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# A space decomposition method for minimization problems

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## 1 A Space Decomposition Algorithm

We consider:

$$\min_{v \in V} F(v) \ , \tag{1.1}$$

where functional $F$ is differentiable and convex and space $V$ is a reflexive Banach space. Our intention is to use space decomposition method to get some parallel domain decomposition and multigrid type algorithms for linear partial differential equations of the type

$$\begin{cases} -\nabla \cdot (a \nabla u) = f \text{ in } \Omega \subset R^2 \ , \\ u = 0 \text{ on } \partial\Omega \ , \end{cases} \tag{1.2}$$

and for nonlinear elliptic problems like

$$\begin{cases} - \quad \nabla \cdot (|\nabla u|^{s-2} \nabla u) = f \text{ in } \Omega \subset \mathrm{R}^2 \ (1 < \mathrm{s} < \infty) \ , \\ u = 0 \text{ on } \partial\Omega \ . \end{cases} \tag{1.3}$$

The algorithm given in this work were first proposed in [Tai92], see also [Tai94b], [Tai95a] and [Tai95b]. As the algorithm is proposed for a minimization problem, it is applicable for a wide class of problems, for example, eigenvalue problems, optimal control problems related to partial differential equations and least-squares method associated with linear and nonlinear equations.

A space decomposition method refers to methods that decompose the space $V$ into a sum of subspaces, i.e. there are spaces $V_i, \ i = 0, 1, \cdots, m$ such that

$$V = V_0 + V_1 + \cdots + V_m \ . \tag{1.4}$$

For the decomposed spaces, we assume that there is a constant $C_1 > 0$ such that $\forall v \in V$, we can find $v_i \in V_i$ to satisfy:

$$v = \sum_{i=0}^{m} v_i \ , \qquad \text{and} \qquad \sum_{i=0}^{m} \|v_i\|_V^2 \le C_1^2 \|v\|_V^2 \ . \tag{1.5}$$

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Moreover, assume that there is a $C_2 > 0$ such that there holds

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\langle F''(w_{ij})u_i, v_j\rangle \le C_2\Big(\sum_{i=0}^{m}\|u_i\|_V^2\Big)^{\frac{1}{2}}\Big(\sum_{i=0}^{m}\|v_i\|_V^2\Big)^{\frac{1}{2}}, \quad \forall w_{ij}\in V, \forall u_i\in V_i, \forall v_j\in V_j\ . \tag{1.6}$$

Domain decomposition methods, multilevel methods and multigrid methods can be viewed as different ways of decomposing finite element spaces into sums of subspaces. For the estimation of the constants $C_1$ and $C_2$ for different types of decomposition of finite element methods for linear problems, one can find the proofs or references in Xu [Xu92]. If the space can be decomposed as in (1.4), then the following algorithm can be used to solve (1.1).

**Algorithm 1** *(A multiplicative space decomposition method).*

1. *Choose initial values* $u_i^0 \in V_i$.
2. *For* $n \ge 1$, *find* $u_i^{n+1} \in V_i$ *sequentially for* $i = 0, 1, \cdots, m$ *such that*

$$F\left(\sum_{k<i}^{m} u_k^{n+1} + u_i^{n+1} + \sum_{k>i}^{m} u_k^n\right) \le F\left(\sum_{k<i}^{m} u_k^{n+1} + v_i + \sum_{k>i}^{m} u_k^n\right), \quad \forall v_i \in V_i\ . \tag{1.7}$$

3. *Go to the next iteration.*

In the following, we denote $u^n = \sum_{i=0}^{m} u_i^n$, $\forall n > 0$. By assuming that $F$ is continuously differentiable and

$$K\|w-v\|_V^2 \le \langle F'(w) - F'(v), w - v\rangle \le L\|w-v\|_V^2\ , \quad \forall w, v \in V\ , \tag{1.8}$$

where $K > 0,\ L > 0$, and using $e^n = |\langle F'(u^n) - F'(u), u^n - u\rangle|^{\frac{1}{2}}$, as a measure of the error between $u^n$ and $u$, the following convergence theorem is proved in Tai and Espedal [TE96].

**Theorem 1** *If the space decomposition satisfies (1.5) and the functional $F$ satisfies (1.8), then for Algorithm 1 we have:*

1. *If $F$ is quadratic with respect to $v$ and the norm of $V$ is taken as $\|v\|_V = \langle F'(v), v\rangle$, then*

$$|e^{n+1}|^2 \le \frac{C_s^2}{1+C_s^2}|e^n|^2\ , \quad \forall n \ge 1\ .$$

   *Above and also later, $C_s = C_2C_1$.*

2. *If $F$ is third order continuously differentiable, then*

$$|e^{n+1}| \to 0 \text{ as } \mathrm{n} \to \infty\ , \text{ and } \quad |\mathrm{e}^{\mathrm{n}+1}|^2 \le \beta_{\mathrm{n}}|\mathrm{e}^{\mathrm{n}}|^2\ , \quad \forall \mathrm{n} \ge 1\ ,$$

   *and the error reduction factor $\beta_n$ satisfies $\lim_{n\to\infty}\beta_n = \frac{\frac{C_s{}^2}{K^2}}{1+\frac{C_s{}^2}{K^2}} < 1$, which means the asymptotic convergence rate only depends on $C_s$ and $K$.*

## 2 Application of the Space Decomposition to a Two-level Domain Decomposition Method

We use the space decomposition Algorithm 1 for a two-level overlapping domain decomposition method. For a given domain $\Omega$, we first divide it into coarse mesh subdomains, and then refine each coarse mesh subdomain to get fine mesh divisions for $\Omega$. In the following examples, domain $\Omega$ is taken as $[0,1] \times [0,1]$. Uniform mesh is used both for the coarse mesh division and the fine mesh division. Let $\Omega_i, i = 1, 2, \cdots$ be a coarse mesh division of $\Omega$, see Figure 1, we then enlarge each $\Omega_i$ to $\Omega_i^\delta = \{T \in \mathcal{T}_h, dist(T, \Omega_i) \leq \delta\}$ to get overlapping subdomains. Here $\{\mathcal{T}_h\}$ denotes the fine mesh division for $\Omega$.

The union of $\Omega_i^\delta$ covers $\bar{\Omega}$ with overlaps of size $2\delta$. Let us denote the piecewise linear finite element space with zero traces on the boundaries $\partial\Omega_i^\delta$ as $S_0^h(\Omega_i^\delta)$, and denote $S_h^0$, $S_H^0$ as the coarse and fine mesh finite element spaces respectively. One can show that

$$S_0^h = S_0^H + \sum S_0^h(\Omega_i^\delta) \; . \tag{2.9}$$

For the overlapping subdomains, assume that there are $m$ colors such that each subdomain $\Omega_i^\delta$ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, we have $m = 4$, see Figure 1. Let $\Omega_i'$ be the union of the subdomains of the $i^{\text{th}}$ color, and $V_i = \{v \in S_0^h \big| \;\; v(x) = 0 \text{ if } \;\; \mathrm{x} \notin \Omega_{\mathrm{i}}'\}$. By denoting subspaces $V_0 = S_0^H$ and $V = S_0^h$, we find that decomposition (2.9) means

$$V = V_0 + \sum_{i=1}^{4} V_i \; , \tag{2.10}$$

and so the two-level method is a way to decompose the finite element space. Moreover, let $V = H_0^1(\Omega)$ and $F$ be the corresponding energy function of linear equation (1.2), then the constants in (1.5) and (1.6) are:

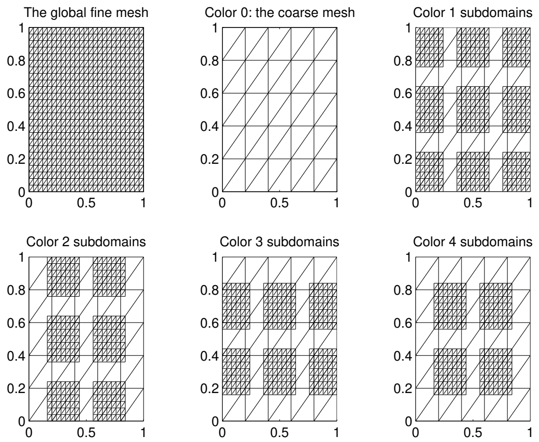
$$C_1 = C\sqrt{1 + \frac{H^2}{\delta^2}}, \; C_2 = Cm. \tag{2.11}$$

The proof for (2.11) can be found in different places, we refer to page 608 of Xu [Xu92] and Tai and Espedal [TE96]. By requiring $\delta = c_0 H$, where $c_0$ is a given constant, we have that $C_1$ and $C_2$ are independent of the mesh parameters $h$ and $H$, and the number of the subdomains. So if the proposed algorithm is used, its error reduction per step does not depend on $h$ and $H$.

## 3 Applications to Linear Elliptic Problems

We apply Algorithm 1 to solving linear problem (1.2). As was shown above, the two-level method is a space decomposition method. With the coarse mesh, the number of the subspaces is $m = 5$. For Algorithm 1, we define $w_i^{n+1} = \sum_{k<i}^{m} u_k^{n+1} + u_i^{n+1} +$

**Figure 1** The coloring of the subdomains and the coarse mesh grid




$\sum_{k>i}^{m} u_k^n$ and $w_{-1}^{n+1} = u^n$. In each subdomain of the $i^{\text{th}}$ color, the subproblem needs to be solved is

$$\begin{cases} (a\nabla w_i^{n+1}, \nabla v_i) = (f, v_i), & \forall v_i \in S_0^h(\Omega_i^\delta) \ , \\ w_i^{n+1} = w_{i-1}^{n+1} \text{ on } \partial\Omega_{\rm i}^\delta \ , \end{cases} \tag{3.12}$$

and $w_i^{n+1} = w_{i-1}^{n+1}$ in $\Omega \backslash \Omega_{\rm i}'$. For the coarse mesh problem, if we let $w_H^{n+1} = u_0^{n+1} - u_0^n$, then it satisfies

$$(a\nabla(u^n + w_H^{n+\frac{1}{2}}), \nabla v_H) = (f, v_H) \ , \quad \forall v_H \in S_0^H(\Omega) \ . \tag{3.13}$$

After the computation of the subdomain problems and the coarse mesh problem, we set $u^{n+1} = w_m^{n+1}$. Note that the subdomains with the same color do not intersect with each other, so in computing the $i^{\text{th}}$ color subdomain solutions, the computation is done in parallel in each of the subdomains of the $i^{\text{th}}$ color. One observes that this is the standard multiplicative Schwarz method for linear elliptic equations. In the literature, this method is often symmetrised and then accelerated by conjugate gradient method, see [SBG96]. In the next example, we try to see the convergence without using the extra acceleration.

**Example 1** *In this example, Algorithm 1 is tested for the case that* $a = e^{xy}, u = \sin(3\pi x)\sin(3\pi y)$. *For a given* $N$*, the coarse mesh size is taken as* $H = Hx = Hy = \frac{1}{N}$. *The fine mesh is then taken as* $h = hx = hy = \frac{1}{N^2}$. *Each subdomain is extended by* $M$ *elements to get overlaps. The initial guess is taken as the coarse mesh solution* $u_H$. *Figure 2 illustrates the computed solution and computed error function. Table 1 shows the convergence property. For different tests with* $hx, hy \in [\frac{1}{125}, 1]$, *i.e. with unknowns* $\leq 15625$, *and with overlap size* $\delta \approx \frac{H}{5}$, *the computed solution always converges to the global finite element solution in less than* 8 *steps.*

## 4 Applications to Linear Interface Problems

**Figure 2** The computed solution with H=1/10, h=1/100, M=2.

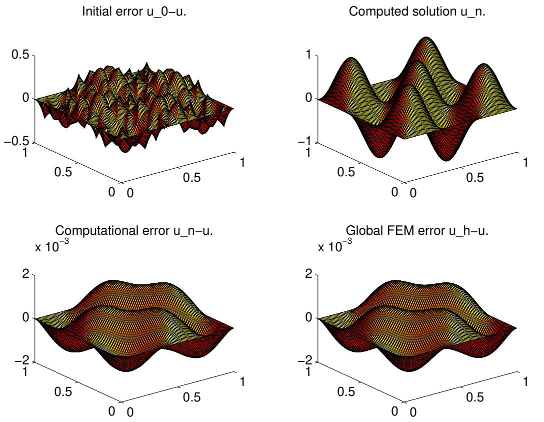
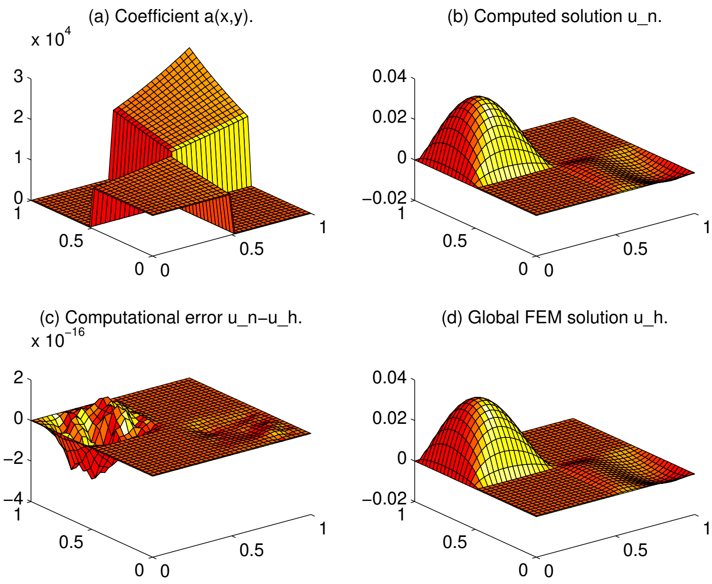



**Table 1** *Maximum error with H=1/10, h=1/100, M=2.*

| *Iteration* | *max-error* | *reduction* |
|---|---|---|
| *1* | *0.0729* | *0.24* |
| *2* | *0.0166* | *0.23* |
| *3* | *0.0036* | *0.22* |
| *4* | *0.0017* | *0.46* |
| *5* | *0.0015* | *0.88* |

**Figure 3** Computational results for a linear interface problem.




**Example 2** *We solve a linear interface problem. The coefficients are taken as* $a = c(x)e^{xy}$ *where* $c(x)$ *is piecewise constant and* $c(x) = 1$ *or* $10^4$*, (see Figure 3). The global fine finite element solution is first computed. After that, the problem is computed by Algorithm 1 and the error between the iterative domain decomposition solution and the global fine mesh solution is calculated. The mesh sizes are* $hx = hy = \frac{1}{100}$*,* $Hx = Hy = \frac{1}{10}$*. The algorithm converges for arbitrary initial guesses. Each subdomain is extended by 2 elements to get overlap. The convergence is similar as the smooth problem.*

## 5 Applications to Nonlinear Elliptic Problems

In the literature, domain decomposition methods and multilevel methods have been intensively studied for linear elliptic problems. For nonlinear problems, it is hard to get some general convergence estimates. For literature results related to nonlinear problems, see [CD94], [CGKT94], [LSL89], [MX95], [Tai92]–[Tai94a], [Xu94], etc. The proposed algorithm of this work can be applied to linear problems (1.2) as well as nonlinear problems (1.3).

The Gauss-Newton method (Matlab subroutine fiminu) is used to solve the minimisation problem (1.7). Without using the domain decomposition, the original problem is simply too large and costly to be solved.

**Example 3** *We use an analytical solution* $u = \sin(2\pi x)\sin(2\pi y)$ *for (1.3) to test Algorithm 1. Figure 4 and Table 2 show the computational results with fine mesh* $hx = hy = \frac{1}{100}$*, and coarse mesh* $Hx = Hy = \frac{1}{10}$*. Each subdomain is extended by 2*

**Figure 4** The computational results for the nonlinear problem by Algorithm 1.

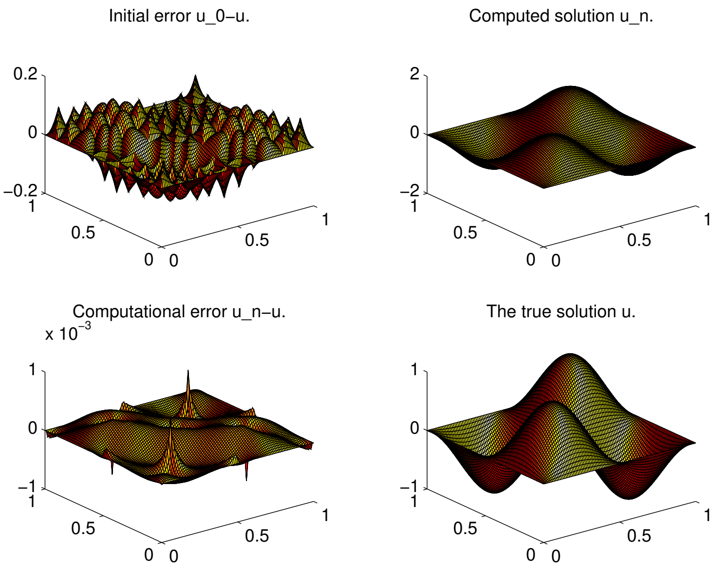



**Table 2** Maximum error for the nonlinear problem by Algorithm 1.

| *Iteration* | *max-error* | *reduction* |
|---|---|---|
| *1* | *0.1345* | |
| *2* | *0.0257* | *0.19* |
| *3* | *0.0049* | *0.19* |
| *4* | *0.0012* | *0.24* |
| *5* | *0.0007* | *0.60* |
| *6* | *0.0007* | *1.05* |

*elements to get overlaps. The initial guess is the coarse mesh solution. The value of $s$ is 3. Numerical tests show that the algorithm converges for arbitrary initial guess and the error reduction does not depend on the initial guess. The dependency of the convergence on the overlapping size and on the number of subdomains is the same as for the linear problem (1.2).*

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