ON ONE PROBLEM OF C. DAVIS

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In this paper we present a rather general method of the proof of the existence of functions with prescribed extreme values. The method is based on the theory of extremal problems.

1. Introduction

In 1956 C. Davis ([1], [2]) posed the following problem: For a real polynomial $p(\cdot)$ of degree N, denote the zeros of $p'(\cdot)$ (multiplicity counted) by τ_i, $i = 1, \dots, N-1$. Assume that τ_i are real, and let $\tau_1 \le \cdots \le \tau_{N-1}$. Now to what extent are the numbers $p(\tau_i)$ arbitrary? More precisely, give necessary and sufficient conditions on an $(N-1)$-tuple of real numbers v_1, $\dots$, v_{N-1} in order that there exists a polynomial $p(\cdot)$ such that $p'(\tau_i) = 0$, $p(\tau_i) = v_i$, $i = 1, \dots, N-1$.

C. Davis solved this problem in 1957 ([2]) for algebraic polynomials. Later many investigators (see [3]–[8]) gave other solutions of this problem and solutions of an analogous setting for Chebyshev systems of polynomials and splines. It should be pointed out that in [6], [7] more general results are obtained.

These results are formulated below (see Theorem 1.1 and 1.2).

2. Theorems on existence of an algebraic polynomial and a spline with prescribed extreme values

Let $\Delta := [0,1]$, $m, N, n \in \mathbb{N}$, $T := \{t_i\}_{i=1}^{n} \subset \Delta$, and $0 \le t_1 < \cdots < t_n \le 1$. Denote by $\mathcal{P}_N$ the family of all algebraic polynomials of order N and by $\mathcal{S}_m^n$ the family of m-order polynomial splines of defect 1 on Δ with nodes at the points $t_i \in T$.

Definition. A sequence

$$V = \{v_i\}_{i=0}^{N}, \quad v_i \in \mathbb{R},\ i = 0, 1, \dots, N,$$

satisfying the condition

$$(v_{i+1} - v_i)\,(v_i - v_{i-1}) < 0, \quad i = 1, \dots, N-1, \tag{1.1}$$

is called *a sequence of extreme values* (s. e. v.).

Theorem 1.1. *Let $V = \{v_i\}_{i=0}^N$ be a s. e. v. Then there exists a unique algebraic polynomial $\widehat{p}(\cdot) \in \mathcal{P}$ and a unique system of points $\{\widehat{\tau}_i\}_{i=0}^N \subset \Delta$,*

$$0 = \widehat{\tau}_0 < \widehat{\tau}_1 < \cdots < \widehat{\tau}_N = 1,$$

such that

$$\widehat{p}(\widehat{\tau}_i) = v_i, \quad i = 0, 1, \dots, N, \tag{1.2}$$

$$\widehat{p}'(\widehat{\tau}_i) = 0, \quad j = 1, \dots, N-1. \tag{1.3}$$

Theorem 1.2. *Let $m, n, N \in \mathbb{N}$, $m + n = N$, and let $V = \{v_i\}_{i=0}^N$ be a s. e. v. Then there exists a unique spline $\widehat{s}(\cdot) \in \mathcal{S}_m^n$ and a unique system of points*

$$\{\widehat{\theta}_i\}_{i=0}^N \subset \Delta, \quad 0 = \widehat{\theta}_0 < \widehat{\theta}_1 < \cdots < \widehat{\theta}_N = 1,$$

such that

$$\widehat{s}(\widehat{\theta}_i) = v_i, \quad i = 0, 1, \dots, N, \tag{1.4}$$

$$\widehat{s}'(\widehat{\theta}_j) = 0, \quad j = 1, \dots, N-1, \tag{1.5}$$

$$\widehat{\theta}_i < t_i < \widehat{\theta}_{i+m}, \quad i = 1, \dots, n. \tag{1.6}$$

3. Setting of extremal problems

Denote by

$$\Sigma := \Big\{ \mu = (\mu_1, \dots, \mu_{N-1}) \colon \mu_i \ge 0, \sum_{i=1}^{N-1} \mu_i = 1 \Big\}$$

the $(N-1)$-dimensional simplex. Put

$$\Sigma_T := \Big\{ \eta = (\eta_1, \dots, \eta_{N-1}) \in \Sigma \colon \sum_{j=1}^{i} \eta_j < t_i < \sum_{j=1}^{m+i} \eta_j, \ t_i \in T \Big\}.$$

Let us introduce the following notation:

$$\tau_0(\mu) := 0,\ \tau_i(\mu) := \sum_{j=1}^{i} \mu_j,\ \theta_0(\eta) := 0,\ \theta_i(\eta) := \sum_{j=1}^{i} \eta_j,\ i = 1, 2, \ldots, N.$$

Given any s. e. v.

$$V = \{v_i\}_{i=0}^{N},$$

we construct a class of algebraic polynomials $\mathcal{P}$ and a class of splines $\mathcal{S}$ as follows

$$\mathcal{P} := \Big\{p(\cdot) \in \mathcal{P}_N : p\big(\tau_i(\mu)\big) = v_i,\ \mu \in \operatorname{int}\Sigma,\ v_i \in V\Big\},$$
$$\mathcal{S} := \Big\{s(\cdot) \in \mathcal{S}_m^n : s(\theta_i\big(\eta)\big) = v_i,\ \eta \in \operatorname{int}\Sigma_T,\ v_i \in V\Big\}.$$

Let us consider the value $p(2)$ $(p(\cdot) \in \mathcal{P})$ as a function of a point $\mu = (\mu_1, \ldots, \mu_{N-1}) \in \operatorname{int}\Sigma$, that is, $p(2) := p(2, \mu)$, and the value $s(2)$ $(s(\cdot) \in \mathcal{S})$ as a function of a point $\eta = (\eta_1, \ldots, \eta_{N-1}) \in \operatorname{int}\Sigma_T$, that is, $s(2) := s(2, \eta)$. Now we can formulate the following extremal problems.

Problem 3.1.

$$p(2) := p(2, \mu) \to \inf, \quad p(\cdot) \in \mathcal{P}, \quad \mu \in \operatorname{int}\Sigma, \quad \text{if } v_{N-1} < v_N,$$

and

$$p(2) := p(2, \mu) \to \sup, \quad p(\cdot) \in \mathcal{P}, \quad \mu \in \operatorname{int}\Sigma, \quad \text{if } v_{N-1} > v_N.$$

Problem 3.2.

$$s(2) := s(2, \eta) \to \inf, \quad s(\cdot) \in \mathcal{S}, \quad \eta \in \operatorname{int}\Sigma_T, \quad \text{if } v_{N-1} < v_N,$$

and

$$s(2) := s(2, \eta) \to \sup, \quad s(\cdot) \in \mathcal{S}, \quad \mu \in \operatorname{int}\Sigma_T, \quad \text{if } v_{N-1} > v_N.$$

4. Solution of extremal problems 3.1 and 3.2

Theorem 4.1. *There exists*

$$\widehat{\mu} = (\widehat{\mu}_1, \dots, \widehat{\mu}_{N-1}) \in \operatorname{int} \Sigma$$

such that

$$\inf_{p(\cdot)\in\mathcal{P}} p(2) = \inf_{\mu\in\Sigma} p(2,\mu) = p(2,\widehat{\mu}) \quad \text{if } v_{N-1} < v_N, \tag{4.1}$$

and

$$\sup_{p(\cdot)\in\mathcal{P}} p(2) = \sup_{\mu\in\Sigma} p(2,\mu) = p(2,\widehat{\mu}) \quad \text{if } v_{n-1} > v_n. \tag{4.2}$$

Theorem 4.2. *There exists* $\widehat{\eta} = (\widehat{\eta}_1, \dots, \widehat{\eta}_{n-1}) \in \operatorname{int} \Sigma_T$ *such that*

$$\inf_{s(\cdot)\in\mathcal{S}} s(2) = \inf_{\eta\in\Sigma_T} s(2,\eta) = s(2,\widehat{\eta}) \quad \text{if } v_{n-1} < v_n, \tag{4.3}$$

and

$$\sup_{s(\cdot)\in\mathcal{S}} s(2) = \sup_{\eta\in\Sigma_T} s(2,\eta) = s(2,\widehat{\mu}) \quad \text{if } v_{n-1} > v_n. \tag{4.4}$$

The idea of proofs of Theorems 4.1 and 4.2 is based on the Weierstrass compactness principle. Since $p(2) = p(2,\mu)$ $\big(p(\cdot) \in \mathcal{P}\big)$ and $s(2) = s(2,\eta)$ $\big(s(\cdot) \in \mathcal{S}\big)$ are continuous functions on Σ and Σ_T, respectively, and

$$\lim_{\operatorname{int}\Sigma \ni \mu \to \mathring{\mu} \in \partial\Sigma} p(2,\mu) = \begin{cases} +\infty, & \text{if } v_{n-1} < v_n \\ -\infty, & \text{if } v_{n-1} > v_n, \end{cases}$$

and

$$\lim_{\operatorname{int}\Sigma_T \ni \eta \to \mathring{\eta} \in \partial\Sigma_T} s(2,\eta) = \begin{cases} +\infty, & \text{if } v_{n-1} < v_n \\ -\infty, & \text{if } v_{n-1} > v_n, \end{cases}$$

then by the Weierstrass theorem there exist points $\widehat{\mu} \in \operatorname{int}\Sigma$ and $\widehat{\eta} \in \operatorname{int}\Sigma_T$ such that identities (4.3) and (4.4) are valid.

5. The scheme of proofs of Theorems 1.1 and 1.2

Beginning with the points $\widehat{\mu} = (\widehat{\mu}_1, \dots, \widehat{\mu}_{n-1}) \in \operatorname{int}\Sigma$ and $\widehat{\eta} = (\widehat{\eta}_1, \dots, \widehat{\eta}_{n-1}) \in \operatorname{int}\Sigma_T$ (which were found in Theorems 4.1 and 4.2), we construct the points

$$\widehat{\tau} = (\widehat{\tau}_0, \widehat{\tau}_1, \dots, \widehat{\tau}_n) \subset \Delta \quad \text{and} \quad \widehat{\theta} = (\widehat{\theta}_0, \widehat{\theta}_1, \dots, \widehat{\theta}_n) \subset \Delta$$

so that

$$\widehat{\tau}_0 := 0, \widehat{\tau}_i = \sum_{j=1}^{i} \mu_i, \quad \widehat{\theta}_0 := 0, \quad \widehat{\theta}_i := \sum_{j=1}^{i} \widehat{\eta}_i, \quad i = 1, \dots, N.$$

Then, we find an interpolation polynomial $\widehat{p}\,(\cdot) \in R_N$ and an interpolation spline $\widehat{s}(\cdot) \in \mathcal{S}_m^n$ satisfying the conditions:

$$\widehat{p}(\widehat{\tau}_i) = v_i, \quad i = 0, 1, \dots, N;$$
$$\widehat{s}(\widehat{\theta}_i) = v_i, \quad i = 0, 1, \dots, N.$$

The polynomial $\widehat{p}(\cdot) \in \mathcal{P}_N$ and the spline $\widehat{s}(\cdot) \in \mathcal{S}_m^n$ are uniquely defined and they satisfy the assumptions of Theorems 1.1 and 1.2.

6. Generalizations

The represented method enables to solve various generalizations of the Davis problem in many directions.

References

[1] C. Davis, *Problem 4714*, Amer. Math. Monthly **63** (1956), no. 10, 729.

[2] C. Davis, *Extrema of a polynomial*, Amer. Math. Monthly **64** (1957), no. 9, 679–680.

[3] J. Mucielski and S. Paszkowski, *A generalization of Tschebysheff polynomials*, Bull. Acad. Polon. Sci. Ser. Math. Astron. Phys. **8** (1960), no. 7, 433–438.

[4] W. J. Kammerer, *Polynomial approximations of finitely oscillating functions*, Math. Comp. **15** (1961), no. 74, 115–119.

[5] J. Kudela, *Construction des certains polynômes extrémaux*, Prace Matematyczne **LXXVII** (1963), no. 9, 31–36.

[6] V. S. Videnskii, *The existence and uniqueness of a solution of an interpolation problem*, Sb. Nauchn. Tr. Leningr. Mech. Inst. (1965), no. 50, 29–41.

[7] S. H. Fitzgerald and L. L. Shumaker, *A differential equation approach to interpolation at extremal points*, J. Analyse Math. **22** (1969), 117–139.

[8] M. B. Korobkova, *A theorem on the existence of a spline-polynomial with a given sequence of extrema*, Mat. Zametki **11** (1972), 251–258.