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The EAFE Scheme and CWDD Method for Convection-dominated Problems

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1 Introduction

In this paper, we derive a monotone finite element scheme for convection diffusion equations and then discuss a special domain decomposition method for the solution of the resulting algebraic system. The work was partially motivated by Brezzi, Marini and Pietra [BMP89] and Markowich and Zlamal [MZ89] where a Scharfetter-Gummel type ([SG69]) of finite element scheme was derived for symmetric positive definite equations in two spatial dimensions. The finite element scheme in this paper is derived by a completely novel technique for a very general class of convection diffusion equations (see (2.1) below). Some error estimates like those in [MZ89] can be obtained in a very straightforward fashion under the new derivation (see Xu and Zikatanov [XZed]). This finite element scheme is monotone for some very general (unstructured) meshes such as Delauney triangulations in two dimensions. A monotone finite element scheme is important not only from the viewpoint of better stability and approximation but also from the fact that the resulting linear algebraic system may be more effectively solved. In this paper, we also discuss an efficient domain decomposition method for the resulting scheme.

We would like to point out that there is a vast literature on numerical methods for convection dominated problems and many special techniques have been developed, to name a few, for example, we refer to [BBFS90], [BMP89], [DEO92], [Hug95], [Joh87], [RST96] and the references cited therein. It is fair to say that all the different methods in the aforementioned papers are related in certain sense. In particular, the monotone schme described in this paper is also essentially similar to many other monotone schemes in the literature. But we emphasis that our scheme is derived in a quite different and elegant manner and it can be neatly applied to rather general unstructured grids in any spatial dimensions.

The rest of the paper is organized as follows. In we discuss a model problem and some properties of finite element discretization for the Poisson equation. In we derive

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an edge-average finite element scheme and discuss the monotonicity property of the scheme. In , we present some numerical example and discuss a *cross-wind strip* domain decomposition technique. We make some concluding remarks in .

2 Model Problems and Finite Element Spaces

We shall mainly consider the following model convection diffusion problem:

$$Lu \equiv -\nabla \cdot (\alpha(x)\nabla u + \beta(x)u) + \gamma(x)u = f(x) \; x \in \Omega \quad \text{and} \quad u = 0 \; x \in \partial\Omega \tag{2.1}$$

where Ω is a polygonal domain in $R^n (n \geq 1)$ with boundary $\partial\Omega$, $f \in L^2(\Omega)$. We assume that α, β and γ are bounded and piecewise smooth functions and $\alpha(x) \geq \alpha_{min} > 0, \gamma(x) \geq 0$ for every $x \in \Omega$.

The weak formulation of the problem (2.1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u, v) = \int_\Omega (\alpha(x)\nabla u + \beta(x)u) \cdot \nabla v + \gamma(x)uv \, dx, \qquad f(v) = \int_\Omega f(x)v dx. \tag{2.3}$$

It is well-known that (2.2) is uniquely solvable for any $f \in L^2(\Omega)$. Furthermore L^{-1} is a positive operator, namely (see Gilbarg and Trudinger [GT83])

$$(L^{-1}f)(x) \geq 0 \text{ for all } x \in \Omega, \text{ if } f(x) \geq 0 \text{ for all } x \in \Omega. \tag{2.4}$$

The above condition will be loosely referred as the *monotonicity property.*

We are interested in the convection dominated case, namely $\beta(x)/\alpha(x) \gg 1$ for all $x \in \Omega$. For the convection dominated case, it is well known that a standard finite element discretization scheme will not give satisfactory result (in fact the discrete solution will have a lot of oscillations), but a scheme possessing a discrete monotonicity property like (2.4) would have much better approximation and stability properties. One aim of this paper is to derive such kind of finite element scheme. To be more specific, if $V_h \subset H_0^1(\Omega)$ is a finite element space and L_h is the corresponding discretized operator for L given in (2.1), we are looking for special finite element scheme such that

$$(L_h^{-1} f_h)(x) \geq 0 \text{ for all } x \in \Omega, \text{ if } f_h \in V_h \text{ and } f_h(x) \geq 0 \text{ for all } x \in \Omega. \tag{2.5}$$

A finite element scheme satisfying the above condition will be known as *a monotone* finite element scheme in this paper.

M-matrix property of the stiffness matrix for Poisson equation As we shall see late that our monotone finite element scheme is closely related to the Poisson equation:

$$-\Delta u = f(x) \; x \in \Omega \quad \text{and} \quad u = 0 \; x \in \partial\Omega. \tag{2.6}$$

For completeness, we now give some details concerning the finite element discretization for this simple equation. In particular we discuss a necessary and sufficient geometric condition for the stiffness matrix to be an M-matrix.

Let $\mathcal{T}_h$ be a usual finite element triangulation of Ω consisting of simplices and $V_h \subset H_0^1(\Omega)$ be the corresponding finite element space consisting of continuous piecewise linear functions. Given an element $\tau \in \mathcal{T}_h$ with vertices q_j $(1 \le j \le n+1)$, let (a_{ij}^τ) be the element stiffness matrix of the Poisson equation, namely $(\nabla u_h, \nabla v_h)_\tau = \sum_{i,j} a_{ij}^\tau u_h(q_i) v_h(q_j)$. Noting that $a_{ii}^\tau = -\sum_{j \ne i} a_{ij}^\tau$, we deduce that

$$(\nabla u_h, \nabla v_h) = \sum_{\tau \in \mathcal{T}_h} \sum_{\mathrm{E} \subset \tau} \omega_{\mathrm{E}}^\tau \delta_{\mathrm{E}} u_h \delta_{\mathrm{E}} v_h. \tag{2.7}$$

where $\omega_{\mathrm{E}}^\tau = -a_{ij}^\tau$ with E connecting the vertices q_i and q_j and $\delta_{\mathrm{E}} = u_h(q_i) - u_h(q_j)$.

For $i \ne j$, let $|\kappa_{ij}|$ denote the measure of $n-2$ dimensional simplex, κ_{ij}, opposite to the edge $\mathrm{E} = (q_i, q_j)$ and θ_{ij} is the angle between the two $n-1$ dimensional simplexes whose intersection forms κ_{E}. Then it can be proved that (see Barth [Bar92] for the case $n = 3$)

$$a_{ij}^\tau = -\frac{1}{n(n-1)} |\kappa_{\mathrm{E}}| \cot \theta_{\mathrm{E}}. \tag{2.8}$$

Consequently the stiffness matrix for the Poisson equation is an M-matrix if and only if the for any fixed edge E the following condition holds:

$$\sum_{\tau \supset \mathrm{E}} |\kappa_{\mathrm{E}}| \cot \theta_{\mathrm{E}} \ge 0, \tag{2.9}$$

where $\sum_{\tau \supset \mathrm{E}}$ means the sum over all simplices containing E.

We would like to remark that, in two dimensions, the condition (2.9) means that $\mathcal{T}_h$ is a *Delauney triangulation* in the sense that the sum of any the two angles opposite to any edge is less than or equal to π.

3 Derivation of a Monotone Finite Element Scheme

In this section, we give a derivation of a monotone finite element scheme for convection diffusion equation (2.1). Unlike many other schemes, this scheme does not make explicit reference to the flow (convection) directions.

Given a triangulation $\mathcal{T}_h$ as described in previous section, let us introduce a function $\psi_{\mathrm{E}}(s)$ defined locally over any fixed edge $\mathrm{E} = (q_i, q_j)$ in $\mathcal{T}_h$ by

$$\frac{\partial \psi_{\mathrm{E}}}{\partial \tau_{\mathrm{E}}} = \frac{1}{|\tau_{\mathrm{E}}|} \alpha^{-1} (\beta \cdot \tau_{\mathrm{E}}). \tag{3.10}$$

Here $\tau_{\mathrm{E}} = q_i - q_j$ and $\frac{\partial}{\partial \tau_{\mathrm{E}}}$ denotes the tangential derivative along E.

Set $J(u) = \alpha \nabla u + \beta u$. Multiplying $J(u)$ by α^{-1}, taking the Euclidean inner product with the directional vector τ_{E} and using the definition of ψ_{E} in (3.10) we obtain:

$$e^{-\psi_{\mathrm{E}}} \frac{\partial (e^{\psi_{\mathrm{E}}} u)}{\partial \tau_{\mathrm{E}}} = \frac{1}{|\tau_{\mathrm{E}}|} \alpha^{-1} (J(u) \cdot \tau_{\mathrm{E}}). \tag{3.11}$$

After integration over edge E, we obtain the following identity:

$$\delta_{\mathrm{E}}(e^{\psi_{\mathrm{E}}}u) = \frac{1}{|\tau_{\mathrm{E}}|}\int_{\mathrm{E}} \alpha^{-1} e^{\psi_{\mathrm{E}}}(J(u)\cdot\tau_{\mathrm{E}})ds. \tag{3.12}$$

The crucial step of our derivation is to assume that $J(u)$ is approximated by a constant vector, say $J_\tau(u)$, over each simplex τ. Once this (only) extra assumption is made, then from (3.12), this constant vector may be related by

$$J_\tau(u)\cdot\tau_{\mathrm{E}} \approx \tilde{a}_{\beta,\mathrm{E}}\delta_{\mathrm{E}}(e^{\psi_{\mathrm{E}}}u), \qquad \tilde{a}_{\beta,\mathrm{E}} = \left[\frac{1}{|\tau_{\mathrm{E}}|}\int_{\mathrm{E}} \alpha^{-1}e^{\psi_{\mathrm{E}}}ds\right]^{-1}. \tag{3.13}$$

Now using the representation from (2.7) we have the following identity for $\phi_h \in V_h$

$$\int_\tau J_\tau(u)\cdot\nabla\phi_h = \sum_{\mathrm{E}} \omega_{\mathrm{E}}^{\tau}(J_\tau(u)\cdot\tau_{\mathrm{E}})\delta_{\mathrm{E}}\phi_h. \tag{3.14}$$

This leads to the following modified bilinear form:

$$a_h(u_h,\phi_h) = \sum_{\tau\in\mathcal{J}_h}\left\{\sum_{\mathrm{E}\subset\tau}\omega_{\mathrm{E}}^{\tau}\tilde{a}_{\beta,\mathrm{E}}\delta_{\mathrm{E}}(e^{\psi_{\mathrm{E}}}u_h)\delta_{\mathrm{E}}\phi_h + \gamma_\tau(u_h\phi_h)\right\}, \tag{3.15}$$

where a lumped mass quadrature rule is used for the zero order term in $a(\cdot,\cdot)$:

$$\gamma_\tau(u\phi_h) = \frac{|\tau|}{n+1}\sum_{i=1}^{n+1}\gamma(q_i)u(q_i)\phi_h(q_i).$$

The resulting finite element discretization is: Find $u_h \in V_h$ such that

$$a_h(u_h,\phi_h) = f(\phi_h) \qquad \text{for all} \quad \phi_h \in V_h. \tag{3.16}$$

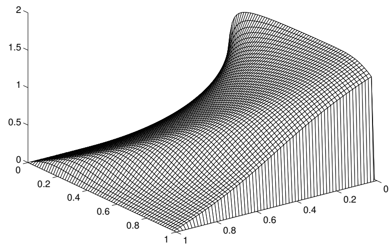
The above derivation appears quite simple, but the resulting scheme has a remarkable (but obvious) monotonicity property.

Lemma 3.1 *The stiffness matrix corresponding to the bilinear form (3.15) is an M-matrix for any $\alpha > 0$, $\gamma \geq 0$ and β, if and only if the stiffness matrix for the Poisson equation is an M-matrix, namely if and only if the condition (2.9) holds.*

This means that our finite element scheme (3.16), regardless the behavior of the convection coefficient β, satisfies a desirable monotonicity property for a very general class of meshes such as Delauney triangulations in two dimensions. Note that our derivation makes little use of any specific property of β. In the very special case that $\beta/\alpha = \nabla\psi$ for some function ψ (which means that (2.1) is symmetrizable) and $n = 2$, the scheme (3.16) is reduced to the Scharfetter-Gummel scheme derived in [MZ89], although our derivation technique is completely different.

We would like remark that the exponential functions in the (3.16) would not cause any numerical problems if they are handled with caution. For detailed discussions, we refer to [XZed] and [WX].

Figure 1 Surface plot of the discrete solution to (4.18).



Formal convergence rate It is well known that monotone schemes can only have first order accuracy in general. It can be easily shown that our scheme admits the following *formal* error estimate:

$$|u_I - u_h|_{H^1(\Omega)} \le Ch(|J(u)|_{W^{1,p}(\Omega)} + |\gamma u|_{W^{1,p}(\Omega)}, \tag{3.17}$$

where u_I is the usual finite element interpolant of the solution of the problem (2.1) and $p > n$. (Details of this error analysis are reported in [XZed]). The error estimate (3.17) is formal since the constant C and the norm $|J(u)|_{1,p,\Omega}$ there depend on α and β. It is interesting to note that this monotone finite element scheme gives a first order (formal) accuracy in H^1 norm.

4 Numerical Examples and the CWS Method

In this section we first give a simple but not trivial example of convection dominated problem to demonstrate the behavior of our finite finite element scheme and then briefly discuss a domain decomposition strategy.

First numerical example We consider the following test problem:

$$-\nabla \cdot (\varepsilon \nabla u + (y, -x)u) = 1, \tag{4.18}$$

subject to the homogeneous Dirichlet boundary conditions on unit square $(0,1)\times(0,1)$. Note that the partial differential operator here is not symmetrizable.

Shown in Figure 1 is a finite element solution obtained on a uniform triangular grid on the unit square with $\varepsilon = 10^{-6}$ and $h = 2^{-6}$ $(h/\varepsilon = 15625)$. The quality of the numerical solution looks quite good and no spurious oscillations or smearing near the boundary layer are observed.

A cross-wind strip domain decomposition strategy It is known that monotone schemes may be solved effectively by Gauß-Seidel methods if the unknowns are properly ordered see, for example, [BY92], [EC93], [Far89], [BWnt]. But for unstructured grids, an optimal ordering can be very difficult to realize. Here we give a brief discussion of a special domain decomposition strategy that proves to

Figure 2 Crosswind blocking with $\beta = (y, -x)$ and an unstructured grid.

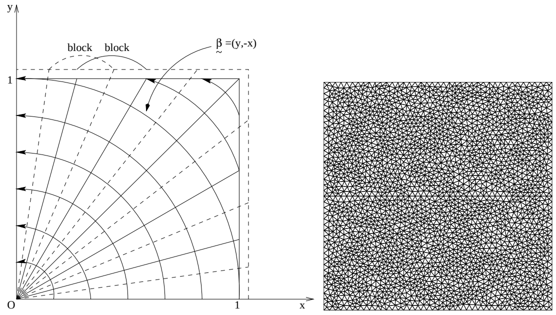


Table 1 Performance of CWS method for (4.18 on unstructured meshes

ε	h	#nodes	CWS method	
			iteration count	CPU time (seconds)
10^{-5}	1/20	729	1	0.004
	1/40	2839	2	0.27
	1/80	11193	3	1.7

be quite effective for unstructured grids with variable convections. More details of this algorithm can be found in Wang and Xu ([WX]).

We here describe the strategy for the two dimensional case. The idea is to decompose the computational domain into very thin strips which are orthogonal to the convection direction given by β. Figure 2 illustrates the case for $\beta = (y, -x)$ as in the previous numerical example. Corresponding to each of these strips, a subdomain can be defined by the union of supports of the basis functions associated with the nodes belonging to the strip. We then carry out the multiplicative Schwarz or successive subspace correction ([Xu92]) domain decomposition method to the aforementioned domain decomposition (possibly with overlapping) with an order following the direction of β. We use banded Gaussian elimination on each subdomain since the strip is very thin and the stiffness matrix associated with each subdomain is a banded matrix with a very small bandwidth. We call this kind of method to be the *cross-wind strip* (CWS) domain decomposition method.

Table 1 shows the performance of CWS method for problem (4.18) discretized by our monotone finite element scheme on unstructured grids on the unit square. The computation was carried on an DEC Alpha-Workstation 5000/240. The iteration stops when the maximum norm of residual vector reaches 10^{-8}. The mesh size h shown in the table is the characteristic size of each mesh. The right hand of plot in Figure 2 is the unstructured grid with characteristic size $h = 1/40$. As we see that the CWS converges extremely fast.

5 Concluding Remarks

The EAFE (edge-average finite element) scheme derived in this paper proves to be an effective approach to discretizing convection dominated problems. The derivation of this scheme is simple and is valid in all spatial dimensions. The monotonicity property of the scheme is uniformly valid for any feasible diffusion and convection coefficients and any size of meshes (as long as satisfying some mild geometric constraints such as being Delauney triangulations in two dimensions).

The CWS (cross-wind strip) domain decomposition method provides a very efficient approach to solving the algebraic systems resulting from a monotone scheme. Instead of using elaborative techniques for ordering the unknowns, the CWS method makes more but simple use of geometric property of the underlying meshes in association with convection coefficients. For strongly convection dominated problems, this method converges in very few iterations. For mildly convection dominated problems or for problems with sharply variable convections, the CWS method may be used as an effective smoother in a multigrid process (see [WX]).

Like any other schemes for convection dominated problems, the convergence analysis for EAFE method is a very technical task. The theoretical justification of the efficiency of CWS method for general unstructured grids is also lacking. These theoretical issues will be addressed in our future work. More importantly, our main goal is to develop efficient multigrid methods and some satisfactory corresponding theory for convection dominated equations and hyperbolic problems in general, which in fact is the main motivation of the current work.

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