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One-level Krylov-Schwarz Domain Decomposition for Finite Volume Advection-diffusion

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1 Introduction

We consider the two-dimensional advection-diffusion equation discretized with cell-centered finite volumes and the trapezoidal time integration scheme. Several aspects of Krylov-Schwarz domain decomposition will be discussed. The name Krylov-Schwarz refers to methods in which Schwarz domain decomposition is used as a preconditioner for a Krylov subspace method, see the preface of [KX95].

Our interests in domain decomposition are more practical than theoretical. Our goals are towards solving large-scale 'real-life' problems. As such, it is important to work with a method that is not too difficult to implement. One-level Schwarz methods belong to this class. Our attention is drawn by nonoverlapping Schwarz methods (sometimes called Schwarz with minimal overlap), because the use of nonoverlapping subdomains facilitates the implementation. Of course, there is a price to be paid; the number of ddm-iterations will grow if the number of subdomains increases. With a two-level method this might be improved. However, it is not easy to formulate and implement the coarse grid correction. In a parallel environment it is even an open question whether the elapsed time will actually go down if a coarse grid correction is included. Moreover, we are dealing with a time-dependent problem and there is some evidence, see [Cai91], that the behaviour of one-level methods is quite acceptable in this case. Our numerical experiments will confirm this statement to some extent.

In this paper the emphasis is on the description and performance of the domain decomposition iteration. We can only briefly touch upon our experiences with the present method for solving practical problems on parallel computers. We refer the interested reader to [VWMF96] for details on this issue.

The outline of this paper is as follows: Section 2 describes the equations and

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the discretization. In Section 3 we formulate the interface equations, which establish a reduction of the Schwarz-preconditioned system to a small set of equations involving interface variables only. Section 4 describes our main field of application, single phase tracer flow in a porous medium. In Section 5 we present some numerical results concerning the scalability properties of the ddm-iteration. Finally, Section 6 draws some conclusions and makes some remarks concerning future research.

2 Equations and Discretization

We consider a scalar conservation law of the form

$$’\frac{\partial c}{\partial t} + \nabla . \left[vc - D \nabla c \right] = 0 \ , \ x \in \Omega \in \mathbb{R}^2 \ , \ t > 0. \tag{2.1}$$

The coefficients ’, v and D are time-independent. We are interested in the advection-dominated case, i.e. the diffusion tensor D depends on small parameters. For the spatial discretization we employ the cell-centered finite volume method. This leads to the semi-discrete system

$$M \frac{dc}{dt} = Bc. \tag{2.2}$$

M is a diagonal matrix, containing the cell values of the coefficient ’ multiplied with the area of the cell.

B is a sparse matrix with nonzero entries at places defined by the molecule of the discretization; for simple discretizations B is a constant matrix. In order to be able to capture regions with high gradients, advanced nonlinear approximations of the advection terms are often needed, involving sensors and switch functions or limiters. In such a case the entries of B depend on c. We assume that B is a differentiable function of c, which enables us to reach the highest level of time accuracy. We set

$$J = \frac{\partial Bc}{\partial c}. \tag{2.3}$$

For the time-discretization of (2.2) we employ the linearly implicit trapezoidal rule in delta- or incremental formulation, i.e.

$$\left(\frac{M}{\tau_n} - \frac{1}{2} J^n \right) \delta^n = (Bc)^n \ , \ \delta^n = c^{n+1} - c^n. \tag{2.4}$$

Here, τ_n denotes the time step. The scheme is second-order accurate in time.

Omitting indices, (2.4) is denoted with

$$A\delta = b \ , \quad A = \frac{D}{\tau_n} - \frac{1}{2} J^n \ , \quad b = (Bc)^n \ . \tag{2.5}$$

The purpose of this paper is to discuss the iterative solution of (2.5) by means of an one-level Krylov-Schwarz domain decomposition method with ILU-preconditioned Bi-CGSTAB for the inversion of the subdomain problems.

3 The Interface Equations

The domain Ω is divided into p nonoverlapping subdomains $\Omega_q, \quad q = 1, ..., p$. The interfaces between the subdomains are an ensemble of some edges of the cells of the finite volume mesh. The nodal points are in the center of the cells (cell-centered finite volumes) and this means that there are no nodal points on the interfaces between subdomains. The set of nodal points is denoted with $z_i, \quad i \in I$. Define the disjunct index sets I_q such that $i \in I_q$ if $z_i \in \Omega_q$. We may partition the matrix A according to these index sets and next define the block Jacobi iteration matrix N. For the sake of simplicity we take $p = 2$ (two subdomains) and in this case there holds

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} , \quad N = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} . \tag{3.6}$$

The block Jacobi matrix N is put forward as the preconditioner, i.e. we consider

$$N^{-1} A \delta = N^{-1} b \, . \tag{3.7}$$

It is well-known, e.g. see [SBG96], that nonoverlaping additive Schwarz and block Jacobi, such as presented above, present identical preconditioners.

In classical substructuring or Schur complement methods the final equations to be solved are formulated in terms of the unknowns associated with nodal points on the interfaces. It is known and used for theoretical purposes that some of the substructuring methods allow for a formulation as a Schwarz method, see [SBG96] and quoted references. It is also possible to go the other way around and to reduce the Schwarz system (3.7) to a smaller set of equations. This has some practical advantages, e.g. the memory requirements (put forward by GMRES) are minimized and the parallel implementation is easier, see [VWMF96]. In the remainder of this section we will briefly describe the main idea behind the reduction, further details may be found in [BW95].

Interface cells are defined as those cells of the finite volume mesh of which the discretization molecule crosses one or more interfaces between subdomains. All other cells are called interior cells. In Figure 1 the interface cells are sketched in the case of a structured quadrilateral finite volume mesh with a 9-point discretization molecule. Figure 2 does the same for an unstructured triangular finite volume mesh with a 10-point discretization molecule. Variables associated with nodal points from the interface

Figure 1 Interface cells, 9-point molecule, quadrilaterals.

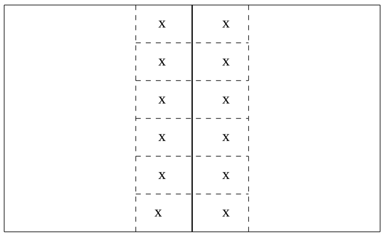
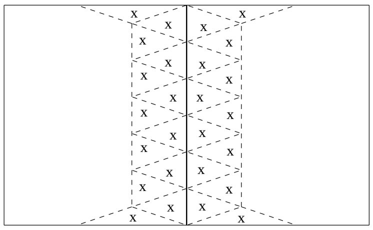


Figure 2 Interface cells, 10-point molecule, triangles.



cells are called interface variables and all other variables are called interior variables.

With α we denote the vector of interface variables, with β the vector of interior variables, and we partition the vector δ of unknowns accordingly, i.e. $\delta = [\alpha,\ \beta]^T$. Let $R = [I\ 0]$ and $Q = [0\ I]$ be the restriction matrices mapping δ to α, respectively δ to β; R^T and Q^T are the corresponding trivial injection matrices.

Now, let δ be arbitrary and set $\alpha = R\delta,\ \beta = Q\delta$. The block Jacobi matrix N and the original matrix A are identical for rows corresponding to interior cells. This means that

$$(N - A)Q^T\beta = 0\ , \quad (N - A)\delta = (N - A)R^T\alpha\ . \tag{3.8}$$

As a consequence (3.7) is equivalent with

$$\begin{bmatrix} P & 0 \\ T & I \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}, \tag{3.9}$$

with

$$P = RN^{-1}AR^T\ , \quad T = QN^{-1}AR^T\ , \quad g = RN^{-1}b\ , \quad h = QN^{-1}b\ .$$

From (3.9) we see that (3.7) can be reduced to the interface equations

$$P\alpha = g\ . \tag{3.10}$$

GMRES is applied to the reduced system (3.10).

The interface equations (3.10) have been described starting from a basic situation. Generalizations are possible, see [BW95]. Here, we only mention that vertex-centered finite volumes can be treated as well. It suffices to double the unknowns on the interfaces and to augment the system (2.5) similar to [Tan92].

It can be shown that there is equivalence between the interface equations (3.10) and a certain preconditioned Schur system, see [WB95]. The Schur system encountered here is of a more general type than the one encountered in the classical substructuring method. Similar equivalence properties have been established in [BW89], [CG92].

4 Tracer Flow in a Porous Medium

Modeling of two-component single-phase miscible flow is an important issue in reservoir engineering, e.g. see [Ewi83], [Lak89], both per se or as a step towards understanding the numerical properties of more complex multi-phase and multi-component models. The mathematical model describing incompressible miscible flow consists of the transport equation (2.1) for the concentration of the solvent coupled with an elliptic equation for the pressure:

$$-\nabla.(a\nabla p) = 0\ ,\ x \in \Omega\ ,\ t > 0. \tag{4.11}$$

Darcy's law states that the advective velocity v in (2.1) can be obtained from $v = -a\nabla p$.

In the case of tracer flow the coefficient a in (4.11) does not depend on the concentration and/or time and the Darcy velocity is computed only once. Our numerical experiments concern a tracer flow. There holds $a = k/\mu$ with $k(x)$ the permeability of the reservoir and μ the viscosity. In the computations we have used strongly heterogeneous real-life permeability data provided by Agip S.p.A.

In [FQ96] it has been argued that a convenient numerical approximation can be based upon a triangular mixed finite method for (4.11) combined with a cell-centered unstructured finite volume approach for the transport equation (2.1), using the same grid. We employ the linear Brezzi-Douglas-Marini (BDM) element with piecewise constant pressure, e.g. see [BF92]. A variant of the JST-scheme, [JM86], [WFM94], has been adopted for the approximation of the advective terms. The JST-scheme is a central scheme stabilized with a nonlinear artificial dissipation term containing both harmonic and biharmonic operators. The scheme has originally been developed for the solution of nonlinear hyperbolic equations with emphasis on capturing shock fronts and high gradient internal layers with a moderate level of numerical diffusion, see [JST81].

Figure 3 Velocity field.

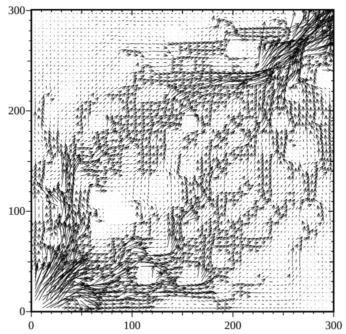
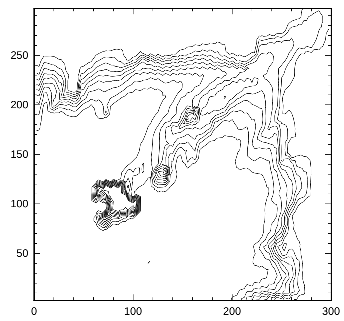


Figure 4 Concentration at .6 PVI.



Our test problem is the quarter of five-spots, a square region with an injection well in the lower left corner and a production well in the upper right corner. In Figure 3 we present the computed velocity field. Figure 4 shows the computed concentration: a front is moving from the injection well to the production well and the wiggles along some horizontal parts of the contour lines are caused by the plotting procedure. A maximal Courant number of $O(40)$ was used in the transport solver. Further physical details can be found in [WFM94]. Both figures illustrate nicely the complexity due to heterogeneity and it shall be clear that fine grids are indispensable in this type of applications.

5 Numerical Results

The square region Ω is divided into p equal sized block shaped subdomains. We fix the number of unknowns N in each subdomain, i.e. $N = 3200$. The total size n of the discrete problem is $n = pN$, i.e. n is growing linearly with the number of subdomains. In a parallel context this is called a memory constrained scaling methodology. Increasing the number of subdomain implies a decreasing spatial grid size and in such a case it is important to take application parameters into account. From the theory of hyperbolic difference schemes it is well-known that the Courant number is the vital similarity parameter and, therefore, we scale such that Courant numbers are fixed. This means that both the spatial grid size and the time-step τ will change if the number of subdomains varies. Domain decomposition methods are not studied often in such a context. Normally, the total size n of the discrete problem is fixed. Nevertheless, questions related to a growing problem size are important if the goal is to use a distributed parallel environment for enlarging the size of the problems treated.

Let us consider a single time step. With M_p we denote the number of matrix-vector multiplications with the matrix R from the interface equation (3.10). There holds $M_p = O_p + 1$, with O_p the number of GMRES ddm-iterations (outer iteration). The subdomain problems are solved iteratively with ILU-preconditioned Bi-CGSTAB (inner iteration). With $I_p(k)$ we denote the number of inner iterations in the $k-$th matrix-vector multiplication and with $\bar{I}_p$ the average number of such iterations, i.e.

$$\bar{I}_p = \frac{1}{M_p} \sum_{k=1}^{M_p} I_p(k).$$

Table 1 presents the measured O_p and $\bar{I}_p$, averaged over the full time interval. The number of ddm-iterations is moderate and increases at an acceptable rate if the number of subdomains grows. It should be remarked, that the overlap parameter H/h is a constant in Table 1 as a consequence of the scaling procedure. ILU-preconditioned Bi-CGSTAB turns out to be very effective, which confirms earlier investigations done in [WFM94].

Table 1 *O_p, the number of outer iterations and $\bar{I}_p$, the averaged number of inner iterations.*

	$p=4$	$p=9$	$p=16$	$p=25$
O_p	10.1	10.8	12.5	13.1
$\bar{I}_p$	2.6	3.4	3.9	4.5

Let T_1 denote the elapsed time per time step on a sequential computer. The final effect of the numbers found in Table 1 can be measured by means of the normalized sequential time T_1/p. The results can be found in Table 2. The loss of performance, caused by a degrading numerical efficiency, is quite moderate.

Table 2 *T_1/p, the normalized sequential time per time step.*

	p=4	p=9	p=16	p=25
	.63	.71	.87	.93

6 Final Remarks

One-level nonoverlapping Schwarz is one of the easiest to implement domain decomposition preconditioners. We have investigated the method for time-dependent advection-diffusion and the performance turns out to be satisfactory for practical purposes. The number of ddm-iterations, found in Table 1, is still somewhat high. A more advanced preconditioner, like adaptive Robin-Neumann, will certainly lead to an improvement at this point.

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