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Hybrid Newton-Krylov/Domain Decomposition methods for Compressible Flows

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1 Introduction

Newton-Krylov methods have been shown to be very efficient for the solution of compressible flows [CGKT94], [Tid95a]-[Tid96]. On the other hand, domain decomposition methods provide efficient algorithms suitable for the parallel computing environment. In this study we are interested to two important classes of domain decomposition methods. The first class corresponds to the classical Schwarz-based domain decomposition methods. These methods reduce the solution of a given global problem into the solution of local problems and have potential applications on parallel computing environment. The second class is more recent and corresponds to the domain decomposition time marching algorithm [TT94]-[JFB96] and [Tid92]-[Tid95b]. This method was introduced initially to solve complex physical problems in which many different phenomenon occur. In this report we study the combination of these two classes of domain decomposition methods with Newton-Krylov matrix-free algorithms.

In the next section we study the first hybrid method. The study of the second hybrid method is performed in section 3. The last section is devoted to some conclusions.

2 First Hybrid Newton-Krylov/Domain Decomposition Methods

Euler Solver

The bidimensional Euler Equations in conservative form are written

$$W_t + F(W)_x + G(W)_y = 0,$$

where $W = (\rho, \rho u, \rho v, e)^T$, $F = (\rho u, \rho u^2 + p, \rho uv, u(e+p))^T$, and $G = (\rho u, \rho uv, \rho v^2 + p, v(e+p))^T$. In these expressions ρ is the density, u and v are the velocity components,

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

e is the internal energy, p is the pressure defined by $p = (\gamma - 1)(e - \rho(u^2 + v^2)/2)$, and finally, γ is a constant with $\gamma \approx 1.4$ for air. After transforming the variables into the curvilinear coordinates

$$\tau = t, \quad \xi = \xi(x, y), \quad \eta = \eta(x, y),$$

we obtain the following set of equations

$$\tilde{W}_\tau + (\tilde{F})_\xi + (\tilde{G})_\eta = 0, \tag{2.1}$$

where $\tilde{W}$ and the contravariant flux vectors, $\tilde{F}$ and $\tilde{G}$, are defined in terms of the Cartesian fluxes and the Jacobian determinant of the coordinate system transformation, through $\tilde{W} = J^{-1}W$, $\tilde{F} = J^{-1}(\xi_t W + \xi_x F + \xi_y G)$, $\tilde{G} = J^{-1}(\eta_t W + \eta_x F + \eta_y G)$, and $J = \frac{\partial(\xi,\eta,\tau)}{\partial(x,y,t)}$. An implicit finite volume discretization of the equation (2.1) together with a flux splitting approach yields the following nonlinear system

$$f(W^{n+1}) = 0. \tag{2.2}$$

A linearization of first order in time yields the standard defect-correction method

$$\mathbf{A}\delta W^n = b. \tag{2.3}$$

The different fluxes involved above are computed using Roe's approximate Riemann solver [Roe81]. In (2.3), the Jacobians are evaluated using Van Leer's scheme. In the fully implicit form the boundary conditions are implemented through: $\frac{\partial f_b}{\partial W}\delta W = -f_b(W)$. In this case, the CFL number may be adaptively advanced according to: $\mathrm{CFL}^{n+1} = \mathrm{CFL}^n \cdot \frac{||f(W)||^{n-1}}{||f(W)||^n}$, where the superscript refers to the iteration in time. For more details we refer to [MBW88] and [Tid95a].

Description of the Preconditioned Newton-Krylov matrix-free algorithms

The preconditioned Newton-Krylov matrix-free method [BS90], applied to the fully implicit nonlinear system (2.2), yields the following algorithm

- Define δW_0^n, an initial guess.
- For $k = 0, 1, 2, \cdots$ until convergence do

$$\text{Solve} \quad M^{-1}\frac{f(W_k^n + \epsilon\delta W_k^n) - f(W_k^n)}{\epsilon} = -M^{-1}f(W_k^n). \tag{2.4}$$

$$\text{Set} \quad W_{k+1}^n = W_k^n + \delta W_k^n.$$

The selection of the parameter ϵ is discussed in [Tid95b]. The preconditioner M^{-1} is constructed using an approximation similar to that used to derive the matrix $\mathbf{A}$ of the defect-correction procedure (2.3). This results in a combined discretization in which for each linear step (2.4) of the Newton iteration the preconditioner is not derived from the actual higher-order system. Instead, this preconditioner is derived using an approximation of the Jacobian matrix that employs a lower-order discretization in a similar fashion to defect-correction procedure. In this study the preconditioner corresponds to the parallel Schwarz domain decomposition preconditioner which will be described in the next section and the Krylov methods correspond to GMRES [SS86].

Additive and Multiplicative Schwarz methods

Considering an overlapping decomposition of the physical polygonal domain, the multiplicative Schwarz algorithm for the solution of the linear system (2.3) or (2.4) corresponds to:

$$(I - O_I)v = g, \tag{2.5}$$

with an appropriate g. Above, $O_I = (I - P_{N_{sd}}) \cdots (I - P_1)$, N_{sd} is the number of subdomains, and $P_i = R_i^T A_i^{-1} R_i A$. A_i are the local matrices and R_i are the algebraic restrictions while R_i^T are the algebraic extensions. The additive Schwarz method corresponds to

$$\sum_{i=1}^{N_{sd}} P_i v = g, \tag{2.6}$$

with an appropriate g.

Numerical Results

The test problem on which we study the performance of the methodology described above corresponds to a NACA0012 steady transonic airfoil at an angle of attack of 1.25 degrees and a freestream Mach number of 0.8 using the C-grids 128×32 cells. All calculations in this section are performed on the same Sparc10 machine. Since we are dealing with different methods which require varying amounts of work at each time step we believe that the CPU time is the only true measure for comparing them. The steady state regime is declared when the nonlinear residual norm reaches a value of (or less than) 10^{-5}. The Schwarz-based domain decomposition solver uses the PETSc library that was developed at Argonne National Laboratory [GS93]. In Table 1, we present the iteration count (number of nonlinear iterations) and CPU time (in seconds) for steady transonic flow at convergence using Schwarz algorithms in combination with defect correction procedures. The treatment of the boundary conditions is implicit and the CFL number is equal to 100. In Table 2, we present the iteration count and CPU time (in seconds) for steady transonic flow at convergence using Schwarz algorithms in combination with Newton-Krylov matrix-free methods. The treatment of the boundary conditions is also implicit and the starting CFL number is 30. Comparing the two tables, we observe that the additive Schwarz algorithm combined with Newton-Krylov matrix-free method reduces the CPU time by almost 50% for the various decompositions studied here, as compared to its combination with defect correction procedures (see [Tid96]).

Table 1 Schwarz methods combined with defect-correction procedures.

	Block Jacobi		Add. Schwarz		Mult. Schwarz	
Decomp.	Iter	CPU	Iter	CPU	Iter	CPU
2×2	547	8911	553	11096	566	9123
4×4	540	9114	552	11899	577	9717
8×8	539	11430	546	16215	574	11482

Table 2 Schwarz methods combined with preconditioned Newton-Krylov matrix-free methods.

	Block Jacobi		Add. Schwarz		Mult. Schwarz	
Decomp.	Iter	CPU	Iter	CPU	Iter	CPU
2×2	31	5474	31	6102	33	8409
4×4	32	5384	28	5708	30	4759
8×8	32	6594	35	7493	25	4106

3 Second Hybrid Newton-Krylov/Domain Decomposition Methods

Navier-Stokes Equations

Let us consider the compressible Navier-Stokes equations which we formally write either as

$$\frac{\partial W}{\partial t} + div[F(W)] = 0 \text{ on } \Omega \text{ (conservative form)}$$

or as

$$\frac{\partial U}{\partial t} + T(U) + D(U) = 0 \text{ on} \Omega \text{ (non conservative form)}$$

with $W = (\rho, \rho v, \rho E)$ and $U = (\rho, v, \theta)$ as the conservative and nonconservative variables, $F = F_C + F_D$ as the total flux (convective and viscous part), and T and D the convective and viscous terms in the nonconservative form of the Navier-Stokes equations. The problem consists in computing a steady solution of these equations, with boundary conditions

$$\rho v, \rho E \text{ given on } \Gamma_e \text{ (exterior limit of the domain)},$$

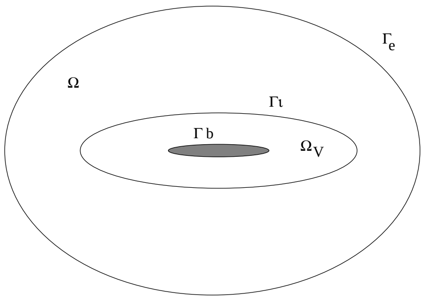
$$\rho \text{ given on } \Gamma_e \cap \{x, v(x) \cdot n \leq 0\} \text{ (inflow)},$$

$$v = 0 \text{ on the body } \Gamma_b, \text{ (no slip)},$$

$$\theta = \theta_b \text{ on the body } \Gamma_b.$$

The strategy discussed below couples a *global conservative scheme*, defined on the whole domain, and based on a finite volume space discretization [RS88], and *a local*

Figure 1 The global geometry



approximation, defined in the neighborhood of the body, which is presently based on a mixed Finite Element approximation of the nonconservative Navier-Stokes equations [BGD+89].

The General Coupling Strategy

For coupling external Navier-Stokes equations, with local Navier-Stokes equations, we introduce two domains, a global one Ω, a local one Ω_V included in Ω, and an interface Γ_i (Fig. 1). The global solution W on Ω and the local solution U_{loc} on Ω_V, which both satisfy the Navier-Stokes equations, are matched by the following boundary conditions, inspired of Schwarz overlapping techniques :

$$\begin{cases} W = \text{ given imposed value on } \Gamma_e, \\ n \cdot \sigma(W) \cdot \tau = n \cdot \sigma(U_{loc}) \cdot \tau \;\text{ on the body } \Gamma_b, \text{ (equality of friction forces)} \\ q(W) \cdot n + n \cdot \sigma(W) \cdot v = q(U_{loc}) \cdot n \text{ on } \Gamma_b, \text{ (equality of total heat fluxes)} \\ v \cdot n = 0 \text{ on } \Gamma_b, \\ U_{loc} = 0 \text{ on } \Gamma_b \; U_{loc} = W \text{ on the interface } \Gamma_i. \end{cases}$$

Above, $n.\sigma.n$ and $n.\sigma.\tau$ respectively denote the normal and the tangential force exerted by the body on the flow, with n the unit normal vector to the body oriented towards its interior.

The calculation of U_{loc} and W satisfying the above boundary conditions is then obtained by the time marching algorithm, which was introduced in [TT94]-[JFB96] and [Tid92]-[Tid95b]) and which leads to the following algorithm : **Initialization**

1. Guess an initial distribution of the conservative variable W in the global domain

Ω ;

2. Advance in time this distribution by using the global Navier-Stokes solver on N_1 time steps, with *Dirichlet* type boundary conditions on the body Γ_b ;

3. Deduce from this result an initial distribution of the local variable U_{loc} on the interface Γ_i and in the local domain Ω_V ;

4. Advance in time this distribution by using the local solver on N_2 time steps with Dirichlet boundary conditions on Γ_i and Γ_b.

Iterations

5. From U_{loc}, compute the friction forces $n \cdot \sigma(U_{loc}) \cdot \tau$ and heat flux $q(U_{loc}) \cdot n$ on the body Γ_b ;

6. Advance the global solution in time (N_1 steps) by using the global Navier-Stokes solver with the above viscous forces as boundary conditions on Γ_b;

7. From W, compute the value of U_{loc} on the interface Γ_i ;

8. Using this new value as Dirichlet boundary conditions on Γ_i, advance the local solution in time (N_2 steps) and go back to step 5 until convergence is reached.

This algorithm completely uncouples the local and the global problems, which can therefore be solved by independent solvers. A parallel version is also quite possible although it is generally wiser to use parallel solvers within steps 6 and 8.

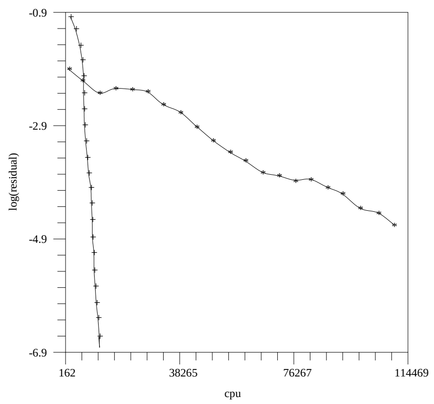
Global and Local Solvers

To solve the global conservative Navier-Stokes equations we use the hybrid finite volume/finite element method in which the convective flux is computed by an Osher approximate Riemann solver. The resulting linear system is solved by a block relaxation method. The local nonconservative Navier-Stokes equations are discretized by mixed finite elements (P_1 for ρ and θ, P_1 on the subdivided P_2 grid for the velocity). The resulting nonlinear local system is solved by using the preconditioned Newton-Krylov matrix-free method described in 2 with diagonal preconditioner. We refer to [Tid95a] for more details.

Numerical Results

The test problem consists of a two dimensional flow around an ellipse, with 0 angle of attack, $M_\infty = 0.85$, Reynolds number $= 100$, and a wall temperature $T_W = 2.82 T_\infty$. First, we have calculated the Navier-Stokes solution employing the global nonconservative solver alone on a mesh that has 4033 nodes and 7942 elements for the P_1 grid and 16184 nodes and 32120 elements for the grid P_2. We then performed a calculation using the coupling algorithm described above on a global mesh that has 1378 nodes and 2662 elements and a local mesh that has 1114 nodes and 4282 elements. On Figure (2) we show a CPU time comparisons of the two calculations which were performed on an apollo DN 10000. This figure shows the excellent performance of Newton-Krylov matrix-free method used to solve the local model through the domain decomposition time marching algorithm as compared to its use in the standard approach (see [Tid95a]).

Figure 2 CPU time comparisons between the uncoupled scheme and the coupled approach. Above the + curv corresponds to the coupled scheme and the * one corresponds to the uncoupled nonconservative scheme.



4 Conclusions

In this study we have studied the combination of Newton-Krylov matrix-free algorithms with two classes of domain decomposition methods. In both cases we have given numerical applications to compressible Euler and Navier-Stokes equations that illustrate the performance of the resulting hybrid algorithms.

Acknowledgement

This work was supported by the National Aeronautics and Space Administration under NASA contract NAS1-19480 while the author was in residence at the Institute for Computer Applications in Science and Engineering.

REFERENCES

[BGD+89] Bristeau M., Glowinski R., Dutto L., G. J. P., and Rogé (1989) Compressible viscous flow calculations using compatible finite element approximations. In *7th Int. Conf. on Finite Element Methods in Flow Problems.* Huntsville, Alabama.

[BS90] Brown P. N. and Saad Y. (1990) Hybrid krylov methods for nonlinear systems of equations. *SIAM J. Sci. Stat. Comp.* 11: 450–481.

[CGKT94] Cai X.-C., Gropp W. D., Keyes D. E., and Tidriri M. D. (1994) Newton-krylov-schwarz methods in cfd. In Rannacher R. (ed) *Proceedings of the International Workshop on the Navier-Stokes Equations, Notes in Numerical Fluid Mechanics.* Braunschweig.

[GS93] Gropp W. D. and Smith B. F. (1993) Simplified linear equations solvers users manual. Technical Report ANL 93/8, Argonne National Laboratory.

[JFB96] J. F. Bourgat P. Le Tallec M. D. T. (1996) Coupling navier-stokes and boltzmann. *J. Comp. Phy.* 127: 227–245.

[MBW88] Mounts J. S., Belk D. M., and Whitfield D. L. (September 1988) Program eagle user's manual, vol. iv – multiblock implicit, steady-state euler code. Technical Report TR-88-117, Air Force Armament Laboratory", TR-88-117, Vol. IV, September 1988.

[Roe81] Roe P. L. (1981) Approximate riemann solvers, parameter vector, and difference schemes. *J. Comp. Phy.* 43: 357–372.

[RS88] Rostand P. and Stoufflet B. (July 1988) Finite volume galerkin methods for viscous gas dynamics. Technical Report RR-863, INRIA.

[SS86] Saad Y. and Schultz M. H. (1986) Gmres: A generalized minimal residual algorithm for solving nonsymmetric linear. *SIAM J. Sci. Stat. Comp.* 7: 865–869.

[Tid92] Tidriri M. D. (1992) *Couplage d'approximations et de modèles de types différents dans le calcul d'ecoulements externes.* PhD thesis, Université de Paris-Dauphine.

[Tid95a] Tidriri M. D. (1995) Domain decompositions for compressible navier-stokes equations. *J. Comp. Phy.* 119: 271–282.

[Tid95b] Tidriri M. D. (June 1995) Krylov methods for compressible flows. Technical Report CR 95-48, ICASE.

[Tid96] Tidriri M. D. (January 1996) Schwarz-based algorithms for compressible flows. Technical Report CR 96-4, ICASE.

[TT94] Tallec P. L. and Tidriri M. D. (October 1994) Convergence of domain decomposition algorithms with full overlapping for the advection-diffusion problems. Technical Report RR-2435, INRIA.