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Domain Decomposition Methods Applied to Sedimentary Basin Modeling

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1 Introduction

Basin modeling aims to reconstruct the evolution of a sedimentary basin taking into account compaction of the porous medium, hydrocarbon formation, and migration. A sedimentary basin is a heterogeneous porous medium consisting of stacked stratigraphic layers that have been deposited from the start of the basin up to the present. Over time, solid organic material contained in some stratigraphic layers is transformed into mobile hydrocarbons under the effect of increased temperature. Then, hydrocarbons flow in the porous medium to accumulate in reservoirs. This migration takes place as a two-phase or three-phase flow.

Up to now, most of basin simulators have been able to handle relatively simple geometries resulting from deposition, erosion, and compaction of the porous medium [UBD+90]. However, most of real life basins are cut by faults along which block displacements can occur. The aim of Ceres project[1] is to model three-phase flow in a 2D section of a basin, whose geometry changes due to deposition, compaction, erosion of sediments, and block displacements along faults. In order to handle these complex geometries, Domain Decomposition (DD for short) techniques have been chosen. Indeed, faults cut the basin into domains that naturally define computational subdomains. In this first stage of the project, a simplified problem is considered in which only one-phase flow can occur and where faults are vertical. Equations that govern the physical phenomena are then discretized using a Finite Volume method and the subdomains are coupled by a nonoverlapping alternating method with interface relaxation [QV91].

The paper is organized as follows. We first review the mathematical formulation of the physical phenomena that are taken into account. Next the discretization and the DD method that have been chosen are presented. Finally, numerical results are shown.

1 Ceres is a joint project between the oil companies Amoco, British Petroleum, Elf, and the Institut Français du Pétrole.

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2 Governing Equations

At present, we consider only incompressible one-phase (water) fluid flow in a 2D section of a sedimentary basin. The set of partial differential equations that govern the phenomena is the following one — subscripts w and s designate the water phase and the solid phase, respectively.

Mass conservation of solid and of water are written as:

$$\frac{\partial}{\partial t}\Big(\rho_s(1-\phi)\Big) + \mathrm{div}\Big(\rho_s(1-\phi)\overrightarrow{V}_s\Big) = q_s, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho_w\phi) + \mathrm{div}(\rho_w\phi\overrightarrow{V}_w) = q_w, \tag{2.2}$$

where ρ_α is the density of phase α, ϕ is the porosity of the medium, $\overrightarrow{V}_\alpha$ is the velocity of phase α, q_α the quantity of phase α deposited on the top of the basin during sedimentation (or removed during erosion).

Darcy's law is written as:

$$\overrightarrow{U_w} = \phi(\overrightarrow{V_w} - \overrightarrow{V_s}) = -\frac{\overline{\overline{K}}}{\mu}\left(\overrightarrow{grad}P - \rho_w\overrightarrow{g}\right), \tag{2.3}$$

where P is the pressure of water, $\overrightarrow{g}$ is the gravitational acceleration vector, μ_w the viscosity of water. $\overline{\overline{K}}$ is the intrinsic permeability tensor of the porous medium, and depends heavily on the lithology under consideration. It can vary by several orders of magnitude — up to four — from one layer to the other.

Compaction of the porous medium is supposed to be merely vertical (the horizontal component of the solid velocity is zero). Mechanical equilibrium is then written as:

$$\frac{\partial\sigma}{\partial z} = (\phi\rho_w + (1-\phi)\rho_s)g, \tag{2.4}$$

where σ is the total vertical stress. Compaction is described by the following rheology:

$$\phi = \phi^o exp(-(\sigma - P)/\sigma_e^o). \tag{2.5}$$

The problem is therefore defined by a system whose main unknowns are ϕ, P, σ, $V_{s,z}$ and $\overrightarrow{U_w}$. The principal equations are the two conservation laws, the rheology, the mechanical equilibrium and Darcy's equations. Introducing (2.3) and (2.5) in (2.2) show that the equations are nonlinear parabolic with respect to the pressure P.

3 Discretization

Although the model we are considering is rather simple, we want to choose a method that can be easily extended to more complex fluid flow model and especially compressible three-phase flow. Therefore, the method chosen to discretize the conservation equations is a Finite Volume method [FWS96].

We give in this section, first the main characteristics of the discretization, then some details about the water conservation equation discretization and especially the flux approximation. In this first stage of the project, we consider only subdomains which are separated by vertical interfaces and as compaction is vertical, the coupling between two subdomains only affects the water conservation equation, that is to say, the flux approximation.

Main Characteristics

A grid is chosen that follows the stratigraphic layers. Each cell is homogeneous and the boundaries between two layers correspond to a series of interfaces of adjacent cells. A cell is a quadrangle whose vertices are located along vertical lines. As the grid deforms with the solid skeleton, vertical compaction constrains cell vertices to move only vertically and thus to remain along vertical lines.

The discretization methods are listed in the Table 1, where the discrete unknowns are also specified. We denote by the subscript k an unknown located at the center of the cell Ω_k and by the subscript δ an unknown associated to a vertical edge δ. P_δ is an auxiliary unknown which is given by the flux continuity on each vertical edge as a function of the pressures in the neighboring cells. At each time step, the discretized equations form a system which is nonlinear for the discrete unknowns ϕ_k, P_k, σ_δ, ϕ_δ. It is solved using a Newton method where at each iteration, the system reduces to a linear system in the unknowns P_k, σ_δ.

Table 1 Discretization methods and discrete unknowns.

Equation	Discretization	Discrete unknown
Solid Conservation (2.1)	Finite Volume	ϕ_k
Water Conservation (2.2)	Finite Volume	P_k
Mechanical Equilibrium (2.4)	Finite Difference	σ_δ
Rheology (2.5)		ϕ_δ
Darcy's law and Flux continuity	Finite Difference	P_δ (auxiliary unknown)

Flux Approximation

Following the Finite Volume principle [EGHon], the water conservation equation is integrated over each cell Ω_k. As the cell evolves at the velocity $\overrightarrow{V}_s$, it gives:

$$\frac{d}{dt}\int_{\Omega_k} \phi d\omega + \sum_{\delta\subset\delta\Omega_k}\int_\delta \overrightarrow{U}_w \cdot \overrightarrow{n} d\gamma = \int_{\Omega_k} q_w d\omega \tag{3.6}$$

where $\overrightarrow{n}$ is the outward normal to cell Ω_k and $\overrightarrow{U}_w$ is given by (2.3).

In order to give some details about the flux approximation, we assume for the sake of simplicity that the mesh is composed of rectangles and that the permeability tensor is diagonal. We denote by K_s and K_a the two diagonal coefficients of $\overline{\overline{K}}$ and by F_δ the flux approximation $-l_\delta \frac{\overline{\overline{K}}}{\mu}(\overrightarrow{grad}P - \rho_w \overrightarrow{g}).\overrightarrow{n}$, where l_δ is the edge length. Let us distinguish the case where the edge is located inside a subdomain and the case where it is on the boundary:

- *Edge located inside a subdomain*
  We consider an edge δ and Ω_a, Ω_b its two adjacent cells (see Fig. 1(a)). P_a

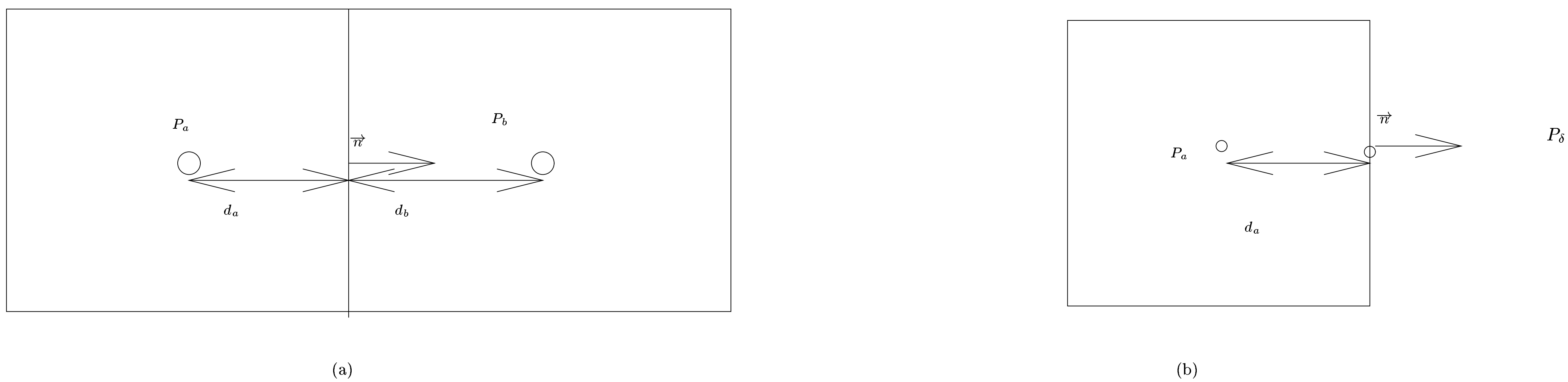
and P_b are the unknown pressures in cells Ω_a, Ω_b. Taking into account the eventual discontinuity of K_s from cell Ω_a to cell Ω_b and expressing the flux continuity on edge δ leads to the following approximation [FWS96]:

$$F_\delta = -l_\delta \overline{\frac{K_s}{\mu}} \frac{P_b - P_a}{d_b + d_a},$$

where $\overline{\frac{K_s}{\mu}}$ is the harmonic mean of $\frac{K_s}{\mu}$ weighted by the distances d_a and d_b.

- *Edge located on the boundary of a subdomain*
  We consider an edge δ located on the boundary of the subdomain, Ω_a its adjacent cell and P_δ the pressure at the center of δ (see Fig. 1(b)).
  If a Dirichlet boundary condition is set on δ, the value of P_δ is given by the boundary condition and the flux is approximated by:
  $$F_\delta = -l_\delta \left(\frac{K_s}{\mu}\right)_a \frac{P_\delta - P_a}{d_a}.$$
  If a Neumann boundary condition is set on δ, the value of F_δ is given by the boundary condition and the pressure is approximated by:
  $$P_\delta = -\frac{d_a}{l_\delta} \left(\frac{\mu}{K_s}\right)_a F_\delta + P_a.$$

Figure 1 (a) Edge inside a subdomain, (b) Edge on the boundary.



4 Domain Decomposition Method

As for the discretization method, we want to implement a DD method that can be easily extended to more complex model. Moreover, even for the model considered here, the set of equations is non linear. Therefore, we have chosen a method which does not require too many properties of the equations such as linearity, symmetry, etc. This method is nonoverlapping DD method with Dirichlet-Neumann sweep and interface relaxation of the Dirichlet condition as suggested in [QV91]. It is directly applied to the nonlinear system of discretized equations. We describe first the main steps of the algorithm, then the choice of the relaxation parameter.

Algorithm

Let us consider a basin divided into nonoverlapping subdomains Ω_1 and Ω_2 and denote by Γ the interface between the two subdomains. It can be shown that the problem on the global domain is equivalent to the problem on the two subdomains if and only if pressure and flux are continuous through the interface. Once discretized, the global domain and subdomains problems are equivalent if and only if

$$\forall \delta \subset \Gamma, \quad P_\delta^1 = P_\delta^2 \text{ and } F_\delta^1 = -F_\delta^2,$$

where the superscript i refers to a quantity related to subdomain Ω_i for $i = 1, 2$.

The algorithm takes the values of the pressure on the interface as main unknowns: λ_δ, $\delta \subset \Gamma$. It is an iterative algorithm in which, for each iteration k, the following stages are executed:

- Solve $\begin{cases} \text{PDE system in } \Omega_1 \\ \text{BC on } \delta\Omega_1 \backslash \Gamma \\ P_\delta^1 = \lambda_\delta^k, \ \forall \delta \subset \Gamma \end{cases}$ then solve $\begin{cases} \text{PDE system in } \Omega_2 \\ \text{BC on } \delta\Omega_2 \backslash \Gamma \\ F_\delta^2 = -F_\delta^1, \ \forall \delta \subset \Gamma \end{cases}$
- Update λ: $\quad \lambda_\delta^{k+1} = (1-\theta)\lambda_\delta^k + \theta P_\delta^2 \quad \forall \delta \subset \Gamma,$
  where θ is the relaxation parameter.

Relaxation Parameter

As shown by the numerical tests (see Section 5), the optimal value of the relaxation parameter, i.e., the value for which the number of iterations is minimal, strongly depends on the basin characteristics (permeability heterogeneities). Therefore, it is necessary to implement an algorithm which automatically computes this parameter. The algorithm chosen here is the one suggested by A. Quarteroni and presented in [HK92], which computes the relaxation parameter θ^k at each iteration k in the following way. Defining error functions:

$$\begin{aligned} e_\delta^{1,k} &= P_\delta^{1,k} - P_\delta^{1,k-1}, \quad e_\delta^{2,k} = P_\delta^{2,k} - P_\delta^{2,k-1}, \\ z_\delta^k(\theta) &= \theta P_\delta^{2,k} + (1-\theta) P_\delta^{1,k} \quad \text{for} \quad \delta \subset \Gamma, \end{aligned}$$

the unique number which minimizes $||z^k(\theta) - z^{k-1}(\theta)||^2 = \sum_{\delta \subset \Gamma} (z_\delta^k(\theta) - z_\delta^{k-1}(\theta))^2$

is

$$\theta_{opt}^k = \frac{\sum_{\delta \subset \Gamma} e_\delta^{1,k}(e_\delta^{1,k} - e_\delta^{2,k})}{\sum_{\delta \subset \Gamma} (e_\delta^{1,k} - e_\delta^{2,k})^2}.$$

5 Numerical Results

The results presented here have been obtained for a basin which is already deposited and that is compacting under the effect of a vertical stress applied at the top of the basin. We first consider basins with a simple lithologic composition for which we study the behavior of the DD method. Then, we are interested in more complex basins divided in several subdomains.

Simple Basins

Two basins of 10 layers, divided in 2 subdomains (10 interface edges), are considered:

- a homogeneous basin only composed of shales, that is to say, of impervious sediments.
- a basin composed of two homogeneous subdomains; the first one is made of shales and the second one of sandstones, that is to say, of pervious sediments. There are four orders of magnitude between the permeability in the two subdomains. The shales subdomain is the one on which a Dirichlet boundary condition is set.

The results, in terms of the maximum number of iterations needed during the different time steps of the simulation, are presented in Table 2. A star indicates that the algorithm does not converge. Each column gives the results for a certain value of the relaxation parameter. When a real value is written, it means that this value has been kept constant during all the simulation. The column corresponding to θ_{opt} gives the results obtained with the algorithm presented in the previous section. For these simple cases, a satisfying value of θ would have been 0.5. The next set of results shows that this is no longer the case for complex basins. Moreover, the results obtained with the dynamical computation of θ are also very good. For the second basin, it should however be noticed that, if a Dirichlet boundary condition is set on the sandstones domain, the DD algorithm does not converge, and this, even for very small value of θ. The Neumann boundary condition has to be set on the more pervious subdomain.

Table 2 Simple basins: number of iterations for different values of θ.

θ	1	0.5	0.25	0.05	θ_{opt}
Homogeneous Shales	*	3	17	117	4
Shales-Sandstones	2	19	42	210	4

Complex Basins

We now consider more complex basins composed of blocks separated by faults. Blocks and faults are computational subdomains and there is only one column of cells in each fault. The blocks consist in alternated shales and sandstones layers while the faults are made of sandstones. The following basins are considered:

- 2 blocks of 5 layers, separated by 1 fault (3 subdomains, 10 interface edges)
- 2 blocks of 10 layers, separated by 1 fault (3 subdomains, 20 interface edges)
- 3 blocks of 10 layers, separated by 2 faults (5 subdomains, 40 interface edges)
- 3 blocks of 20 layers, separated by 2 faults (5 subdomains, 80 interface edges)
  This last basin is represented in Figure 2.

For all these tests, Neumann boundary conditions have been set on the two boundaries of each fault. The results are presented in Table 3, in the same way as in Table 2. The evolution of the relaxation parameter and of the corresponding error for the basin of Figure 2 is represented on Figure 3. The results show that the dynamical computation of the relaxation parameter represents an important gain in the number of iterations compared to a constant value of θ. Moreover, the number of iterations increases slowly with the number of subdomains and of interface unknowns.

6 Conclusion

A DD method has been implemented for modeling one-phase flow in a sedimentary basin. The equations are discretized by a Finite Volume method and the DD method is a nonoverlapping DD method with Dirichlet-Neumann sweep and interface relaxation. This method gives satisfactory results for simple and rather complex basins. It is currently being extended to non-matching meshes and to three-phase fluid flow.

Table 3 Maximum number of iterations for different values of θ.

θ	0.25	0.05	θ_{opt}
5 layers, 3 subdomains	25	137	7
10 layers, 3 subdomains	26	144	10
10 layers, 5 subdomains	*	125	16
20 layers, 5 subdomains	*	126	23

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Figure 2 Complex basin: 3 blocks of alternated shales and sandstones layers separated by 2 faults of sandstones.

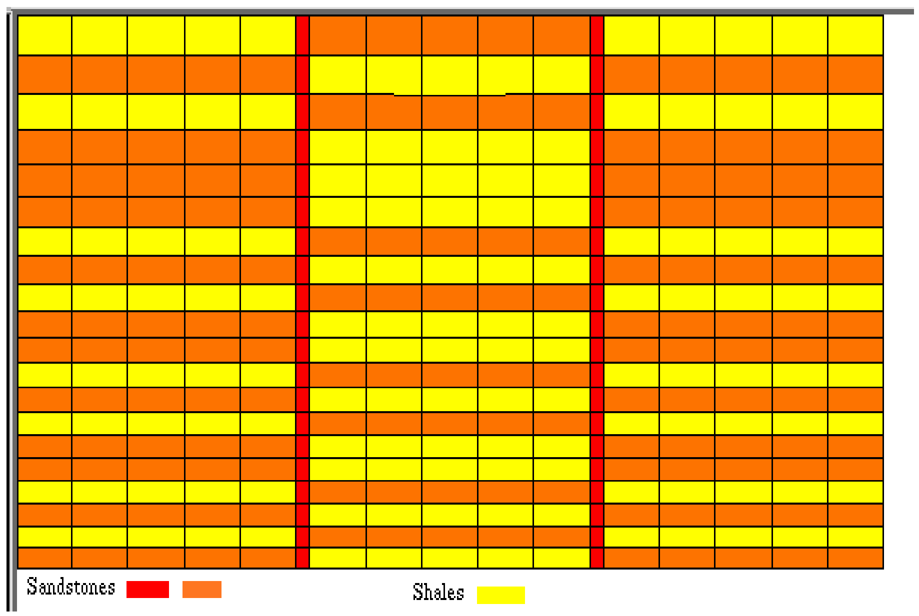
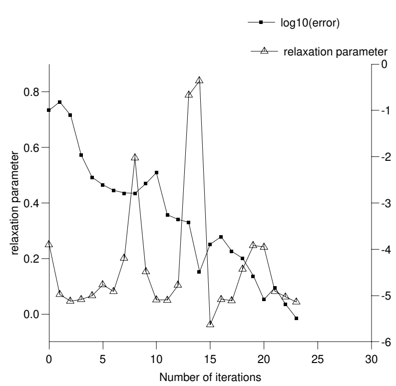


Figure 3 Evolution of the relaxation parameter and of the corresponding error during one time step of the simulation for the basin shown on Fig. 2.



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