CONTRIBUTION OF SVATOPLUK FUČÍK

The series of Spring Schools "Nonlinear Analysis, Function Spaces and Applications" is part of a series of seminars taking place every year. The seminar was founded almost twenty years ago, in 1976, by Svatopluk Fučík and Alois Kufner. This Spring School was devoted, as already mentioned in the preface, to the memory of Svatopluk Fučík. Since I was one of his students at Charles University in Prague in the seventies, I would like to use this opportunity to write a few words about Fučík's personality and his contribution to the development of nonlinear analysis.

In spite of the fact that Fučík was an extremely gifted scientist, he was first of all a human being. I will never forget his enthusiasm and his ability to attract people and to win their friendship. I was only a student of the second course when, under his supervision I began to study the properties of the Nemytskii operator in Hőlder spaces. I remember very well his willingness to spend plenty of time with me in order to guide me into the subject. I am sure that every student of Fučík has the same memory of him. On the other hand, it should be mentioned that Fučík did not like slovens and lazy students. He was very strict and sarcastic with them. Now being a university teacher myself and looking back to my student days, I have difficulty to understanding how he was able to find time both for his students and for his research.

His contribution to the development of nonlinear analysis is remarkable and it is emphasized by the fact that he succeeded to do everything in a relatively short period of time. The importance of his work does not consist only in extensive list of publications which includes several books and many papers published in prestigious journals. In my opinion, it consists mainly in the fact that after more than 15 years after his death, his work is still being referred to and some of his results came to be classical sources for present-day research. For the reader's convenience, I refer to the obituary published by J. Mawhin, J. Nečas and B. Novák in Časopis Pěst. Mat. (now Math. Bohemica) 105 (1980), 91–101 for the list of publications of Fučík and for the brief review of his research activities. His scientific heritage is, however, contained in his book "Solvability of Nonlinear Equations and Boundary Value Problems" which was published after his death by D. Reidel

Publ. Company in Holland in cooperation with the Union of Czechoslovak Mathematicians and Physicists in 1980.

I want to mention, in this article, two important concepts widely used nowadays in the mathematical literature, which were in some sense originated by Svatopluk Fučík. The first one is the *Fučík spectrum* which already has become widely accepted term in the theory of nonlinear differential equations. It plays a very important role in the solvability of semilinear problems and generalizes the concept of eigenvalues in a very natural way.

Let us consider the second order boundary value problem (BVP) for ordinary differential equation (ODE) of the second order:

$$-u''(x) - \mu u^+(x) + \nu u^-(x) = 0, x \in (0,1), \quad u(0) = u(1) = 0, \tag{1}$$

$u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$. The Fučík spectrum $\sum$ is the set of all couples $(\mu, \nu) \in \mathbb{R}^2$ for which the problem (1) has at least one nontrivial (i.e. nonzero) solution. In the early seventies, S. Fučík and independently of him also the Australian mathematician E.N. Dancer showed that $\sum$ is formed by a set of curves in the μ, ν-plane which can be expressed analytically. It is even more interesting that they discovered that the knowledge of the shape of these curves is a very useful tool in the study of the solvability of the BVP of the type

$$-u''(x) + g\big(u(x)\big) = f(x), x \in (0,1), \quad u(0) = u(1) = 0,$$

where the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ has finite limits $g(\pm\infty) = \lim_{s \to \pm\infty} g(s)$. Let us point out that, in particular, we have $(\lambda_n, \lambda_n) \in \sum$ for any eigenvalue λ_n of the problem

$$-u''(x) - \lambda u(x) = 0, x \in (0,1), \quad u(0) = u(1) = 0.$$

This is one of the reasons that the knowledge of $\sum$ provides more information than the structure of the set of eigenvalues and, moreover, offers the possibility to extend some results which follow from the classical Fredholm alternative. It is relatively easy to find a description of $\sum$ in the case of the second order ODE (1), but it is, however, a very difficult problem in the case of the partial differential equations (PDE) or ODE of higher order. For example, a complete description of the Fučík spectrum $\sum$ for the BVP

$$-\Delta u(x) - \mu u^+(x) + \nu u^-(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,$$

where Ω is the square $(0,1) \times (0,1)$ is still an open problem. Nevertheless, several papers have been published since Fučík's death trying to extend the classical results of him and Dancer to BVP for both PDEs and higher order ODEs and many interesting partial results have been obtained.

I want to mention here a second concept investigated by Fučík in the early seventies. It is the *p-Laplace operator.* Fučík was the supervisor of my diploma thesis dealing with the nonlinear BVP of the type

$$-(|u'|^{p-2}u')' + g(x,u) = f \quad \text{in} \quad (0,1), \quad u(0) = u(1) = 0. \tag{2}$$

with real number $p \geq 2$. Several existence results for (2) were obtained there using the properties of the Fučík spectrum of the quasilinear problem

$$-(|u'|^{p-2}u')' - \mu|u|^{p-2}u^{+} + \nu|u|^{p-2}u^{-} = 0 \quad \text{in} \quad (0,1),\ u(0) = u(1) = 0.$$

During the early eighties very intensive studies of the p-Laplacian had started. Its general form in higher dimensions is the following:

$$\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1. \tag{3}$$

In the case of ODEs it reduces to the principal part of (2) and for $p = 2$ it is nothing but the linear Laplace operator. To date a large body of work devoted almost exclusively to the theoretical study of the p-Laplacian has been published. Many of these papers emphasize its importance in applications, such as non-Newtonian fluid mechanics, glaceology, climatology, differential geometry etc. Nevertheless, there are still many open problems connected with the basic properties of the operator (3). Let us mention, among others, the following two of them. According to my best knowledge a complete description of the set of all eigenvalues of the problem

$$-\Delta_p u - \lambda|u|^{p-2}u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

is not known if $\Omega \subset \mathbb{R}^N$, $N \geq 2$. There are results concerning the principal eigenvalue $\lambda_1 > 0$ and several authors proved properties analogous to those for the Laplace operator ($p = 2$). The second open problem is connected with finding an analog of the Fredholm alternative for the p-Laplacian, considering the solvability of the BVP

$$-\Delta_p u - \lambda_1|u|^{p-2}u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{4}$$

The characterisation of the set of all right hand sides f for which (4) has at least one solution (in a suitable sense) is an open problem even in the one-dimensional case $\Omega = (0,1)$.

In addition, it should be mentioned that M. Konečný, also a student of S. Fučík, founded the Educational Svatopluk Centre, the High School and the Foundation named by Svatopluk Fučík in a small town Brušperk in Moravia.

All the facts mentioned above demonstrate that Fučík performed a good job. But the most important thing is that his work is still alive. This is also one of the reasons why his colleagues and pupils will never forget him.

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