s of the Pin group a transformation of $\mathbb{R}_{pq}$ is defined by

$$\chi(s)a = sas'^{-1}.$$

This clearly defines a group action. $\chi(s)$ leaves the space of k-vectors invariant for every k, and is usually only applied to $\mathbb{R}^{pq}$. As an example we calculate $\chi(\vec{x})$ for a vector $\vec{x}$ as it acts on $\mathbb{R}^{pq}$. A vector $\vec{y}$ can be split into a part $\lambda\vec{x}$ parallel to $\vec{x}$ (λ real), and a part $\vec{y}_\perp$ orthogonal to $\vec{x}$, and hence anticommuting with $\vec{x}$. Since $\vec{x}' = -\vec{x}$ one obtains

$$\begin{aligned}
\chi(\vec{x})\vec{y} &= \vec{x}\vec{y}\vec{x}'^{-1} = -\vec{x}(\lambda\vec{x} + \vec{y}_\perp)\vec{x}^{-1} \\
&= -\lambda\vec{x}(\vec{x}\vec{x}^{-1}) + \vec{y}_\perp(\vec{x}\vec{x}^{-1}) \\
&= -\lambda\vec{x} + \vec{y}_\perp.
\end{aligned}$$

So $\chi(\vec{x})$ is the orthogonal reflection into the hyperplane orthogonal to $\vec{x}$, and $\chi(a)$ is an orthogonal transformation for any $a \in \mathrm{Pin}(p,q)$. It is easily seen that $\chi(1)$ and $\chi(-1)$

both are the identity, and it is possible to prove that no other a results in the identity. This way $\mathrm{Pin}(p,q)$ gives a double covering of the orthogonal group $O(p,q)$, while $\mathrm{Spin}(p,q)$ gives a double covering of $SO(p,q)$.

When dealing with conformal mappings use will be made of mappings of the form

$$\vec{x} \to a\vec{x}a^*,$$

where a is an element of the Clifford group. For such a the element $b = a/\sqrt{|a\bar{a}|}$ is in the Pin group, and $b'^{-1} = \varepsilon b^{-1} = \varepsilon\bar{a}\sqrt{a\bar{a}}/(a\bar{a})$, where $\varepsilon = \pm 1$. Hence $a\vec{x}\bar{a}$ is up to a constant $\chi(b)$, and the mapping $\vec{x} \to a\vec{x}\bar{a}$ is the product of a dilatation and an orthogonal transformation.

Monogenic functions

Definitions. Let f be a C^1, $\mathbb{R}_{0n}$-valued function on a domain in $\mathbb{R}^{0n}$. The Dirac operator D for Euclidean space is given by

$$Df(\vec{x}) = \sum_{i=1}^{n} e_i \partial_i f(\vec{x}).$$

It is a linearisation of the Laplacian, $D^2 = -\Delta$.

Let spherical coordinates of Euclidean space be given by the equation $\vec{x} = r\xi$, where $r = |\vec{x}|$ and ξ is a unit vector. The Dirac operator can be split into a radial and a spherical part:

$$D = \xi\left(\partial_r + \frac{\Gamma}{r}\right)$$

where $\partial_r = \xi \cdot D = \sum_i (\xi \cdot e_i)\partial_i$ and $\Gamma = \xi \wedge D = \sum_i (\xi \wedge e_i)\partial_i$. Writing out both sides of the equation $-D^2 = \Delta$ in spherical coordinates gives

$$\partial_r^2 + \frac{\Gamma - \xi\Gamma\xi}{r}\partial_r - \frac{\Gamma + \xi\Gamma\xi\Gamma}{r^2} = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{\Delta_{LB}}{r^2},$$

where Δ_{LB} is the Laplace-Beltrami operator on the sphere of $\mathbb{R}^{0n}$. This results in

$$\begin{aligned} \Gamma - \xi\Gamma\xi &= n-1 \\ -\xi\Gamma\xi\Gamma - \Gamma &= (2-n+\Gamma)\Gamma = \Delta_{LB}, \end{aligned} \tag{2}$$

2 Dirac Operators

It is possible, for the definition of the Dirac operator, to allow milder assumptions on the continuity of the functions involved (such as the embedding mapping) than C^∞, as only derivatives of some small order are needed. However this would need *ad hoc* adaptation of the continuity demands, and this would be at the expense of clarity.

Definition. A metric manifold M is said to be *conformally embedded* in $\mathbb{R}_{pq}$ if there exists a one to one C^∞ mapping ϕ of M to a submanifold $\phi(M)$ of $\mathbb{R}^{pq}$ such that there exists a strictly positive C^∞ function α on $\phi(M)$ describing the metric of M.

This description of the metric is to be understood as follows: as a submanifold of $\mathbb{R}^{pq}$ the tangent space $T_{\phi(\vec{x})}$ is described in a natural way in $\mathbb{R}^{pq}$, leading to the metric $\langle\cdot,\cdot\rangle_{\phi(\vec{x})}$

on the tangent space $T_{\phi(\vec{x})}$, while ϕ induces a mapping $T_{\vec{x}} \to T_{\phi(\vec{x})}$, which will be denoted by $\phi_{\vec{x}}$. We say that α describes the metric of M on $\phi(M)$ whenever

$$\alpha(\phi(\vec{x}))^2 \langle t, s\rangle_{\vec{x}} = \langle \phi_{\vec{x}}(t), \phi_{\vec{x}}(s)\rangle_{\phi(\vec{x})}$$

for any $\vec{x}$ in M and t and s in $T_{\vec{x}}$. The natural metric on $\phi(M)$ will in the sequel be denoted by N if there is no confusion about the conformal embedding used, and the manifold M will then be identified with $\phi(M)$. That the metric is described by α will then be abbreviated by saying that the metric on M is given by N/α.

It is of course true that the embedding is not unique. Indeed, a conformal mapping of $\mathbb{R}_{pq}$ into itself (we assume for simplicity that no point of M is mapped to infinity) gives a new conformal embedding. As an example we can take the punctured sphere $S^{n-1}_{\vec{s}}$, which is the sphere S^{n-1} with the south pole $\vec{s}$ excluded, with its natural metric as a subset of $\mathbb{R}^n$. Using a Möbius transformation (the so-called Cayley transformation) this sphere can be mapped conformally to the hyperplane $\mathbb{R}^{n-1}$, and the metric in terms of the natural metric N_{n-1} of $\mathbb{R}^{n-1}$ becomes $N_{n-1}(\vec{x})/(1+|\vec{x}|^2)$. As will be shown however, the relation between the relevant terms for different conformal embeddings, such as integration over cycles with codimension 1, satisfy simple relations which make it easy to relate the different models. It should be remarked however that the notion of constant function can*not* be transferred this way and is peculiar to the conformal embedding used.

2.1 Integration over cycles

Functions on M or on a cycle C will always be assumed to take values in an algebra A containing $\mathbb{R}_{pq}$ as a subalgebra, and having an antiautomorphism $a \to \bar{a}$ which coincides on $\mathbb{R}_{pq}$ with the classical main antiautomorphism. If one has one conformal embedding of M in $\mathbb{R}_{pq}$, then of course A can be taken to be $\mathbb{R}_{pq}$ itself. However, if we want to compare two embeddings of M, say in $\mathbb{R}_{pq}$ and $\mathbb{R}_{p'q'}$, then it is useful to consider functions with values in a Clifford algebra containing both $\mathbb{R}_{pq}$ and $\mathbb{R}_{p'q'}$.

Moreover, if any function space is mentioned, it is always implicitly supposed to be a space of functions with values in A, and to be regarded as a right module over $\mathbb{R}_{pq}$, where the action is defined by pointwise multiplication.

Let M be embedded in $\mathbb{R}^{pq}$ with metric N/α and let C be a k-cycle in M. Then the notion of oriented elementary k-dimensional surface element can be described in terms of the Clifford algebra, and so the notion

$$\int_C F(\vec{x})dM_k(\vec{x})G(\vec{x})$$

is clear.

It is possible to give the definition in terms of the classical integral over C. Assume that $t = (t_1, t_2, \ldots, t_k)$ is a set of coordinates on C with orientation compatible with that of C (global coordinates are assumed here; patching together local coordinates is left to the reader), such that ψ maps a domain Ω in $\mathbb{R}^k$ to C. For t inside Ω fixed the vectors $\partial_i\psi(t)$ are well defined and

$$\int_C F(\vec{x})dM_k(\vec{x})G(\vec{x}) = \int_\Omega F(\psi(t))\left(\bigwedge_{i=1}^{k} \partial_i\psi(t)\right) G(\psi(t))\frac{d_k(t)}{\alpha(\psi(t))^k}$$

where d_k is k-dimensional Lebesgue measure. With the integral over C it is possible to define an A-valued inner product on the space of A-valued functions on C (which is not defined for all functions; we refrain from specifying a domain for this inner product as it may be used for duality relations) by the formula

$$(F,G)_C = \int_C \overline{F(\vec{x})} dM_k(\vec{x}) G(\vec{x}).$$

Whether this inner product is symmetric or antisymetric depends on k.

2.2 The Dirac operator

The Dirac operator of a manifold M, D_M is defined using Stokes' theorem as a basis. Notice that, for an $(m-1)$ cycle C we define the orientation by a vector $\vec{w}$ not parallel to the cycle in a point $\vec{x}$ as follows: let a set of local coordinates $(t_2, \ldots, t_m)$ of C be given, and t_1 an extra coordinate such that $(t_1, \ldots, t_m)$ is a set of local coordinates on M such that t_1 increases in the direction of $\vec{w}$. Then $(t_2, \ldots, t_m)$ agrees with the orientation of C if and only if $(t_1, \ldots, t_m)$ agrees with the orientation of M. For a cycle C its boundary has the inherited orientation which is defined in any point by a vector pointing outwards.
Definition. Let Ω be a domain in the m-dimensional manifold M. A real-linear operator $P : C^1(\Omega) \to C^0(\Omega)$ such that, for any f and g in $C^1(\Omega)$, Stokes' theorem holds is called a *pre-Dirac operator* on M in Ω.

Stokes' theorem says that, for any m-dimensional bounded cycle C in Ω having boundary ∂C with the inherited orientation,

$$\int_{\partial C} \overline{f(\vec{y})} dM_{m-1}(\vec{y}) g(\vec{y}) \tag{3}$$

$$= \int_C \overline{Pf(\vec{x})} dM_m(\vec{x}) g(\vec{x}) + (-1)^m \int_C \overline{f(\vec{x})} dM_m(\vec{x}) Pg(\vec{x}), \tag{4}$$

or, in inner product notation $(f,g)_{\partial C} = (Pf,g)_C + (-1)^m (f,Pg)_C$.

It is clear that P is $\mathbb{R}_{pq}$-linear. Indeed, if one takes a function g with invertible values everywhere in Ω and replaces in Stokes' formula f by $f\lambda$, then necessarily $P(f\lambda) = (Pf)\lambda$. It is not necessarily true that there is only one pre-Dirac operator in a domain Ω. Later on we shall describe all possible pre-Dirac operators of a manifold in terms of the Dirac operator. In order to distinguish a special operator, which will be the Dirac operator of the manifold, we have to introduce an extra condition.
Definition. A pre-Dirac operator D_M which is such that for any real valued g and each $\vec{x}$ in Ω, $D_M g(\vec{x})$ is a vector tangent to M in $\vec{x}$, is called a *Dirac operator*.
Remark
This definition of a Dirac operator differs rather considerably from other definitions, since a curvature term here is missing. As a result the notion of Dirac operator is not completely independent of the embedding of the manifold. However this rather wilful dependence of the Dirac operator on the embedding will serve to clarify the inherent differences of Spin representations and of possible inner products to make these unitary in the case where different embeddings are available.

Theorem 2.1 *The Dirac operator, if it exists, is unique.*

Proof.
Let L and T be two Dirac operators. It is sufficient to prove that $Tf = Lf$ for real valued functions. Take such f and let Z_f the interior of the set where f is zero. If Z_f is not empty, then for any cycle C within Z_f and any function g both $\int_{\partial C} \overline{f(\vec{y})} dM_{m-1}(\vec{y}) g(\vec{y})$ and $\int_C \overline{f(\vec{x})} dM_m(\vec{x}) P g(\vec{x})$ are zero, and so, by continuity $Pf(\vec{x})$ is zero in Z_f for any pre-Dirac operator. Hence $Lf(\vec{x}) = Tf(\vec{x})$ in Z_f. Take now a cycle C outside Z_f, and subtract Stokes' formula for T and L. This gives

$$\int_C \overline{(T-L)f(\vec{x})} dM_m(\vec{x}) g(\vec{x}) + (-1)^m \int_C \overline{f(\vec{x})} dM_m(\vec{x})(T-L)g(\vec{x}) = 0.$$

It is true that $\overline{(T-L)f(\vec{x})} dM_m(\vec{x}) = (-1)^m dM_m(\vec{x})(T-L)f(\vec{x})$, because $dM_m(\vec{x})$, like the pseudoscalar of the tangent space in $\vec{x}$, commutes with any tangent vector if m is odd, and anticommutes with any such vector if m is even. Hence the equation can be written, if we take $g = f$, as $2\int_C dM_m(\vec{x}) f(\vec{x})(T-L)f(\vec{x}) = 0$. Since this holds for any C outside Z_f, and $f \neq 0$ almost everywhere there, this means that $(L-T)f = 0$. ∎

Remark
The definition of the Dirac operator is locally independent of the domain Ω chosen. Indeed, let Ω_1 and Ω_2 be two domains in M, with the two Dirac operators D_1 and D_2. If the intersection is non-empty, and f is a function in $C^1(\Omega_1 \cup \Omega_2)$, then by the uniqueness theorem, $D_1 f = D_2 f$ in $\Omega_1 \cap \Omega_2$, while $C^1(\Omega_1 \cup \Omega_2) = C^1(\Omega_1) \oplus C^1(\Omega_2)$ if the intersection is empty. In both cases the extension of the Dirac operator to $C^1(\Omega_1 \cup \Omega_2)$ is straightforward.

Using the Dirac operator it is now possible to describe all pre-Dirac operators.

Theorem 2.2 *Let M be a manifold with Dirac operator D_M, and let T be a pre-Dirac operator. Then there exists a C^0 function λ satisfying $\overline{\lambda(\vec{x})} dM_m(\vec{x}) = (-1)^{m-1} dM_m(\vec{x}) \lambda(\vec{x})$ such that*

$$Tf(\vec{x}) = (D_M - \lambda(\vec{x})) f(\vec{x}).$$

Proof.
Take the constant function $g(\vec{x}) = 1$, and define $\lambda(\vec{x}) = (D_M - T)g(\vec{x})$. If for any C and any f, we subtract Stokes' formula for T from that for D_M we obtain

$$0 = \int_C \overline{(D_M - T)f(\vec{x})} dM_m(\vec{x}) + (-1)^m \int_C \overline{f(\vec{x})} dM_m(\vec{x}) \lambda(\vec{x}).$$

For $f = 1$ this gives $\overline{\lambda(\vec{x})} dM_m(\vec{x}) = (-1)^{m-1} dM_m(\vec{x}) \lambda(\vec{x})$; then for arbitrary f it gives $\overline{(D_M - T)f(\vec{x})} = \overline{f(\vec{x})}\, \overline{\lambda(\vec{x})}$. ∎

The following theorem is a generalisation of the chain rule.

Theorem 2.3 *Let g be real valued and let T be a pre-Dirac operator on M. Then, for any f,*

$$T(gf) = (Tg - g(T1))f + g(Tf)$$

Proof.
We first prove the theorem for the Dirac operator. It is obvious that $(fg,1)_{\partial C} = (f,g)_{\partial C}$. Applying Stokes' theorem to both sides of the equation shows that

$$(D_M(fg),1)_C + (-1)^m(fg, D_M 1)_C = (D_M f, g)_C + (-1)^m (f, D_M g)_C$$

Since $D_M g$ is vector valued, we get $\overline{D_M g} dM_n = (-1)^m dM_n D_M g$, and the same holds for $D_M 1$. The equation can thus be rewritten as $([D_M(fg) + (D_M 1)fg - g(D_M f) - (D_M g)f], 1)_C = 0$. Since this holds for all C, the term between square brackets is zero everywhere. From the expression $Tf(\vec{x}) = D_M f(\vec{x}) - \lambda(\vec{x})f(\vec{x})$ the general case easily follows. ∎

We finally prove that, even if the conformal embedding chosen influences the explicit form of the Dirac operator, different embeddings essentially lead to the same operator. Assume that for a given manifold we have two conformal embeddings, ϕ in $\mathbb{R}^{uv}$ and ψ in $\mathbb{R}^{sr}$. To be able to make comparisons we define the Dirac operator for functions with values in $\mathbb{R}_{pq}$, where $p = \max\{u,s\}$ and $q = \max\{v,r\}$. Of course we can then consider both ϕ and ψ as conformal embeddings in $\mathbb{R}^{pq}$, so in the following theorem we can immediately assume that both embeddings are in the same space, and moreover that ψ is the identical mapping. For any point $\vec{x}$ of M we again consider the mapping ϕ between the tangent spaces $T_{\vec{x}}$ and $T_{\phi\vec{x}}$. This mapping can be extended to a mapping $R_{\vec{x}}$ on $\mathbb{R}^{pq}$ which is up to a constant an orthogonal transformation (even for fixed $\vec{x}$ this is not uniquely defined, unless $m = p+q$). Since however ϕ is an isometry, the constant in consideration must be 1, and since the orientations of M and $\phi(M)$ are the same, one can take the orthogonal transformation to be direct. Hence such a transformation can be described by an element of the Spin(p,q) group, $S_{\vec{x}}$. Now, while the mapping $\vec{x} \to R_{\vec{x}}$ can be chosen in such a way that it is continuous, this is not necessarily true for the mapping $\vec{x} \to S_{\vec{x}}$ (locally it can be defined to be continuous, but after travelling around it might be that we end up with the sign changed), and we must make an extra assumption to relate Stokes' formula on M and on $\phi(M)$:

Theorem 2.4 *Assume that the Spin field $S_{\vec{x}}$ can be chosen such as to be continuous. Define the $\mathbb{R}_{pq}$ linear mapping $\Phi : F(M) \to F(\phi(M))$ (here $F(M)$ means the space of all A-valued functions on M) by*

$$\Phi f(\vec{y}) = S_{\vec{x}} f(\vec{x})$$

where $\phi(\vec{x}) = \vec{y}$. Then Let C be any cycle on M and g and f be two functions. Then if, for any of the two pairs of integrals

$$\int_{\partial C} \overline{f(\vec{x})} dM_{m-1}(\vec{x}) g(\vec{x}) \qquad \int_{\partial\phi C} \overline{\Phi f(\vec{y})} d(\phi(M))_{m-1}(\vec{y}) \Phi g(\vec{y})$$

$$\int_{C} \overline{f(\vec{x})} dM_{m}(\vec{x}) g(\vec{x}) \qquad \int_{\phi C} \overline{\Phi f(\vec{y})} d(\phi M)_{m}(\vec{y}) \Phi g(\vec{y})$$

one of the integral makes sense, then so does the other. Moreover they are either equal, or differ in sign (this does not depend on C, only on the chosen field S).

Proof.
The action of $S_{\vec{x}}$ can be extended from $\mathbb{R}^{pq}$ to the Clifford algebra and obviously we have (with a bit of abuse of notation) that, if $\vec{y} = \phi(\vec{x})$, then

$$\begin{aligned} d(\phi(M))_{m-1}(\vec{y}) &= S_{\vec{x}} dM_{m-1}(\vec{x}) S'^{-1}_{\vec{x}} \\ d(\phi(M))_{m}(\vec{y}) &= S_{\vec{x}} dM_{m}(\vec{x}) S'^{-1}_{\vec{x}} \end{aligned}$$

Now, since $S_{\vec{x}}$ is in the Spin group, the expression $\varepsilon = \overline{S_{\vec{x}}} S_{\vec{x}} S'^{-1}_{\vec{x}} S_{\vec{x}}$ is either $+1$ or -1, and from the continuity of S it follows that this expression is independent of $\vec{x}$, so

$$\begin{aligned} \overline{S_{\vec{x}}} d(\phi(M))_{m-1}(\vec{y}) S_{\vec{x}} &= \varepsilon dM_{m-1}(\vec{x}) \\ \overline{S_{\vec{x}}} d(\phi(M))_{m}(\vec{y}) S_{\vec{x}} &= \varepsilon dM_{m}(\vec{x}) \end{aligned}$$

∎

Remark
The condition that a continuous function S exists is of course closely linked to the condition of the existence of a Spin structure on the manifold. Moreover it should be noticed that the mapping Φ cannot be defined in classical theory, since it does not map sections of the Clifford bundle of M to sections of the Clifford bundle of $\phi(M)$.

Corollary 2.5 *If M has a pre-Dirac operator T, then the operator T_ϕ defined on $\phi(M)$ by*

$$T_\phi f = \Phi T \Phi^{-1} f$$

is a pre-Dirac operator on $\phi(M)$.

Proof.
¿From the previous theorem it is clear that Stokes' theorem holds for T_ϕ, and so T_ϕ is a pre-Dirac operator. ∎

Let now D_M be the Dirac operator on M, and let D_ϕ be defined by

$$D_\phi f = \Phi D_M \Phi^{-1} f.$$

For any real valued function g we have that

$$\begin{aligned} \left[D_\phi(g(\vec{y})S_{\vec{x}})\right](S_{\vec{x}})'^{-1} &= \left[(D_\phi g)S_{\vec{x}} + g(D_\phi S_{\vec{x}}) - (D_\phi 1)gS_{\vec{x}}\right](S_{\vec{x}})'^{-1} = \\ S_{\vec{x}}(D_M G(\vec{x}))(S_{\vec{x}})'^{-1} &= (D_\phi g)\varepsilon + gS_{\vec{x}}(D_M 1)(S_{\vec{x}})'^{-1} - (D_\phi 1)g\varepsilon \end{aligned}$$

where $\varepsilon = S_{\vec{x}}(S_{\vec{x}})'^{-1} = \pm 1$, and $G(\vec{x}) = g(\vec{y})$. Since G is real valued, $D_M G(\vec{x})$ is in the tangent space $T_{\vec{x}}$, and the expression at the right hand side is in $T_{\vec{y}}$. Similarly the second term at the right hand side is in $T_{\vec{y}}$, and the remaining terms are either both in $T_{\vec{y}}$, or not. Hence, if there is *one* g, vanishing almost nowhere, such that $D_\phi g(\vec{y})$ is in the tangent space, then $D_\phi 1(\vec{y})$ is in the tangent space, and so $D_\phi g$ is in the tangent space for *all* real g, and D_ϕ is the Dirac operator of $\phi(M)$.

2.3 Examples of Dirac operators

2.3.1 Euclidean space

In the case where the m-dimensional manifold M is embedded in $\mathbb{R}^n$, $m = n$, and the orientation of M coincides with the orientation given by the orthonormal basis $e_1, \ldots, e_n$, Stokes' formula can be written in a different form. Fix C, let $e_N = e_1 \ldots e_n$ be the pseudoscalar of $\mathbb{R}_{0n}$ and let $n_{\vec{y}}$ be the outer unit normal on ∂C. Then the equalities

$$\begin{aligned} dM_n(\vec{x}) &= \frac{e_N d\vec{x}}{\alpha^n(\vec{x})} \\ dM_{n-1}(\vec{y}) &= \frac{-n_{\vec{y}} e_N d\sigma(\vec{y})}{\alpha^{n-1}(\vec{y})} = (-1)^n \frac{e_N n_{\vec{y}} d\sigma(\vec{y})}{\alpha^{n-1}(\vec{y})}, \end{aligned}$$

hold, where again N/α is the metric of M, $d\vec{x}$ is the Lebesgue measure on $\mathbb{R}^n$ and $d\sigma(\vec{y})$ is the $n-1$-dimensional surface measure. For real valued f we have that $D_M f e_N = (-1)^{n-1} e_N D_M f$ and Stokes' formula becomes

$$\begin{aligned} &\int_{\partial C} \overline{f(\vec{y})} \frac{n_{\vec{y}} d\sigma(\vec{y})}{\alpha^{n-1}(\vec{y})} g(\vec{y}) \\ &= -\int_C \overline{D_M f(\vec{x})} \frac{d\vec{x}}{\alpha^n(\vec{x})} g(\vec{x}) + \int_C \overline{f(\vec{x})} \frac{d\vec{x}}{\alpha^n(\vec{x})} D_M g(\vec{x}) \end{aligned}$$

which still is $\mathbb{R}_{0n}$-linear in f and so is equivalent to Stokes' formula even for f which are not real. This is (almost) the form of Stokes' theorem which is used e.g. in [3] for Euclidean space itself, i.e. for $\alpha(\vec{x}) = 1$. The proof there that the Dirac operator defined there satisfies Stokes' theorem thus can be considered as a proof that this operator is the Dirac operator as we have defined it:

Theorem 2.6 *The Dirac operator in Euclidean space is defined by*

$$Df(\vec{x}) = \sum_{i=1}^{n} e_i \partial_i f(\vec{x}).$$

¿From this operator it easy to derive the Dirac operator for the more general metric N/α:

Theorem 2.7 *Let the m-dimensional manifold M be embedded in $\mathbb{R}^n$, with $m = n$, as a domain with metric N/α. Then the Dirac operator D_M is given as*

$$D_M f(\vec{x}) = \alpha(\vec{x})^{\frac{1+n}{2}} D\left(\alpha(\vec{x})^{\frac{1-n}{2}} f(\vec{x})\right).$$

Proof.
It is immediately clear that if f is real valued, then $D_M f$ is vector valued, so we only have to prove that D_M is a pre-Dirac operator. We put $F(\vec{x}) = \alpha(\vec{x})^{(1-n)/2} f(\vec{x})$ and $G(\vec{x}) = \alpha(\vec{x})^{(1-n)/2} g(\vec{x})$ and obtain

$$\begin{aligned} &\int_{\partial C} \overline{f(\vec{y})} dM_{n-1}(\vec{y}) g(\vec{y}) \\ &= \int_{\partial C} \overline{F(\vec{y})} dN_{n-1}(\vec{y}) G(\vec{y}) \end{aligned}$$

$$
\begin{aligned}
&= \int_C \overline{DF(\vec{x})} dN_n(\vec{x}) G(\vec{x}) + (-1)^n \int_C \overline{F(\vec{x})} dN_n(\vec{x}) DG(\vec{x}) \\
&= \int_C \overline{\alpha(\vec{x})^{(-1-n)/2)} D_M f(\vec{x})} dN_n(\vec{x}) \alpha(\vec{x})^{(1-n)/2)} g(\vec{x}) \\
&\quad + (-1)^n \int_C \overline{\alpha(\vec{x})^{(1-n)/2)} f(\vec{x})} dN_n(\vec{x}) \alpha(\vec{x})^{(-1-n)/2)} D_M g(\vec{x}) \\
&= \int_C \overline{D_M f(\vec{x})} dM_n(\vec{x}) g(\vec{x}) + (-1)^n \int_C \overline{f(\vec{x})} dM_n(\vec{x}) D_M g(\vec{x}).
\end{aligned}
$$

∎

2.3.2 The sphere

We consider here the sphere $S = S^{n-1}$ embedded in the natural way in $\mathbb{R}^n$. Let C be a cycle in S and let T be the n-dimensional cycle $T = \{r\xi : \xi \in C \text{ and } r_1 \leq r \leq r_2\}$ for some $0 < r_1 < 1 < r_2$. ∂T consists of three parts: $r_2 C$, $r_1 C$ (but with opposite orientation), and $\partial C \times [r_1, r_2]$. We choose the orientation of the sphere defined by the outer unit normal to the sphere and obtain, for dN_{n-1} on the sphere, explicitly that

$$dN_n(\vec{x}) = \xi r^{n-1} dr dN_{n-1}(\xi)$$

If one calculates $\int_{\partial T} \overline{F(\vec{y})} dN_{n-1}(\vec{y}) G(\vec{y})$, where F is defined in T by $F(r\xi) = \xi r^{1-n} f(\xi)$ while $G(r\xi) = g(\xi)$, one sees that the terms over the first two parts annihilate each other while the integral over the third part can be written as an integral over ∂C

$$
\begin{aligned}
\int_{\partial T} \overline{F(\vec{y})} dN_{n-1}(\vec{y}) G(\vec{y}) &= \int_{\partial C} \overline{\xi f(\xi)} \left(\int_{r_1}^{r_2} \frac{-\xi}{r} dr \right) dN_{n-2}(\xi) g(\xi) \\
&= -\int_{\partial C} \overline{f(\xi)} dN_{n-2}(\xi) g(\xi) \left(\frac{1}{r_1} - \frac{1}{r_2} \right)
\end{aligned}
$$

The right hand side of Stokes' equation for T becomes

$$\int_T \overline{\frac{\Gamma f(\xi)}{r}} dN_n(\vec{x}) \frac{1}{r^{n-1}} g(\xi) + (-1)^n \int_T \overline{\xi f(\xi)} dN_n(\vec{x}) \frac{\xi \Gamma g(\xi)}{r^n}$$

where we have calculated $DF(\vec{x})$ by

$$
\begin{aligned}
DF(\vec{x}) &= \xi \left(\partial_r + \frac{\Gamma}{r} \right) \frac{\xi}{r^{n-1}} f(\xi) \\
&= \frac{1-n}{r^n} \xi^2 f(\xi) + \frac{\xi \Gamma \xi f(\xi)}{r^n} \\
&= \frac{(n-1) + (\Gamma - n + 1)}{r^n} f(\xi) = \frac{\Gamma f(\xi)}{r^n}
\end{aligned}
$$

using the formula $\xi \Gamma \xi = \Gamma - n + 1$. If we write the integrals over T as a double integral, first for r going from r_1 to r_2 and then over C we obtain

$$-\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \int_{\partial C} \overline{f(\xi)} dN_{n-2}(\xi) g(\xi)$$

$$= \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$
$$\left(\int_C \overline{\Gamma f(\xi)}\xi dN_{n-1}(\xi)g(\xi) + (-1)^n \int_C \overline{\xi f(\xi)}\xi dN_{n-1}(\xi)\xi\Gamma g(\xi)\right)$$
$$= -\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$
$$\left(\int_C \overline{\xi\Gamma f(\xi)} dN_{n-1}(\xi)g(\xi) + (-1)^{n-1} \int_C \overline{f(\xi)} dN_{n-1}(\xi)\xi\Gamma g(\xi)\right)$$

This proves that $\xi\Gamma$ is a pre-Dirac operator on S. However, from the explicit expression from the Γ operator, $\Gamma = -\xi \wedge D = -\sum_i (\xi \wedge e_i)\partial_i$ it follows that in the expression

$$D_S = \xi\Gamma = -\sum_i \xi(\xi \wedge e_i)\partial_i$$

the coefficients of each ∂_i are orthogonal to ξ and hence parallel to the sphere, and so D_S is the Dirac operator of the sphere.

2.3.3 The nondefinite space $\mathbb{R}^{pq}$

The Dirac operator here is given by

$$\begin{aligned} Df(\vec{x}) &= -\sum_{i=1}^{n} e_i^3 \partial_i f(\vec{x}) \\ &= -\sum_{i=1}^{p} e_i \partial_i f(\vec{x}) + \sum_{i=p+1}^{n} e_i \partial_i f(\vec{x}) \end{aligned}$$

where $n = p + q$. Indeed, for f real valued this obviously leads to values $Df(\vec{x})$ which are vector valued and parallel to $\mathbb{R}^{pq}$, and D is $\mathbb{R}_{pq}$-linear. The only thing left to prove is that D is a pre-Dirac operator, i.e. that Stokes' theorem holds. This is done in the same way as in the case of $\mathbb{R}^n$. Notice, that one cannot rewrite Stokes' theorem here in the same way as was done for $\mathbb{R}^n$ without caution. Indeed, while it is always possible to write

$$dM_{n-1}(\vec{y}) = \frac{-n_{\vec{y}} e_N d\sigma(\vec{y})}{\alpha^{n-1}(\vec{y})}$$

for some vector $n_{\vec{y}}$, which is normal to ∂_C, this needs not be the outer unit normal. Indeed ∂_C may have an isotropic normal vector (for which no exterior direction can even be defined). Moreover, even if $n_{\vec{y}}$ is not isotropic, the sign of $n_{\vec{y}}^2$ must be taken into account.

The Dirac operator is a linearisation of the wave operator:

$$D^2 = \Box = \sum_i e_i^2 \partial_i^2.$$

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