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Algebraic Domain Decomposition Method for Unstructured Grids

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1 Introduction

The Black Box Multigrid method of [Den82] is considered robust for diffusion problems with variable coefficients. However, it is applicable only to structured grids, that is, 3^d-coefficient stencils (where d is the dimension of the problem) and not to more complicated (e.g., unstructured) finite element schemes. Furthermore, it is pointed out in [Sha94] [Sha96a] that Black Box Multigrid stagnates for certain diffusion problems with high diffusion areas separated by thin strips. Surprisingly, this stagnation occurs when the discontinuity curves are aligned with all the coarse grids, case which can be handled easily by either standard multigrid or the method of [BPWX91]. The AutoMUG method of [Sha94] [Sha96a] avoids this stagnation but diverges for other examples. In [Sha96b] this stagnation is studied and a modified version of Black Box Multigrid which avoids it is introduced. This version is related to the method of [KM81] and is based on 'throwing' certain matrix elements onto the main diagonal when constructing the prolongation operator from coarse to fine grids. It is shown in [Sha96b] that this version is robust both theoretically (for a certain class of problems) and numerically (for the above example and others). The method is generalized in [Sha97] to finite element schemes on locally refined meshes.

In this work we further generalize the method to finite element schemes on unstructured meshes which do not necessarily arise from local refinement. The method is based on a given domain decomposition, usually determined by the physical or the geometrical nature of the problem (Figure 1). The idea is to choose suitable vertex variables on the interface between subdomains for serving as a coarse grid. The prolongation operator extending a coarse grid function to the whole grid consists of two steps: first solve low order systems for defining the function on the interfaces between subdomains; then solve the original scheme on each subdomain separately for defining the function in the subdomain interiors. (For simplicity we consider the 2-d case; in 3-d, three steps are needed.) The coarse grid equation is obtained from a

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Figure 1 The unstructured grid and the domain decomposition.

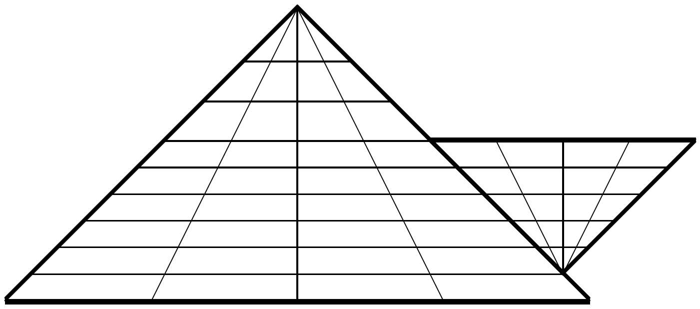
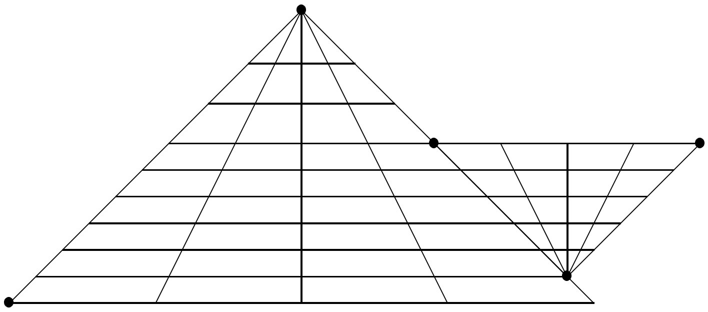


Figure 2 The coarse grid variables are denoted by '•'.



Galerkin scheme; it is solved either directly or iteratively using some preconditioning method or multigrid. Both the formulation of the coarse grid equation and the actual restriction and prolongation use only local operations which can be done in parallel. The method may be supplemented with presmoothing and postsmoothing as in multigrid or with an outer acceleration method. Unlike in [BPWX91], it is not assumed here that the discontinuities in the coefficients of the PDE are aligned with the coarse mesh. The method and the analysis are algebraic in the sense that, once the domain decomposition and the coarse grid are determined, only the coefficient matrix is used and not the PDE or the finite element mesh. Therefore, the method is named Algebraic Domain Decomposition (ADD).

2 The Algebraic Domain Decomposition Method

Consider a finite element scheme for an elliptic boundary value problem on a mesh of the type used in [BPWX91] (illustrated in Figure 1). Assume that the underlying linear system is given by

$$Ax = \mathbf{b}, \tag{1}$$

where x is the vector of unknowns corresponding to the nodes in the mesh, $\mathbf{b}$ is the right-hand side vector and A is the nonsingular coefficient matrix.

Consider a domain decomposition as in Figure 1, where nodes on the thick lines correspond to interface or boundary unknowns. Let some of these unknowns (typically, vertex variables such as those denoted by '•' in Figure 2) serve as coarse grid variables. In the following we denote by c the set of coarse grid variables, by b the set of the other boundary and interface variables and by s the set of all other variables (corresponding to nodes in subdomain interiors). This induces a partitioning of the coefficient matrix A as

$$A = \begin{pmatrix} A_{ss} & A_{sb} & A_{sc} \\ A_{bs} & A_{bb} & A_{bc} \\ A_{cs} & A_{cb} & A_{cc} \end{pmatrix}. \tag{2}$$

In the sequel we use this partitioning also for other matrices of the same order and (unless specified otherwise) refer by 'blocks' to the blocks in such a partitioning. We also denote $f = s \cup b$ (the set of fine grid points, namely, all variables but the coarse grid ones). This induces another block partitioning of A:

$$A = \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix},$$

where

$$A_{ff} = \begin{pmatrix} A_{ss} & A_{sb} \\ A_{bs} & A_{bb} \end{pmatrix}. \tag{3}$$

This partitioning is used for other matrices of the same order as well.

For any set g, let $|g|$ denote its cardinality (the number of elements in g). For any positive integer k, let I_k denote the identity matrix of order k. For any set $g \subset c \cup f$, let $J_g : l_2(c \cup f) \to l_2(g)$ denote the injection operator defined by

$$(J_g w)_j = w_j, \; w \in l_2(c \cup f), \; j \in g.$$

For any matrix M, $M = (m_{i,j})_{1 \le i \le K,\ 1 \le j \le L}$, define the absolute value of M by $|M| = (|m_{i,j}|)_{1 \le i \le K,\ 1 \le j \le L}$ and the diagonal matrix of row-sums of M by

$$rs(M) = diag \left(\sum_{j=1}^{L} m_{i,j} \right)_{1 \le i \le K}.$$

Let us define a matrix $T(A)$ which is obtained from A by 'throwing' certain matrix elements onto the main diagonal. More specifically, $T(A)$ is of the same order as A and is upper block-triangular (with respect to the partitioning (2)) with $T(A)_{cc} = I_{|c|}$. The structure of $T(A)$ is thus

$$T(A) = \begin{pmatrix} T(A)_{ss} & T(A)_{sb} & T(A)_{sc} \\ 0 & T(A)_{bb} & T(A)_{bc} \\ 0 & 0 & I_{|c|} \end{pmatrix}.$$

Furthermore, $T(A)_{bb}$ is block-diagonal, with blocks corresponding to interface or boundary segments. Consider, for example, the unknowns on the thick line in Figure 3. Denote by C the set of the two variables denoted by '•', by B the set of the other

Figure 3 First prolongation step; from C, the set of variables denoted by '•', into B, the set of all the other variables on the thick line.

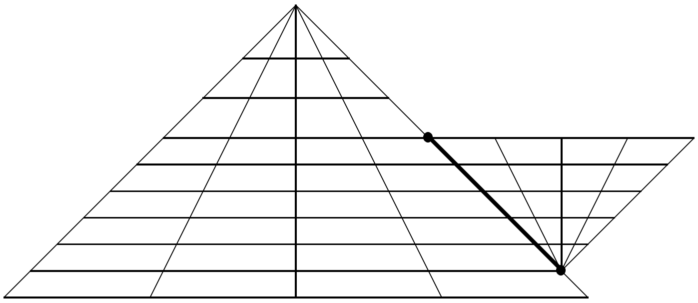
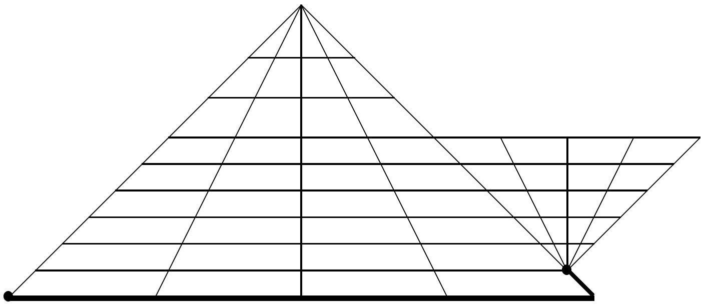


Figure 4 First prolongation step; from C, the set of variables denoted by '•', into B, the set of all the other variables on the thick lines.



variables on the thick line and by F the set of all other variables. The rows in A corresponding to B can be partitioned in the form

$$\begin{pmatrix} A_{BF} & A_{BB} & A_{BC} \end{pmatrix}.$$

The corresponding rows in $T(A)$ are defined by

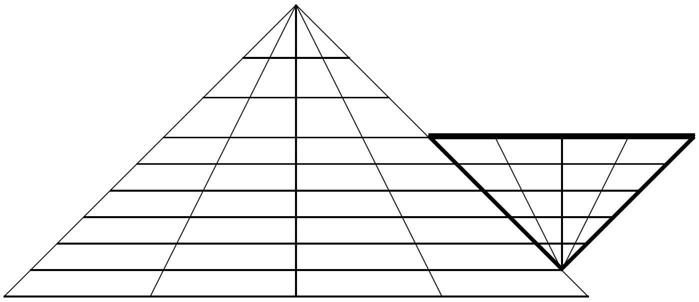
$$\begin{pmatrix} 0 & A_{BB} - rs(|A_{BF}|) & A_{BC} \end{pmatrix}.$$

Consequently, the unknowns in B are coupled in $T(A)$ with themselves and with those in C only. It is assumed here that $T(A)_{BB}$ is nonsingular. A sufficient condition for this is that A is diagonally dominant and A_{BB} is irreducible, which implies that $T(A)_{BB}$ is irreducibly diagonally dominant. The rows of $T(A)$ corresponding to other boundary or interface segments of the form $B \subset b$ (such as that of Figure 4) are defined in a similar way. Note that for usual finite element schemes there is no coupling in A between B and $c \setminus C$ and, therefore,

$$T(A)_{bc} = A_{bc}. \tag{4}$$

Similarly, $T(A)_{ss}$ is block diagonal, with blocks corresponding to sets $S \subset s$ corresponding to interiors of subdomains. For example, denote by B the unknowns corresponding to nodes on the thick lines in Figure 5, by S the unknowns corresponding to the nodes in the interior of the subdomain bounded by these lines and by F the rest

Figure 5 Second prolongation step; from B, the set of variables on the thick lines, into S, the set of variables in the interior of the subdomain bounded by the thick lines.



of the unknowns. The rows in A corresponding to S can be partitioned in the form

$$\left(A_{SF} \quad A_{SS} \quad A_{SB} \right).$$

The corresponding rows in $T(A)$ are defined by

$$\left(0 \quad A_{SS} - rs(|A_{SF}|) \quad A_{SB} \right).$$

Consequently, the unknowns in S are coupled in $T(A)$ with themselves and with those in B only. Rows of $T(A)$ corresponding to other subdomain interiors are defined in a similar way. It is assumed hereafter that $T(A)_{ss}$ is nonsingular. For usual finite element schemes (such as a linear finite element scheme on the mesh in Figure 1) $A_{SF} \equiv 0$ and, therefore,

$$T(A)_{ss} = A_{ss}, \quad T(A)_{sb} = A_{sb} \quad \text{and} \quad T(A)_{sc} = A_{sc}. \tag{5}$$

Therefore, nonsingularity of $T(A)_{ss}$ is guaranteed whenever the differential operator is symmetric positive definite (SPD). (The above more general definition of $T(A)_{ss}$ is introduced for cases where interiors of subdomains might be coupled with each other in A, e.g., when subdomains are not aligned with finite elements. It guarantees that such a coupling cannot exist in $T(A)$, which allows efficient and parallelizable restriction and prolongation operations.)

Define the prolongation operator P and the restriction operator R by

$$P = T(A)^{-1} \quad \text{and} \quad R = P^*,$$

where '$*$' denotes the conjugate with respect to the usual inner product in $l_2(c \cup f)$. Since $T(A)$ is block triangular, the application of P and R is performed easily by block back substitution and block forward elimination, respectively. Furthermore, since the subdomain interiors (such as S in Figure 5) are decoupled from each other in $T(A)$ and the boundary segments (such as B in Figure 3) are also decoupled from each other in $T(A)$, the applications of R and P are highly parallelizable.

Finally, the coarse grid operator Q is a matrix of the same order as A defined by

$$Q = \begin{pmatrix} W & 0 \\ 0 & J_c RAPJ_c^t \end{pmatrix}, \tag{6}$$

where W is a nonsingular matrix of order $|f|$. The reasonable choices for W are

$$W = I_{|f|}$$

or, in the spirit of [Den82],

$$W = R_{ff} diag(A_{ff}) blockdiag(P_{ff}) \tag{7}$$

(where '*blockdiag*' corresponds to the partitioning (3)). The choice (7) yields better numerical results in [Sha97]. It is assumed hereafter that $J_c RAP J_c^t$ is nonsingular; this is guaranteed, e.g., when A is SPD.

The two-level iteration for the solution of (1) is defined by

$$x_{out} = x_{in} + PQ^{-1}R(\mathbf{b} - Ax_{in}). \tag{8}$$

(8) may be supplemented with relaxations before and after it in the spirit of multigrid. (This approach is used in [Sha96b] for uniform grids and in [Sha97] for locally refined grids.) Alternatively, a Lanczos type acceleration may be applied to it. (This approach is used in [Sha97] for uniform grids.) For both approaches, the condition number of the preconditioned matrix $PQ^{-1}RA$ is an important measure for the rate of convergence. In the following, an upper bound on this condition number is given for a class of SPD problems.

3 Analysis in the SPD Case

Here $(\cdot,\cdot)$ denotes the usual inner product in $l_2(c \cup f)$ and $\|\cdot\|$ denotes the corresponding vector and matrix norms. The terms 'symmetry', 'SPD' and 'orthogonality' used below are interpreted with respect to this inner product. The following lemma is used in the proof of Theorem 1.

Lemma 1 *Let M be a symmetric and positive semidefinite matrix of the same order as A. Then, for any vector $x \in l_2(c \cup f)$,*

$$(x, Mx) \leq 2(x, \left(J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c\right) x).$$

Proof: Let $\tilde{x} = J_f^t J_f x - J_c^t J_c x$. Then we have

$$0 \leq (\tilde{x}, M\tilde{x}) = (x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x) - (x, (J_f^t J_f M J_c^t J_c + J_c^t J_c M J_f^t J_f)x).$$

The lemma follows from

$$\begin{aligned}(x, Mx) &= (x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x) + (x, (J_f^t J_f M J_c^t J_c + J_c^t J_c M J_f^t J_f)x) \\ &\leq 2(x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x).\end{aligned}$$

For any matrix M which is SPD with respect to some inner product, define its condition number by

$$\kappa(M) = \rho(M)\rho(M^{-1}),$$

where ρ stands for the spectral radius of a matrix.

Theorem 1 *Assume that A is symmetric and diagonally dominant, $T(A)$ is nonsingular and W is SPD. Then*

$$\begin{aligned}
&\kappa(PQ^{-1}RA) \\
&\quad\leq\ 2\max\left\{\rho(W^{-1}J_fRAPJ_f^t),1\right\}\left(1+2\|P_{ff}\|\sqrt{\eta\|A\|}+\eta\|RAP\|+\eta\|W\|\right), \quad (9)
\end{aligned}$$

with $\eta = \left(\sqrt{2}+1\right)^2\|A\|$.

Proof: Since A is symmetric and diagonally dominant, it follows from Gershgorin's theorem that it is positive semidefinite. Let $x \in l_2(c \cup f)$ satisfy $\|x\| = 1$ and denote $\varepsilon = (x, Ax)$. Since A is symmetric and positive semidefinite, x may be written as a linear combination of the orthogonal eigenvectors of A. Consequently, $\|Ax\|^2 \leq \|A\|\varepsilon$.

Define

$$A_1 = \begin{pmatrix} (A-T(A))_{ss} & (A-T(A))_{sb} & (A-T(A))_{sc} \\ (A-T(A)^*)_{bs} & rs(|(A-T(A)^*)_{bs}|) & 0 \\ (A-T(A)^*)_{cs} & 0 & rs(|(A-T(A)^*)_{cs}|) \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} rs(|A_{sb}|) & A_{sb} & 0 \\ A_{bs} & (A-T(A))_{bb} & (A-T(A))_{bc} \\ 0 & (A-T(A)^*)_{cb} & rs(|(A-T(A)^*)_{cb}|) \end{pmatrix}.$$

Note that, when (4) and (5) hold, this simplifies to read

$$A_1 \equiv 0 \quad \text{and} \quad A_2 = \begin{pmatrix} rs(|A_{sb}|) & A_{sb} & 0 \\ A_{bs} & (A-T(A))_{bb} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since A_1, A_2, $A - A_1$ and $A - A_2$ are symmetric and diagonally dominant, it follows from Gershgorin's theorem that they are positive semidefinite. Using the same argument as in the beginning of the proof, one obtains

$$\|A_nx\|^2 \leq \|A_n\|(x, A_nx) \leq \|A_n\|\varepsilon, \quad n = 1, 2.$$

For convenience we use here the notation $f_1 = s$ and $f_2 = b$. Note that

$$J_{f_n}A_n = J_{f_n}(A - P^{-1}), \quad n = 1, 2.$$

Consequently,

$$\begin{aligned}
|\ \|J_fAx\| - \|J_fP^{-1}x\|\ |^2 \ &\leq\ \|J_f(A-P^{-1})x\|^2 = \sum_{n=1}^{2}\|J_{f_n}A_nx\|^2 \\
&\leq\ \sum_{n=1}^{2}\|A_nx\|^2 \leq (\|A_1\| + \|A_2\|)\,\varepsilon,
\end{aligned}$$

which implies that

$$\|J_fP^{-1}x\| \leq \sqrt{\eta\varepsilon}, \quad \text{where} \quad \eta = \left(\sqrt{\|A_1\|+\|A_2\|}+\sqrt{\|A\|}\right)^2 \leq \left(\sqrt{2}+1\right)^2\|A\|.$$

As a result, we have

$$\begin{aligned}(x, R^{-1}QP^{-1}x) &= (P^{-1}x, QP^{-1}x)\\ &= \left(J_c^tJ_cx + J_f^tJ_fP^{-1}x, Q\left(J_c^tJ_cx + J_f^tJ_fP^{-1}x\right)\right)\\ &\le (J_c^tJ_cx, RAP(J_c^tJ_cx)) + \eta\|W\|\varepsilon\\ &= \left(P\left(P^{-1}x - J_f^tJ_fP^{-1}x\right), AP\left(P^{-1}x - J_f^tJ_fP^{-1}x\right)\right) + \eta\|W\|\varepsilon\\ &\le \left(1 + 2\|P_{ff}\|\sqrt{\eta\|A\|} + \eta\|RAP\| + \eta\|W\|\right)\varepsilon,\end{aligned}$$

which implies that the function $(x, R^{-1}QP^{-1}x)/(x, Ax)$ is bounded. On the other hand, we have from Lemma 1 that, for any $y \in l_2(c \cup f)$,

$$\begin{aligned}(y, RAPy) &\le 2(y, \left(J_f^tJ_fRAPJ_f^tJ_f + J_c^tJ_cRAPJ_c^tJ_c\right)y)\\ &\le 2\max\left\{\rho(W^{-1}J_fRAPJ_f^t), 1\right\}(y, Qy),\end{aligned}$$

which implies that the function $(x, Ax)/(x, R^{-1}QP^{-1}x)$ is bounded. This completes the proof of the theorem.

Although W in (7) is not SPD, Theorem 1 can be applied with W replaced by an SPD matrix such as $blockdiag(W)$ (or $diag(W)$, or $I_{|f|}$). Using this, Theorem 1 applies also to the nonsymmetric matrix W of (7), provided that the bound in (9) is multiplied by the condition number of

$$P\begin{pmatrix} blockdiag(W)^{-1}W & 0\\ 0 & I_{|c|}\end{pmatrix}P^{-1} \qquad \text{(see [Sha96b] for the details).}$$

Note that the above analysis would still be valid if

1. the inner product used above had been replaced by the inner product $(D\cdot,\cdot)$ induced by some diagonal SPD matrix D;
2. the above vector norm had been replaced by $\|\cdot\|_D = \sqrt{(D\cdot,\cdot)}$;
3. the above matrix norm had been replaced by the matrix norm induced by D, that is, $\|\cdot\|_D = \|D^{1/2}\cdot D^{-1/2}\|$
4. and the terms 'symmetry', 'SPD' and 'orthogonality' and the conjugate '*' had been interpreted with respect to $(D\cdot,\cdot)$.

Consequently, we have

Corollary 1 *Assume that A is symmetric and diagonally dominant and $T(D^{-1}A)$ is nonsingular, where D is a diagonal SPD matrix. Let $\tilde{T}(A)$ be the same matrix as $T(A)$ but with the lower right block $I_{|c|}$ replaced by D_{cc}. Define*

$$P = T(D^{-1}A)^{-1} = \tilde{T}(A)^{-1}D, \qquad R = D^{-1}P^tD = \tilde{T}(A)^{-t}D$$
$$\textit{and} \quad Q = blockdiag(I_{|f|}, J_cRAPJ_c^t).$$

Then

$$\begin{aligned}&\kappa(PQ^{-1}RD^{-1}A)\\ &\le 2\max\left\{\rho(J_fRD^{-1}APJ_f^t), 1\right\}\left(1 + 2\|P_{ff}\|_{D_{ff}}\sqrt{\eta\|D^{-1}A\|_D} + \eta\|RD^{-1}AP\|_D + \eta\right)\\ &= 2\max\left\{\rho(J_f\tilde{R}\tilde{A}\tilde{P}J_f^t), 1\right\}\left(1 + 2\|J_f\tilde{P}J_f^t\|\sqrt{\eta\|\tilde{A}\|} + \eta\|\tilde{R}\tilde{A}\tilde{P}\| + \eta\right),\end{aligned} \qquad (10)$$

where $\tilde{A} = D^{-1/2}AD^{-1/2}$, $\tilde{P} = D^{1/2}\tilde{T}(A)^{-1}D^{1/2}$ *and* $\tilde{R} = D^{1/2}\tilde{T}(A)^{-t}D^{1/2}$

$$\text{and } \eta = \left(\sqrt{2}+1\right)^2 \|D^{-1}A\|_D = \left(\sqrt{2}+1\right)^2 \|\tilde{A}\|.$$

4 Discussion

The bound in (10) depends on $\|\tilde{A}\|$, $\|\tilde{R}\|$ and $\|\tilde{P}\|$. The latter two quantities may be large if a submatrix of the form $T(A)_{BB}$ (corresponding to B in Figures 3 or 4) is nearly singular, which might happen if the variables in B are nearly decoupled in A, or if a submatrix of the form A_{SS} (corresponding to S in Figure 5) is nearly singular, which might happen if $|S|$ is large. Consequently, we have

Corollary 2 *Assume that*

1. *A well-posed problem of the form* $(\mathcal{D}u_x)_x + (\mathcal{D}u_y)_y = f$ *(with suitable boundary conditions and* $\mathcal{D}$ *a uniformly positive function) is considered.*
2. *The scheme is a linear finite element scheme on a triangle mesh.*
3. *The element angles are all less than or equal to* $\pi/2$.
4. *The element angles of the form* $\angle XYZ$ *where the nodes* X *and* Z *lie on one boundary or interface segment are all less than or equal to a mesh-independent constant smaller than* $\pi/2$.
5. *The number of elements per subdomain is bounded mesh-independently.*
6. $D = diag(A)$ *is defined and then (1) is scaled in advance by multiplying it from the left by* D^{-1} *and then the operators* $P = T(A)^{-1}$, $R = D^{-1}P^tD$ *and* Q *as in (6) are defined.*

Then ADD converges with a convergence rate which is independent of both the mesh-size and the jump in the diffusion coefficient $\mathcal{D}$.

Indeed, since (1) is scaled in advance, the coefficient matrix is symmetric with respect to $(D\cdot,\cdot)$ and, therefore, Corollary 1 implies that the convergence rate is independent of the jump in the diffusion coefficient $\mathcal{D}$. Using the other assumptions, it follows from Section 4 in [Sha97] that A is diagonally dominant and that the matrices of the form $T(A)_{BB}$ and $T(A)_{SS}$ are diagonally dominant M-matrices of a bounded order with main diagonal elements bounded away from zero and, therefore, their inverses are of a bounded norm.

Since the definition of ADD relies on the domain decomposition and the coefficient matrix only, it is applicable also to non-conformal schemes and nonmatching grids, provided that the underlying linear system is available. Furthermore, it can be extended in a natural way to the case of overlapping subdomains. The key is the definition of the prolongation operator from the coarse grid to the rest of the nodes. The first prolongation step determining the values at the interface and boundary unknowns is done as before. The second prolongation step determining the values in subdomain interiors is also done as before in each subdomain separately, and then the average of the multiply defined values is taken for nodes in the overlapping area.

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