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# Two-level Schwarz Methods for Indefinite Integral Equations

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## 1 Introduction

In this paper we consider additive Schwarz preconditioners for indefinite linear systems arising from the $h$-version of the boundary element method (BEM) for solving Helmholtz problems. Here we extend the approach introduced by Cai and Widlund [CW92] for finite element discretizations to boundary element discretizations. We report on two-level methods applied to the $h$-version of the Galerkin method for weakly singular and hypersingular integral equations of the first kind on the interval $\Gamma = (-1, 1)$. The Neumann problem for the Helmholtz equation in $\mathbb{R}^2 \backslash \bar{\Gamma}$ leads to the hypersingular integral equation

$$D_k v(x) := -\frac{i}{2} \frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} [H_0^1(k|x-y|)] v(y)\, ds_y = g_1(x), \quad x \in \Gamma. \tag{1.1}$$

Correspondingly the Dirichlet problem leads to the weakly singular integral equation

$$V_k \psi(x) = \int_\Gamma H_0^1(k|x-y|) \psi(y)\, ds_y = g_2(x), \quad x \in \Gamma. \tag{1.2}$$

There $H_0^1$ is the Hankel function of the first kind and of order zero, $\mathrm{Im} k \geq 0$, $k \neq 0$ and $\frac{\partial}{\partial n}$ denotes the normal derivative on $\Gamma$.

It was shown in [SW90, SW84] that for $g_1 \in H^{-1/2}(\Gamma)$ equation (1.1) has a unique solution in $\tilde{H}^{1/2}(\Gamma) := H_{00}^{1/2}(\Gamma)$ whereas for given $g_2 \in H^{1/2}(\Gamma)$ equation (1.2) has a unique solution in $\tilde{H}^{-1/2}(\Gamma)$ (the dual of $H^{1/2}(\Gamma)$). (For definitions of the Sobolev spaces see [LM72]). Note that $D_k$ is a pseudodifferential operator $B_\alpha$ of order $\alpha = 1$ and $V_k$ of order $\alpha = -1$ mapping $\tilde{H}^\alpha(\Gamma)$ into $H^{-\alpha/2}(\Gamma)$ both satisfying $B_\alpha = A_\alpha + K_\alpha$ with a positive definite operator $A_\alpha$ and a compact operator $K_\alpha$.

With $f = g_1$ in (1.1) and $f = g_2$ in (1.2) the boundary element Galerkin schemes for the above integral equations ($\alpha = 1$ or $\alpha = -1$) read as follows:
For a given subspace $X_N^\alpha$ of $\tilde{H}^{\alpha/2}$ find $u_N \in X_N^\alpha$ such that

$$\langle B_\alpha u_N, \phi \rangle_{L^2(\Gamma)} = \langle f, \phi \rangle_{L^2(\Gamma)} \quad \text{for all } \phi \in X_N^\alpha. \tag{1.3}$$

*Ninth International Conference on Domain Decomposition Methods*
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes 

These Galerkin schemes lead to very large indefinite systems of linear equations with dense and ill-conditioned system matrices and therefore iterative methods require good preconditioners [ST97a, ST97b, MS97, MST97]. We report on additive Schwarz methods applied to (1.3) which are efficient preconditioners for the GMRES method. For efficient Schwarz preconditioners for positive definite boundary integral equations see [TS97, HS96, Ste96].

## 2 Preconditioners for the Hypersingular Operator

As subspace $X_N^1$ we use the subspace $S_h^1(\Gamma)$ of continuous, piecewise linear functions on a quasi-uniform mesh which vanish at the endpoints of $\Gamma$. Let $\phi_j^h$, $j = 1, \ldots, N-1$ denote the hat function which takes value 1 at the meshpoint $x_j$ and 0 at other mesh points. These functions form a basis for $S_h^1(\Gamma)$. We then decompose $S = S_h^1(\Gamma)$ as

$$S = S_0 + S_1 + \ldots + S_{N-1} \tag{2.4}$$

where $S_0 = S_H^1(\Gamma)$ is defined as $S_h^1(\Gamma)$ with mesh size $H = 2h$ and $S_j = \operatorname{span}\{\phi_j^h\}$ for $j = 1, \ldots, N-1$.

Let operators $Q_j$ be defined via

$$\langle B_1 Q_j w, v_j \rangle = \langle B_1 w, v_j \rangle \quad \forall w \in S, v_j \in S_j, j = 0, 1, \ldots, N-1. \tag{2.5}$$

Then the additive Schwarz operator is given by $Q = Q_0 + \ldots + Q_{N-1}$ and the additive Schwarz method consists in solving

$$Q u_N = b_N \tag{2.6}$$

with RHS $b_N = \sum_{j=0}^N b_j$ where

$$\langle b_j, v_j \rangle = \langle f, v_j \rangle \; \forall v_j \in S_j, \; j = 0, 1, \ldots, N-1. \tag{2.7}$$

Then as shown in [ST97a] this algorithm when used with the GMRES method gives an efficient solver for the Galerkin scheme (1.3), namely the rates of convergence of the Schwarz operator is bounded from above independently of the number of degrees of freedom if the mesh size of the coarse space $S_0$ is sufficiently small. As proved in [CW92] the rate of convergence of the GMRES method when used to solve (2.6) is given as $1 - \frac{C_0^2}{C_1^2}$ where

$$C_0^2 = \inf_{v \in S} \frac{\langle A_1 v, Qv \rangle}{\langle A_1 v, v \rangle} \text{ and } C_1^2 = \sup_{v \in S} \frac{\langle A_1 Qv, Qv \rangle}{\langle A_1 v, v \rangle}. \tag{2.8}$$

In view of this result we show in [ST97a] that $C_0$ and $C_1$ are independent of the number of degrees of freedom.

To get rid of a large coarse subspace $S_0$ we consider in [MS97] a non-overlapping method where one has a coarse mesh which is almost independent of the fine mesh.

**The coarse mesh:** We divide $\Gamma$ into disjoint subdomains $\Gamma_i$, $i = 1, \ldots, J$, so that $\bar{\Gamma} = \cup_{i=1}^J \bar{\Gamma}_i$. The length of $\Gamma_i$ is denoted by $H_i$.

**The fine mesh:** We further divide each $\Gamma_i$ into disjoint subintervals $\Gamma_{ij}$, $j = 1, \dots, N_i$, so that $\bar{\Gamma}_i = \cup_{j=1}^{N_i} \bar{\Gamma}_{ij}$. The maximum length of the subintervals in $\Gamma_i$ is denoted by $h_i$. For the non-overlapping method, we require that the fine mesh is locally quasi-uniform, i.e., it is quasi-uniform in each subdomain.

The additive Schwarz method is designed via an appropriate decomposition of $S_h(\Gamma)$

$$S_h(\Gamma) = S_H(\Gamma) \oplus \bigoplus_{i=1}^{J} \bigoplus_{j=1}^{N_i} S_1^0(\Gamma_{ij}) \tag{2.9}$$

where

$$S_H(\Gamma) := \{v \in C(\Gamma) : v|_{\Gamma_i} \in \mathcal{P}_1(\Gamma_i) \text{ for } i = 1, \dots, J;\ v(\pm 1) = 0\}$$
$$S_1^0(\Gamma_{ij}) := \{v \in \mathcal{P}_1(\Gamma_{ij}) : v = 0 \text{ at the endpoints of } \Gamma_{ij}\},\ i = 1, \dots, J,\ j = 1, \dots, N_i.$$

Then the corresponding algorithm consists in solving (2.6) with the Schwarz operator $Q = Q_0 + Q_{11} + \dots + Q_{JN_J}$.

Here for $i = 1, \dots, J$; $j = 1, \dots, N_i$ and for any $w \in S_h(\Gamma)$, $Q_{ij}w \in S_1^0(\Gamma_{ij})$ is the solution of the boundary element equation

$$\langle B_1 Q_{ij} w, v_{ij} \rangle = \langle B_1 w, v_{ij} \rangle \quad \forall v_{ij} \in S_1^0(\Gamma_{ij}) \tag{2.10}$$

and $Q_0 w \in S_H(\Gamma)$ solves

$$\langle B_1 Q_0 w, v_0 \rangle = \langle B_1 w, v_0 \rangle \quad \forall v_0 \in S_H(\Gamma). \tag{2.11}$$

In [MS97] we show for $H_0$ sufficiently small and $H_i \le H_0$ that $C_0$, $C_1$ in (2.8) satisfy

$$C_0^{-1} \sim \max_{1 \le i \le J} \left(1 + \log \frac{H_i}{h_i}\right) \text{ and } C_1 = \text{const.} \tag{2.12}$$

## 3 Preconditioners for the Weakly Singular Operator

As subspace $X_N^{-1}$ we use the space $S_h^0(\Gamma)$ of piecewise constant functions on a quasiuniform mesh of $\Gamma$. Let $\phi_j^h$, $j = 1, \dots, N-1$ denote the hat function in Section 2 and let the Haar basis function $\chi_j^h$ be defined as the derivative of $\phi_j^h$. These functions $\chi_j^h$ together with the constant function 1 form a basis for $\bar{S} = S_h^0(\Gamma)$. We then decompose $\bar{S}$ as

$$\bar{S} = \bar{S}_0 + \bar{S}_1 + \dots + \bar{S}_{N-1} \tag{3.13}$$

where $\bar{S}_0$ is defined as $S_h^0(\Gamma)$ with mesh size $H = 2h$ and where $\bar{S}_j = \operatorname{span}\{\chi_j^h\}$, for $j = 1, \dots, N-1$. Then an associate additive Schwarz method can be defined for the weakly singular integral equation via operators $\bar{Q}_j$ which are given by (2.5) with $B_{-1}$ instead $B_1$ and $\bar{S}, \bar{S}_j$ substituting $S, S_j$ respectively. Analogously to (2.9) a non-overlapping subspace decomposition of $\bar{S}$ may be introduced and corresponding operators $\bar{Q}, \bar{Q}_0, \bar{Q}_{ij}$ when using $B_{-1}$ instead of $B_1$ in (2.11).

Then again for (3.13) the additive Schwarz method yields a GMRES method with convergence rates strictly less than 1 independently of the degrees of freedom [ST97b] whereas in the non-overlapping case again the constants $C_0, C_1$ of (2.8) show the behavior (2.12).

## 4 Numerical Results

The numerical experiments for the hypersingular integral equation (1.1) with $g_1(x) \equiv 1$ and wavenumber $k = 2.0$ on a quasiuniform mesh were performed on a SUN-Sparcstation 4/470 at the Institute for Applied Mathematics at University of Hannover. Here we give the eigenvalues and condition numbers which are linked to the rate of convergence by $C_1 = \lambda_{\max}$ and $C_0 = \sqrt{\lambda_{\min}}$. Table 1 gives the absolute values $\lambda_{\min}$, $\lambda_{\max}$ and the condition number of the unpreconditioned Galerkin system (1.3) and of the additive Schwarz preconditioner (2.6) with $H = 2h$.

In Table 2 the condition numbers for the additive Schwarz operator defined by (2.10) and (2.11) are given for different quotients $H/h$.

**Table 1** Hypersingular integral equation (1.1) with $g_1(x) \equiv 1$: quasiuniform $h$-version, wavenumber $k = 2.0$

| | Stiffness matrix | | | 2-level | | |
|---|---|---|---|---|---|---|
| $N$ | $\lambda_{\min}$ | $\lambda_{\max}$ | cond | $\lambda_{\min}$ | $\lambda_{\max}$ | cond |
| 31 | 8.8633d-02 | 1.1308 | 12.7586 | 1.0045 | 2.2324 | 2.2224 |
| 63 | 4.5142d-02 | 1.1333 | 25.1050 | 0.9923 | 2.2312 | 2.2484 |
| 127 | 2.2750d-02 | 1.1339 | 49.8413 | 0.9880 | 2.2304 | 2.2573 |
| 255 | 1.1399d-02 | 1.1340 | 99.4872 | 0.9875 | 2.2302 | 2.2582 |
| 511 | 5.7018d-03 | 1.1341 | 198.9038 | 0.9884 | 2.2300 | 2.2560 |
| 1023 | 2.8509d-03 | 1.1341 | 397.8011 | 0.9896 | 2.2299 | 2.2532 |

**Table 2** Condition numbers for additive Schwarz operator of the hypersingular integral equation (1.1) with $g_1(x) \equiv 1$: quasiuniform $h$-version, wavenumber $k = 2.0$

| $N \setminus H/h$ | 2 | 4 | 8 | 16 | 32 | 64 | 128 |
|---|---|---|---|---|---|---|---|
| 64 | 3.7124 | 3.9360 | 5.0803 | 5.9118 | 6.3642 | | |
| 128 | 3.5003 | 4.0075 | 5.1320 | 6.1031 | 6.7030 | 7.0320 | |
| 256 | 3.3507 | 4.0258 | 5.2133 | 6.4471 | 7.5519 | 8.2994 | 8.3348 |
| 512 | 3.2436 | 4.0305 | 5.2341 | 6.5358 | 7.9045 | 9.1635 | 10.0478 |

## 5 Conclusion

The numerical examples clearly underline the theoretical results, i.e. the condition numbers of both methods are independent of the number of degrees of freedom but the condition numbers for the additive Schwarz operator defined by (2.10) and (2.11) depend logarithmically on $H/h$. Whereas the first method is only of theoretical interest due to the large coarse grid space, the second method can be implemented in an efficient and parallel way if we choose the size of the subspaces appropriately.

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