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A Characteristic Domain Decomposition Algorithm for Two-Phase Flows with Interfaces

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1 Introduction

The mathematical model that describes the process of an immiscible displacement of oil by water in reservoir production or other two-phase fluid flows in porous media leads to a strongly coupled system of a degenerated nonlinear advection-diffusion equation for saturation and an elliptic equation for pressure and velocity. The hyperbolic nature, strong coupling, and nonlinearity of the system and the degeneracy of the diffusion makes numerical simulation a challenging task. Many numerical methods suffer from serious non-physical oscillations, excessive numerical dispersion, and/or a combination of both [CJ86, Ewi84]. Previously, Espedal, Ewing, and coworkers developed a characteristic, operator-splitting technique in effectively solving two-phase fluid flow problems [DEES90, EE87]. In practice, a reservoir often consists of different subdomains with different porosities and permeabilities. In the case of single-phase fluid flows the concentration and total flux are continuous across the interfaces between different subdomains since the diffusion never vanishes. Our earlier work addressed numerical simulation to linear transport equations arising in single-phase flows with interfaces [WDE$^+$94]. However, in the case of two-phase fluid flows the saturation equation itself is nonlinear and different subdomains have different capillary pressure curves. The continuity of capillary pressure across interfaces implies a jump discontinuity of the water saturation at the same locations. The jump discontinuity of the saturation at the interfaces might incur some oscillations around the interfaces, which can be propagated into the domain and destroy the overall accuracy. Hence, great care has to be taken in the development of an effective solution procedure for the simulation of two-phase fluid flows in porous media with interfaces.

This paper describes a characteristic-based, non-overlapping domain decomposition algorithm for solving the saturation equation in two-phase fluid flows with interfaces. First, with the known saturation at the previous time step one obtains an approximate Dirichlet boundary condition at the outflow domain interface by integrating the

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saturation equation (ignoring the capillary pressure term) along characteristics. With the approximate outflow Dirichlet boundary condition at the domain interface and the given boundary condition at the physical inflow boundary one can close the system on the current subdomain and applies the characteristic operator-splitting procedure [DEES90, EE87] to solve the full saturation equation (including the capillary pressure effect). Second, one uses the continuity of capillary pressure across the domain interface to pass the value of saturation as an approximate inflow Dirichlet boundary condition to the next subdomain, one then applies the same characteristic operator-splitting procedure to solve the saturation equation on the current subdomain. Third, according to the overall loss or gain of mass one adjusts the approximate outflow Dirichlet boundary condition at the domain interface to iterate between different subdomains until the algorithm converges. Finally, a mixed method is adopted to solve the pressure equation due to its accurate approximation to the velocity field and its local mass conservation property.

The rest of the paper is organized as follows: In Sections 2 and 3 we formulate the problem and discuss related solution techniques. In Section 4 we present a domain decomposition algorithm for the two-phase fluid flow problems with interfaces. In Section 5 we present some numerical results to show the promise of the method.

2 Problem Formulation

A suitable mathematical model for the total Darcy velocity $\underline{u}$, the total pressure p, and the water saturation $S \in [0,1]$ in an incompressible displacement of oil by water in a porous medium can be described by the following set of partial differential equations [CJ86]:

$$
\begin{aligned}
\nabla \cdot \underline{u}(\underline{x},t) &= q_1(\underline{x},t), && (\underline{x},t) \in \Omega \times [0,T], \\
\underline{u}(\underline{x},t) &= -\underline{K}(\underline{x})(\lambda_o(S)+\lambda_w(S))\nabla p(\underline{x},t), && (\underline{x},t) \in \Omega \times [0,T], \\
\underline{u}(\underline{x},t) \cdot \underline{n}(\underline{x}) &= q_2(\underline{x},t), && (\underline{x},t) \in \partial\Omega \times [0,T],
\end{aligned}
\tag{1}
$$

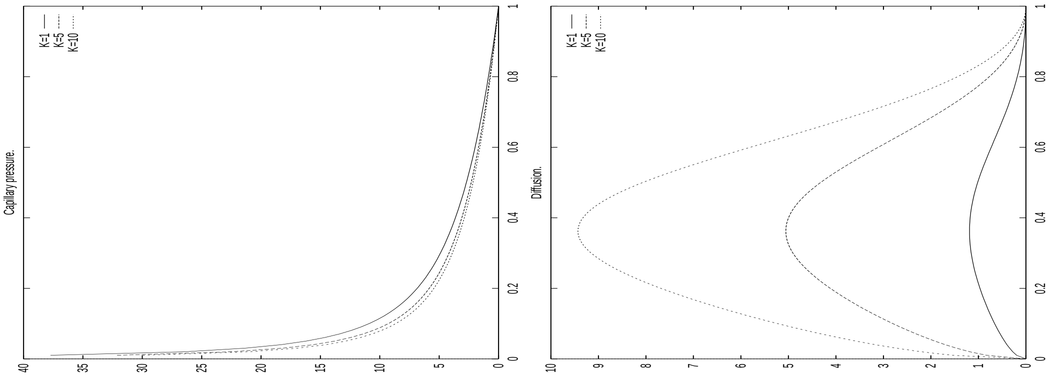
and

$$
\begin{aligned}
\phi(\underline{x})\frac{\partial S}{\partial t} + \nabla \cdot (f(S)\underline{u} - \varepsilon \underline{D}(S,\underline{x})\nabla S) &= q_3(\underline{x},t), && (\underline{x},t) \in \Omega \times [0,T], \\
(f(S)\underline{u} - \varepsilon \underline{D}(S,\underline{x})\nabla S) \cdot \underline{n}(\underline{x}) &= q_4(\underline{x},t), && (\underline{x},t) \in \partial\Omega \times [0,T], \\
S(\underline{x},0) &= S_0(\underline{x}), && \underline{x} \in \Omega,
\end{aligned}
\tag{2}
$$

where Ω is the physical domain, $\underline{K}(\underline{x})$ is the absolute permeability tensor of the medium, λ_i, $i = o, w$, denotes the water and oil mobilities respectively, $q_1(\underline{x},t)$ and $q_3(\underline{x},t)$ are source terms, $q_2(\underline{x},t)$ and $q_4(\underline{x},t)$ are the prescribed boundary conditions, $\underline{n}(\underline{x})$ is the unit outward normal vector, $\varepsilon << 1$ is a scaling factor to the diffusion term, p_c is the capillary pressure, and $f(S)$ and $\underline{D}(S,\underline{x})$ are the fractional flow function and diffusion term given by

$$
\begin{aligned}
f(S) &= \frac{\lambda_w(S)}{\lambda_w(S)+\lambda_o(S)}, \\
\underline{D}(S,\underline{x}) &= \underline{K}(\underline{x})\frac{\lambda_w(S)\lambda_o(S)}{\lambda_o(S)+\lambda_w(S)}\frac{dp_c}{dS}.
\end{aligned}
\tag{3}
$$

Figure 1 Capillary pressure and diffusion as functions of saturation S for permeabilities 1, 5, and 10.



Note that the two equations in (1) form a second-order elliptic equation for the pressure $p(\underline{x}, t)$ and are coupled to the saturation equation (2) through the saturation S in the coefficients. On the other hand, saturation equation (2) is a nonlinear advection-diffusion equation and is coupled to the pressure equation (1) through the Darcy velocity $\underline{u}$. Furthermore, in the mathematical model the diffusion term $\underline{D}(S, \underline{x})$ vanishes at $S = 0$ and $S = 1$, which is an idealized case since physically $\underline{D}(S, \underline{x})$ vanishes for $S \in [0, S^{ir}]$ or $S \in [1 - S^{ir}, 1]$ with S^{ir} being the irreducible saturation value. The fractional flow function $f(S)$ defined in (3) is typically an S-shaped curve of saturation S and degenerates at $S = 0$ (with the same understanding). Because the saturation profile is usually a decreasing function in space, as time evolves $f(S)$ tends to force a shock discontinuity to develop in S while the diffusion term $\underline{D}(S, \underline{x})$ tends to prevent the shock from forming. The dynamic process could be fairly complicated because the diffusion degenerates in front of the steep saturation front. It depends on the interaction between the advection and diffusion terms, in particular, on the rates at which $\underline{D}(S, \underline{x})$ and $f(S)$ tend to zero as S tends to zero.

When the physical domain Ω is composed of different media, the different porosities and permeabilities result in different capillary pressure curves on each subdomain (Figure 1). Across an interface Γ the phase pressures are continuous and mass is conserved, leading to the following interface conditions

$$
\begin{aligned}
p_c(S)|_{\Gamma_-} &= p_c(S)|_{\Gamma_+}, \\
\underline{u} \cdot \underline{n}|_{\Gamma_-} &= \underline{u} \cdot \underline{n}|_{\Gamma_+}, \\
(f(S)\underline{u} - \varepsilon \underline{D}(S, \underline{x}) \nabla S) \cdot \underline{n}|_{\Gamma_-} &= (f(S)\underline{u} - \varepsilon \underline{D}(S, \underline{x}) \nabla S) \cdot \underline{n}|_{\Gamma_+}.
\end{aligned}
\tag{4}
$$

The continuity of capillary pressure p_c across an interface Γ implies the discontinuity of the saturation across the interface (Figure 1). One has to resolve the discontinuity carefully so that no spurious effects will be propagated into the domain.

3 Operator Splitting Techniques

Extensive research has been carried out for the numerical simulation of system (1)–(2) without interfaces. Various techniques have been developed to decouple and linearize the system, including a fully coupled and fully implicit linearization strategy, an IMPES (IMplicitly advances the Pressure and Explicitly updates the Saturation in time) strategy, and a sequential time stepping strategy [Ewi84]. Different numerical methods, including the standard Galerkin finite element method, the cell-centered finite difference method, the finite volume method, and the mixed finite element method, have been used to solve the pressure equation [CJ86, DEES90, DEW83, EE87, Ewi84]. We used the mixed method to solve the pressure equation due to its accurate approximation to the velocity field and its local mass conservation property. Because the normal component of the velocity field is continuous, the discrete algebraic system for the pressure equation is in fact the same as that with no interfaces. Hence, one can solve the global system as usual. Alternatively, one can use a domain decomposition procedure to solve the pressure equation on each subdomain iteratively. We refer the interested readers to [BW86, SBG96] and the references therein for details.

For simplicity of exposition we consider a one-dimensional analogue of equation (2). Notice that equation (2) is almost hyperbolic due to the small parameter $\varepsilon << 1$. An effective solution procedure for solving the dominating advective part of equation (2)

$$\phi(x)\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(f(S)u) = 0 \tag{5}$$

is to discretize equation (5) along the characteristics, which allows large time steps to be used in the numerical simulation. Because equation (5) may have more than one solution due to the shape of the fractional flow function $f(S)$, one cannot directly apply the modified method of characteristics [DR82] to equation (5). We follow the work of Espedal, Ewing, and coworkers [DEES90, EE87] and split the fractional flow function $f(S)$ into two parts by

$$f(S) = \bar{f}(S) + b(S)S, \tag{6}$$

with

$$\bar{f}(S) = \begin{cases} \dfrac{f(S_{BL})}{S_{BL}}S, & \text{if } 0 \leq S \leq S_{BL}, \\ f(S), & \text{if } S_{BL} < S \leq 1. \end{cases} \tag{7}$$

Here the Buckley-Leverett shock saturation S_{BL} is defined by

$$f'(S_{BL}) = \frac{f(S_{BL})}{S_{BL}}. \tag{8}$$

Because $\bar{f}(S)u$ gives the unique physical velocity for an established shock, we use this operator splitting and rewrite equation (2) along the characteristics as

$$\psi(x)\frac{\partial \bar{S}^{n+1}}{\partial \tau} \equiv \phi(x)\frac{\partial \bar{S}^{n+1}}{\partial t} + \bar{f}'(\bar{S}^{n+1})u\frac{\partial \bar{S}^{n+1}}{\partial x} = 0, \tag{9}$$

and

$$\psi(x)\frac{\partial S^{n+1}}{\partial \tau} + \frac{\partial}{\partial x}\Big(b(\bar{S}^{n+1})S^{n+1}u - \varepsilon D(\bar{S}^{n+1}, x)\frac{\partial S^{n+1}}{\partial x}\Big) = q_3(x, t^{n+1}). \tag{10}$$

From the definition of $\bar{f}$ it follows that the characteristic direction is uniquely determined by equation (9) since the shape of $\bar{f}$ allows only a rarefaction wave and a contact discontinuity for a non-increasing saturation profile. Thus, the hyperbolic equation (9) is discretized by integrating backwards along the characteristics

$$x^* = x - \bar{f}^{'}(S^{n*})\Delta t, \tag{11}$$

where $S^{n*} = S(x^*, t^n)$ and $\Delta t = t^{n+1} - t^n$ is the time step.

Note that the characteristics determined by equation (9) are all straight lines in the (x, t) plane. If equation (9) is solved exactly, the only change in the solution along the characteristics is due to diffusion (and possibly the source term which vanishes except at wells). Thus, we solve equation (10) by the modified method of characteristics [DEES90, DR82, EE87]

$$\begin{aligned} &\int_\Omega \phi \frac{S^{n+1} - S^{n*}}{\Delta t} w d\Omega + \int_\Omega \Big(\varepsilon D(S^{n*}, x) \frac{\partial S^{n+1}}{\partial x} - b(S^{n*}) S^{n+1} u \Big) \frac{\partial w(x)}{\partial x} d\Omega \\ &= \int_\Omega q_3 w d\Omega, \forall w(x) \in H_0^1(\Omega). \end{aligned} \tag{12}$$

Here a characteristic tracking is used for the advection term, and the quadratic Petrov-Galerkin method is used for the diffusion term and the residual advection term where the trial functions S are chosen to be hat functions and the test functions $w(x)$ are constructed by adding an quadratic perturbation to the hat functions [DEES90, EE87].

4 A Characteristic Domain Decomposition Algorithm for System (1)–(2) with Interfaces

We now describe a characteristic domain decomposition algorithm for solving the system (1)–(2) with interfaces. We adopt a sequential solution strategy to decouple and linearize the system [DEES90, DEW83]. For the domain decomposition techniques for pressure equations with interfaces we refer the interested readers to [BW86, SBG96] for details. We present the algorithm for a one-dimensional problem on $\Omega = (a, b)$ with one interface at $a < d < b$. Let N be a positive integer, $\Delta t = T/N$, and $t^n = n\Delta t$.

Initialization

Substitute the initial condition $S(x, 0)$ for S in (1) and solve equations (1) at t^0 by the mixed method to obtain the Darcy velocity $u^0(x)$, where $u^n(x) = u(x, t^n)$.

for $n = 0, 1, \dots, N-1$ **do**

for $l = 0, 1, \dots, l_M - 1$ **do**

L1. For $l = 0$, in equation (2) approximate $u^{n+1}(x)$ by $u_0^{n+1}(x) = u^n(x)$ or $2u^n(x) - u^{n-1}(x)$. For $l \geq 1$, substitute S_{l-1,k_M}^{n+1} for S in (1) and solve equations (1) at t^{n+1} by the mixed method to obtain the Darcy velocity u_l^{n+1}.

L2. For $l = 0$, assign $S_{0,0}^{n+1}(d_-) = S^n(d^*)$, where $S_{l,k}^{n+1}(d_-) = \lim\limits_{x \to d, x<d} S_{l,k}(x, t^{n+1})$ and d^* is defined in equation (11) with x being replaced by d. For $l \geq 1$, assign $S_{l,0}^{n+1}(d_-) = S_{l-1,k_M}^{n+1}(d_-)$.

L3. Use the interface condition $p_c^L(S_{l,0}^{n+1}(d_-)) = p_c^R(S_{l,0}^{n+1}(d_+))$ to evaluate $S_{l,0}^{n+1}(d_+)$, where $S_{l,k}^{n+1}(d_+) = \lim\limits_{x \to d, x>d} S_{l,k}^{n+1}(x)$.

for $k = 0, 1, \ldots, k_M - 1$ **do**

if ERROR > TOLERANCE **then**

K1. With the given inflow boundary condition at $x = a$ and $S_{l,k}^{n+1}(d_-)$ as the outflow Dirichlet boundary condition at $x = d$, solve equation (12) on the subdomain (a, d) for $S_{l,k}^{n+1}$ at time t^{n+1}.

K2. With $S_{l,k}^{n+1}(d_+)$ as the inflow Dirichlet boundary condition at $x = d$ and the given outflow boundary condition at $x = b$, solve equation (12) on (d, b) for $S_{l,k}^{n+1}$ at t^{n+1} in parallel to the previous step.

K3. Calculate the mass error $M_{l,k}^{n+1} = \Delta M - \int_\Omega (S_{l,k}^{n+1} - S^n) d\Omega$, where ΔM is the mass injected at the inflow boundary and through the wells during the time period $[t^n, t^{n+1}]$.

K4. Update the Dirichlet boundary condition at the interface $x = d$ by $S_{l,k}^{n+1}(d_-) := S_{l,k}^{n+1}(d_-) + \kappa M_{l,k}^{n+1}$, where κ is a relaxation parameter.

K5. Use the interface condition $p_c^L(S_{l,k}^{n+1}(d_-)) = p_c^R(S_{l,k}^{n+1}(d_+))$ to evaluate $S_{l,k}^{n+1}(d_+)$.

else

$k = k_M$ and $l = l_M$

endif

end

$k = k_M$

end

$l = l_M$

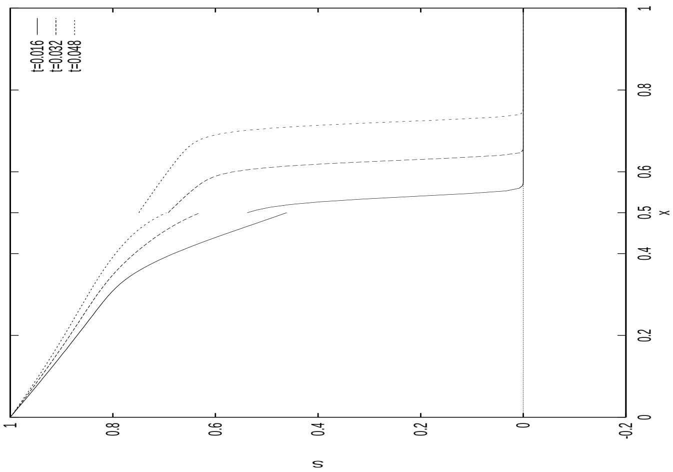
$u^{n+1} := u_{l_M}^{n+1}$ and $S^{n+1} := S_{l_M,k_M}^{n+1}$.

end

Note that the full equation (12) is almost symmetrized and almost well conditioned. Namely, the condition number is of order $\mathcal{O}(D\Delta t/(\Delta x)^2)$. Hence, a diagonal preconditioner works well in practice, in contrast to elliptic equations where the coefficient matrix is ill conditioned and extensive research has been carried out to develop an efficient preconditioner.

We now outline generalizations of the above algorithm in several directions. First, it is easy to see that the above algorithm applies to problems with several interfaces. Second, we note that the procedure applies to multidimensional problems, as long as the adjustment in Step K3 is kept local in space to avoid introducing any spurious nonzero saturation to the location where the saturation is zero. Third, Because the

Figure 2 The saturation "jumps up" across the interface from a higher permeability zone to a lower permeability zone.



coefficient matrix for the pressure equation has a much bigger condition number than that for the saturation equation, it is much more expensive to solve equations in (1) than to solve equation (2) at each time step. Physically the Darcy velocity is much smoother and varies less rapidly than the saturation. Thus, we can use larger time steps for pressure equations in (1) and smaller time steps for the saturation equation (2) (see [DEW83, Ewi84] for details).

5 Numerical Experiments

In this section we present a numerical example to show the promise of the algorithm. More extensive results can be found in [Ers96]. In the example, the space domain $(a, b) = (0, 1)$ with the interface located at $d = 0.5$. The time interval $[0, T] = [0, 0.048]$, $\varepsilon = 0.01$, $\lambda_w(S) = S^2$, $\lambda_o(S) = (1 - S)^2$, $\Delta x = 1/150$, $\Delta t = 0.001$, $K = 10$ on $(0, 0.5)$ and 1 on $(0.5, 1)$. The initial condition is an established shock given by

$$S_0(x) = \begin{cases} 1 - \dfrac{0.3}{0.4}x, & \text{if } 0 \leq x \leq 0.4, \\ 0, & \text{if } 0.4 < x \leq 1. \end{cases} \tag{13}$$

In the numerical experiments, $l_M = 1$ and $k_M = 4$. Namely, we extrapolated the current velocity field u^{n+1} by its values at the previous time steps and did not iterate on equations in (1). With the extrapolated velocity field at the current time step, we iterated four times on the saturation equation (2) at each time step. It was seen in Figure (1) that the permeability has considerable effect on diffusion and capillary pressure. For a fixed saturation the capillary pressure is higher in a lower permeable zone than it is in a high permeable zone. We observe that the continuity of capillary pressure in (4) enforces a jump up in the saturation profile across the interface. The numerical results are free of oscillation or numerical dispersion, and agree with the results in [CY92].

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