FUNCTION SPACES OF LIZORKIN-TRIEBEL TYPE WITH EXPONENTIAL WEIGHTS

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1. Preliminaries

Let $S = S(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing C^∞ functions. All function spaces that occur here are defined on $\mathbb{R}^n$. Therefore, we omit the suffix $\mathbb{R}^n$ in $S(\mathbb{R}^n)$ as well as in the other spaces below. Let D be the space of all compactly supported C^∞ functions equipped with the usual topology. By S' and D' we denote the strong topological dual spaces of S and D, respectively. For $0 < p < \infty$, L_p are the usual Lebesgue spaces. If u is a weight function, i.e., measurable and positive a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$, then $L_p(u)$ is equipped with the norm $\|f \mid L_p(u)\| = \|uf \mid L_p\|$. To each function $\varphi \in S$ we assign its Fourier transform

$$\widehat{\varphi}(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \exp(-ixy) f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Furthermore, $\varphi^\vee$ stands for the inverse Fourier transform of φ. Both the Fourier transform and its inverse are extended to S' by duality. The unweighted spaces F^s_{pq} can be defined as follows (cf. [12], [14]). Let $\varphi_0 \in S$ be such that

$$(1) \qquad \varphi_0(x) = 1 \quad \text{if } |x| \le 1, \quad \text{and} \quad \varphi_0(x) = 0 \quad \text{if } |x| \ge 2.$$

Put $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) = \varphi(2^{-j}x)$, $j \in \mathbb{N}$. Then

$$(2) \qquad f = \sum_{j=0}^{\infty} \big(\varphi_j \widehat{f}\,\big)^\vee, \quad f \in S'.$$

Definition 1. Let $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Then

$$(3)\quad F^s_{pq} = \Big\{ f \in S' \colon \|f \mid F^s_{pq}\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \big| \big(\varphi_j \widehat{f}\,\big)^{\vee}(\cdot) \big|^q \Big)^{1/q} \Bigm| L_p \Big\| < \infty \Big\}.$$

The spaces F^s_{pq} are independent of the function φ_0 which satisfies (1). They are quasi-Banach spaces (Banach spaces if $p \ge 1$ and $q \ge 1$). The continuous embeddings $S \hookrightarrow F^s_{pq} \hookrightarrow S'$ hold. If $q < \infty$, then S is a dense subset of F^s_{pq}. Finally,

$$(4)\qquad F^k_{p2} = W^k_p, \quad k \in \mathbb{N}_0, \quad 1 < p < \infty,$$

where W^k_p stands for the classical Sobolev spaces. As for proofs, see for instance [12].

Let us introduce the following notation. If (k_0, k) is a pair of functions in D, then we put $k^N_0 = k_0$ and $k^N_j(x) = 2^{jn}(\Delta^N k)(2^j x)$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}$, where Δ^N stands for the N-Laplacian.

The following theorem is due to H. Triebel (cf. [13], [14]).

Theorem 1. *Let $1 < p < \infty$, $1 < q \le \infty$, and $s \in \mathbb{R}$. Let $k_0, k \in D$ be such that*

$$(5)\qquad \widehat{k}_0(0) \neq 0 \quad \text{and} \quad \widehat{k}(0) \neq 0.$$

Let $N \in \mathbb{N}_0$ be such that $2N > s$. Then
(6)

$$F^s_{pq} = \Big\{ f \in S' \colon \|f \mid F^s_{pq}\|^N_{(k_0,k)} = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f * k^N_j(\cdot)|^q \Big)^{1/q} \Bigm| L_p \Big\| < \infty \Big\},$$

and $\|\cdot \mid F^s_{pq}\|^N_{(k_0,k)}$ is an equivalent norm on F^s_{pq}.

It is our aim to study weighted Lizorkin-Triebel spaces, i.e., we wish to replace L_p-norms by $L_p(u)$-norms in (3) and (6). In recent literature there occur several approaches to such spaces. First of all, one can study $F^s_{pq}(u)$ in the framework of S'. This is possible if

$$(7)\qquad u(x) = (1 + |x|)^{\alpha} \quad \text{with} \quad \alpha \in \mathbb{R}$$

(cf., for instance, [8], [9], [12], and the references given there). For u from the Muckenhoupt class A_∞ we refer to [2]. In order to deal with weights of type

$$(8)\qquad u(x) = \exp\big(\pm |x|^{\beta}\big), \quad 0 < \beta \le 1,$$

the tempered distributions are not sufficient. In fact, in virtue of (4) we should obtain $F^0_{p2}(u) = L_p(u)$, $1 < p < \infty$. On the other hand, if $u(x) = \exp(-|x|^\beta)$, then $L_p(u) \not\subset S'$. If u is of type (8) with $\beta < 1$, then one can employ the spaces S'_ω of ultra-distributions, which are due to G. Björck [1] (see also [9]). A corresponding theory of the spaces $F^s_{pq}(u)$ is due to H. Triebel [11] (see also [4], [9] and [12]). If $\beta = 1$ in (8), then neither the spaces S'_ω are sufficient. Therefore, we introduce a suitable space of distributions.

2. Distributions

Definition 2. By S_e we denote the set of all $\varphi \in C^\infty$ such that

$$p_N(\varphi) = \sup_{x\in\mathbb{R}^n} \exp(N|x|) \sum_{|\alpha|\le N} |D^\alpha \varphi(x)| < \infty, \quad N \in \mathbb{N}_0.$$

We equip S_e with the locally convex topology which is defined by the system of the semi-norms p_N. Further, S'_e is the collection of all continuous linear forms on S_e. We equip S'_e with the strong topology.

Let us discuss some properties of S_e and S'_e, respectively. The continuous embeddings $D \hookrightarrow S_e \hookrightarrow S$ hold, D is dense in S_e, and S_e is dense in S. Consequently, we have $S' \subset S'_e \subset D'$ in the sense of the usual interpretations. If $f \in S'_e$ and $\varphi \in S_e$, then their convolution $f * \varphi$ is a C^∞ function of at most exponential growth and, hence, is a regular distribution in S'_e. If $\varphi \in S_e$, then $\widehat{\varphi}$ is an analytic function on $\mathbb{R}^n$. On the other hand, we have $D \subset S_e$. Hence, the Fourier transform does not map S_e onto S_e. As a consequence, we cannot define the Fourier transform in S'_e similarly as in S'. We need a suitable counterpart of (2) within the framework of S'_e.

Definition 3. Let $M, N \in \mathbb{N}_0$. By $\mathcal{L}^M_N$ we denote the collection of all pairs (k_0, k) of functions $k_0, k \in S_e$ which have the following properties:

$$\int_{\mathbb{R}^n} k_0(x)\,\mathrm{d}x = 1;$$

there are functions $\kappa_0, \kappa \in S_e$ such that

$$\int_{\mathbb{R}^n} \kappa_0(x)\,\mathrm{d}x = 1$$

and

$$\Delta^N k * \Delta^M \kappa(x) = k_0 * \kappa_0(x) - 2^{-n} k_0 * \kappa_0(x/2), \quad x \in \mathbb{R}^n.$$

Let $(k_0, k) \in \mathcal{L}_N^M$ for some $M, N \in \mathbb{N}_0$. If the pair (κ_0, κ) corresponds to (k_0, k) via Definition 3, then we write $(\kappa_0, \kappa) \sim (k_0, k)$.

Remark 1. Let $M, N \in \mathbb{N}_0$. Then $\mathcal{L}_N^M \neq \emptyset$. In particular, $\mathcal{L}_N^M$ contains compactly supported rotation-invariant pairs (k_0, k) that satisfy (5).

Proposition 1. *Let $M, N \in \mathbb{N}_0$. Let $(k_0, k) \in \mathcal{L}_N^M$ and $(\kappa_0, \kappa) \sim (k_0, k)$. If $f \in S_e'$, then*

$$f = \sum_{j=0}^{\infty} f * k_j^N * \kappa_j^M \quad \text{in } S_e'. \tag{9}$$

3. Function spaces

Definition 4. By W_e we denote the collection of all real-valued continuous functions u on $\mathbb{R}^n$ which satisfy

$$0 < u(x) \leq c u(x-y) \exp(d|y|), \quad x, y \in \mathbb{R}^n,$$

for some $c > 0$ and $d \geq 0$.

Example 1. If u is of type (7) or (8), then u belongs to W_e.

Definition 5. Let $1 < p < \infty$, $1 < q \leq \infty$, $s \in \mathbb{R}$, and $u \in W_e$. Let $M, N \in \mathbb{N}_0$ be such that $2M > -s$ and $2N > s$. Let $(k_0, k) \in \mathcal{L}_N^M$. Then

$$F_{pq}^s(u) = \Big\{ f \in S_e' : \\ \|f \mid F_{pq}^s(u)\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f * k_j^N(\cdot)|^q \Big)^{1/q} \Bigm| L_p(u) \Big\| < \infty \Big\}.$$

If $f \in S_e'$ satisfies $\|f \mid F_{pq}^s(u)\| = 0$, then $f * k_j^N = 0$ for each $j \in \mathbb{N}_0$. By (9) this implies $f = 0$. Hence, $\|f \mid F_{pq}^s(u)\|$ is a norm on $F_{pq}^s(u)$. Let us observe that $F_{pq}^s(u)$ is independent of M, N and (k_0, k) in the sense of equivalent norms.

Theorem 2. *Let $1 < p < \infty$, $1 < q \leq \infty$, $s \in \mathbb{R}$, and $u \in W_e$. Then $F_{pq}^s(u)$ is a Banach space and*

$$S_e \hookrightarrow F_{pq}^s(u) \hookrightarrow S_e' \quad \textit{(continuous embeddings)}.$$

Furthermore, if $q < \infty$, then S_e is dense in $F^s_{pq}(u)$.

For proofs we refer to [10]. The main ingredients are the decomposition formula (9) and the vector-valued maximal inequality

$$\Big\| \Big(\sum_{j=0}^{\infty} |Mg_j|^q \Big)^{1/q} \Big| L_p \Big\| \le c \Big\| \Big(\sum_{j=0}^{\infty} |g_j|^q \Big)^{1/q} \Big| L_p \Big\|, \tag{10}$$

where M stands for the Hardy-Littlewood maximal operator (cf. [3]). Compared to Definition 1, our approach requires from the restrictions $p > 1$ and $q > 1$ which come in when we use (10). In the unweighted case (and even in the case that u is of type (8) with $\beta < 1$) these restrictions can be removed by using some inequalities of the Nikol'skii type (cf., for instance, [9] and [14]). Unfortunately, these inequalities do not hold in $L_p(u)$-spaces with weight $u(x) = \exp(\pm|x|)$ (cf. [9, Subsection 1.4.6]).

Remark 2. Let $u_1(x) = 1$, $x \in \mathbb{R}^n$. Then $F^s_{pq}(u_1) = F^s_{pq}$. This follows from Theorem 1 and Remark 1.

Now we deal with the question whether $\|uf \mid F^s_{pq}\|$ yields an equivalent norm in $F^s_{pq}(u)$. Therefore, we need to modify the class of admissible weight functions.

Definition 6. By W^∞_e we denote the collection of all $u \in W_e \cap C^\infty$ with the following property. For each multi-index α there is a $c > 0$ such that

$$|D^\alpha u(x)| \le cu(x), \quad x \in \mathbb{R}^n.$$

Remark 3. (i) If $u \in W^\infty_e$, then $f \mapsto uf$ yields a linear and continuous map from S'_e into S'_e.

(ii) Let $\varphi \in D$ be such that $\varphi(x) \ge 0$, $x \in \mathbb{R}^n$, and $\varphi(0) = 1$. Let $u \in W_e$. Put $v = u * \varphi$. Then $v \in W^\infty_e$ and $c_1 v(x) \le u(x) \le c_2 v(x)$, $x \in \mathbb{R}^n$, for some positive constants c_1 and c_2.

Theorem 3. *Let $1 < p < \infty$, $1 < q \le \infty$, $s \in \mathbb{R}$, and $u \in W^\infty_e$. Then $f \mapsto uf$ yields an isomorphic map from $F^s_{pq}(u)$ onto F^s_{pq}. In particular, $\|uf \mid F^s_{pq}\|$ is an equivalent norm on $F^s_{pq}(u)$.*

Theorem 3 has been proved in [10]. As a basic tool we used an atomic decomposition of $F^s_{pq}(u)$. Concerning atomic decompositions of (unweighted) F^s_{pq} spaces, we refer to [5], [6], [7] and [14].

Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and $u \in W_e$. Then the weighted Sobolev space $W_p^k(u)$ is the set of all $f \in D'$ such that $D^\alpha f \in L_p(u)$, $|\alpha| \le k$. The norm on $W_p^k(u)$ is given by

$$\|f \mid W_p^k(u)\| = \Big(\sum_{|\alpha| \le k} \|D^\alpha f \mid L_p(u)\|^p \Big)^{1/p}.$$

The following Theorem 4 follows from (4) and Theorem 3.

Theorem 4. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and $u \in W_e$. Then $F_{p2}^k(u) = W_p^k(u)$.*

In particular, we have $F_{p2}^0(u) = L_p(u)$, which is the desired result from the introduction.

Remark 4. Let u be a weight of type (8) with $\beta < 1$. In this special case, Theorems 3 and 4 are due to J. Franke (cf. [4]).

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