50

Some Recent Results on Domain Decomposition Methods for Eigenvalue Problems

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1 Introduction

Domain decomposition methods for partial differential equations have traditionally been classified into two types: the Schwarz type where subdomains overlap and the Schur complement type where subdomains do not overlap. In this paper, we focus mainly on Schwarz algorithms for the eigenvalue problem for self–adjoint operators. Specifically, we study the model problem

$$-\triangle u = \lambda u \text{ on } \Omega$$

with homogeneous Dirichlet boundary conditions. Here, Ω is a bounded open domain with a smooth boundary. We shall discuss two different Schwarz algorithms with a brief mention of Schur algorithms. Some open problems will also be raised.

An incomplete list of papers on domain decomposition methods for the eigenvalue problems is [AGG88], [AG88], [Ben87], [Bou90], [Bou92], [Bd92], [Dri95] [DK92], [D'y96], [FG94], [HMV95], [Kny87], [KS89], [KS94], [Kro63], [Kuz86a], [Kuz86b], [LHL94], [Luo92], [Mal92], [Seh89], [Sim74] and [Sko91]. See also the references in these papers.

2 Schwarz Algorithms

Suppose the domain Ω is a union of $m > 1$ overlapping subdomains $\Omega_1 \cup \cdots \cup \Omega_m$. We discuss two Schwarz alternating methods.

Maliassov's Algorithm

This algorithm works with the variational formulation of the eigenvalue problem. Let (u, v) denote the usual $L^2(\Omega)$ inner product and $\|u\|^2 = (u, u)$. Denote the

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energy inner product in the Sobolev space $H_0^1(\Omega)$ by $[u,v] = \int_\Omega \nabla u \cdot \nabla v$ and let $\|u\|_2 = \left(\int_\Omega (\triangle u)^2\right)^{1/2}$ for $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Let the eigenvalues of $-\triangle$ be $\lambda_1 < \lambda_2 \leq \cdots$ and let ϕ_i be some eigenfunction corresponding to λ_i. For any $u \in H_0^1(\Omega) \setminus 0$, define the Rayleigh Quotient

$$R(u) = \frac{[u,u]}{(u,u)}.$$

Maliassov [Mal92] recently gave the following Schwarz Alternating Method to find the smallest eigenvalue and its associated eigenfunction and 'proved' its convergence.

Let $u^{(0)} \in H_0^1(\Omega) \setminus 0$ with $\lambda_1 \leq R(u^{(0)}) < \lambda_2$. For $n \geq 0$ and $1 \leq i \leq m$, define the sequence

$$\begin{aligned} \lambda^{(n+\frac{i}{m})} &= \inf\{R(u^{(n+\frac{i-1}{2})} + v_i);\, v_i \in H_0^1(\Omega) \setminus 0 \text{ with } v_i = 0 \text{ on } \Omega \setminus \Omega_i\} \\ &\equiv R(u^{(n+\frac{i}{m})}). \end{aligned}$$

(We identify the index $n + \frac{0}{m}$ with n and $n + \frac{m}{m}$ with $n+1$.) Then $\lim_{n\to\infty} \lambda^{(n)} = \lambda_1$ and a subsequence of $u^{(n)}$ converges to ϕ_1.

However, there is a small flaw in the algorithm. The fault lies in his definition of $u^{(n+\frac{i}{m})}$ which may not exist as the following example shows. Consider

$$-u'' = \lambda u \text{ on } (0, \tfrac{3\pi}{2})$$

with homogeneous Dirichlet boundary conditions. The smallest eigenvalue is $\frac{4}{9}$ with $\sin\frac{2x}{3}$ as a corresponding eigenfunction. Let Ω_1 be the interval $(0,\pi)$ and Ω_2 be the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$. Take

$$u^{(0)} = \begin{cases} 2\sin x, & \text{if } 0 \leq x \leq \pi; \\ \sin 2x, & \text{if } \pi \leq x \leq \frac{3\pi}{2}. \end{cases}$$

Then $\lambda^{(0)} = \frac{4}{3}$. Let $D_1 = \{v_1 \in H_0^1(0, \frac{3\pi}{2}) \text{ with } v_1 = 0 \text{ on } [\pi, \frac{3\pi}{2}]\}$. Then,

$$\begin{aligned} \inf\{R(u^{(0)} + v_1);\ v_1 \in D_1\} &= \lim_{|a|\to\infty} R(u^{(0)} + aE\sin x) \\ &= 1, \end{aligned}$$

where $E\sin x$ is the function which is $\sin x$ on $[0,\pi]$ and is 0 on $[\pi, \frac{3\pi}{2}]$. This infimum cannot be attained by any $v_1 \in D_1$.

Despite this defect, Maliassov's main ideas are still valid. We now give a correct algorithm with a proof of convergence. Note that our result holds for an arbitrary eigenvalue, not just the smallest one. We restrict to the two-subdomain case.

Theorem 1 *Fix* $p \in \mathbf{N}$. *Define* $H = \{f \in H_0^1(\Omega) \cap H^2(\Omega);\ (f, \phi_i) = 0,\ i = 1, \cdots, p-1\}$ *and* $D_i = \{f \in H;\ f = 0 \text{ on } \Omega \setminus \Omega_i\}$, $i = 1, 2$. *Let the initial guess be* $u^{(0)} \in H \setminus 0$ *with* $\lambda^{(0)} = R(u^{(0)})$ *smaller than the first eigenvalue strictly larger than* λ_p. *For* $n \geq 0$ *and* $i = 1, 2$, *define the sequence*

$$\begin{aligned} \lambda^{(n+\frac{i}{2})} &= \inf\left\{R(cu^{(n+\frac{i-1}{2})} + v_i); c \in \mathbf{R}, v_i \in D_i, \|cu^{(n+\frac{i-1}{2})} + v_i\|_2 = 1\right\} \\ &\equiv R(u^{(n+\frac{i}{2})}). \end{aligned}$$

The above infimum is always attained. In case it is attained at more than one pair (c, v_i), any one can be taken to define $u^{(n+\frac{i}{2})}$. Then, $\lim_{n\to\infty} \lambda^{(n)} = \lambda_p$ and a subsequence of $u^{(n)}$ converges to ϕ_p in the energy norm.

Proof: We first show that the sequence is well–defined. Fix i and n. Let $c_j \in \mathbf{R}$ and $w_j \in D_i$ such that $z_j = c_j u^{(n+\frac{i-1}{2})} + w_j$ with $\|z_j\|_2 = 1$ and $R(z_j) \to \lambda^{(n+\frac{i}{2})}$ as $j \to \infty$. Since z_j is a bounded sequence in $(H, \|\cdot\|_2)$, there exists a subsequence, which we label by k_j, such that $z_{k_j} \to u^{(n+\frac{i}{2})}$ for some $u^{(n+\frac{i}{2})} \in H \setminus 0$ weakly in the norm $\|\cdot\|_2$ and strongly in the energy norm. Then $R(z_{k_j}) \to R(u^{(n+\frac{i}{2})})$ as $j \to \infty$ and the uniqueness of limit implies $\lambda^{(n+\frac{i}{2})} = R(u^{(n+\frac{i}{2})})$.

Since $\lambda^{(n+\frac{i}{2})}$ is a non–increasing sequence which is bounded below by λ_p, it must converge to some number, say, λ. Since $u^{(n)}$ is a bounded sequence in $(H, \|\cdot\|_2)$, there exists some subsequence which we label by n_j such that

$$u^{(n_j)} \to u$$

in the energy norm for some $u \in H \setminus 0$. Thus

$$\lim_{j\to\infty} R\left(u^{(n_j)}\right) = R(u) = \lambda.$$

We now show that (λ, u) is an eigenpair of $-\triangle$. Observe that for any $t \in \mathbf{R}$, $n \geq 1$ and $v_i \in D_i$, $i = 1, 2$,

$$R\left(u^{(n_j)} + tv_1\right) \geq \lambda^{(n_j+\frac{1}{2})}$$

and

$$R\left(u^{(n_j)} + tv_2\right) \geq R(u^{(n_j-1+\frac{1}{2})}) = \lambda^{(n_j-1+\frac{1}{2})}.$$

Taking the limit as $j \to \infty$ in the above inequalities, we obtain $R(u+tv_i) \geq \lambda,\ i = 1, 2$ which is equivalent to

$$\begin{aligned} t^2([v_i, v_i] - \lambda\|v_i\|^2) + 2t\,([u, v_i] - \lambda(u, v_i)) + [u, u] - \lambda\|u\|^2 &\geq 0 \\ t^2([v_i, v_i] - \lambda\|v_i\|^2) + 2t\,([u, v_i] - \lambda(u, v_i)) &\geq 0. \end{aligned}$$

This is possible only if $[u, v_i] = \lambda(u, v_i)$. Since the subdomains are overlapping, $H = D_1 + D_2$. Now any $v \in H_0^1(\Omega) \cap H^2(\Omega)$ can be represented as $v = v_1 + v_2 + \sum_{i=1}^{p-1} a_i\phi_i$ with $v_i \in D_i$ and $a_i \in \mathbf{R}$. Noting that $(u, \phi_l) = 0,\ l = 1, \cdots, p-1$, we obtain $[u, v] = \lambda(u, v)$. Since $H_0^1(\Omega) \cap H^2(\Omega)$ is dense in $H_0^1(\Omega)$, $[u, v] = \lambda(u, v)$ for all $v \in H_0^1(\Omega)$. Thus u is an eigenfunction with corresponding eigenvalue λ. By the variational principle for eigenvalues and the choice of initial guess, we must have $\lambda = \lambda_p$.

The general multiple–subdomain case is considered in [Lui96c]. ¿From the point of view of parallel computation, the above algorithm is not satisfactory because the computation on subdomain Ω_i must precede that on Ω_{i+1}. We now propose a version in which the computation in each subdomain can be carried out simultaneously. However, the calculation of the eigenvalue λ_p must precede that of λ_{p+1}. We shall consider the general m-subdomain case. The notation will be as in the previous theorem with the exception that we no longer identify an element indexed by $n+1$ with one indexed by $n + \frac{m}{m}$.

Theorem 2 *Fix $p \in \mathbf{N}$. Let the initial guess be $u^{(0)} \in H \setminus 0$ with $\lambda^{(0)} = R(u^{(0)})$ smaller than the first eigenvalue strictly larger than λ_p. For $n \geq 0$ and $1 \leq i \leq m$, define the sequences*

$$\begin{aligned} \lambda^{(n+\frac{i}{m})} &= \inf\{R(cu^{(n)} + v_i);\ c \in \mathbf{R}, v_i \in D_i, \|cu^{(n)} + v_i\|_2 = 1\} \\ &\equiv R(u^{(n+\frac{i}{m})}) \end{aligned}$$

and

$$\lambda^{(n+1)} = R(u^{(n+1)}) \equiv \min\left\{ R\left(\sum_{i=0}^{m} c_i u^{(n+\frac{i}{m})}\right);\ \sum_{i=0}^{m} c_i^2 = 1 \right\}.$$

The above infimum is always attained and in case the infimum is attained at more than one pair (c, v_i), then define $u^{(n+\frac{i}{m})}$ from any one of them. Similarly for the minimization problem. Then, $\lim_{n\to\infty} \lambda^{(n)} = \lambda_p$ and a subsequence of $u^{(n)}$ converges to ϕ_p in the energy norm.

The proof is very similar to that of the previous theorem and can be found in [Lui96c].

Note that in the statement of these theorems, only a subsequence converges. If the eigenvalue that we seek is a multiple eigenvalue, it is quite possible that different subsequences converge to different eigenfunctions corresponding to the same multiple eigenvalue. We are currently investigating whether the entire sequence converges in case the eigenvalue is simple.

If the initial Rayleigh Quotient is sufficiently large, then the Maliassov sequence for a 3-subdomain example converges to a different eigenvalue ([Lui96c]). These occurrences are rare and from our limited experience, the algorithms do converge globally in practice. We conjecture that global convergence holds for the 2-subdomain case. This article does not touch upon implementation issues. For an efficient hierarchical implementation and further theoretical results of Maliassov's algorithm, see the article by Chan and Sharapov elsewhere in this volume.

Another Schwarz Algorithm

We now discuss briefly another Schwarz algorithm which transmits information between the subdomains by boundary functions. Consider Ω to be an union of two overlapping subdomain Ω_1 and Ω_2. Let Γ_1 be $\partial\Omega_1 \cap \Omega_2$ and Γ_2 be $\partial\Omega_2 \cap \Omega_1$. The idea is to solve an eigenvalue problem in each subdomain with Robin boundary conditions. In this section, we are only concerned with the smallest eigenvalue and its associated eigenfunction which is nonzero in Ω.

For any positive integer i, define $u_2^{(i)}$ as the solution of the eigenvalue problem $-\triangle u_2^{(i)} = \lambda_2 u_2^{(i)}$ on Ω_2 with boundary conditions $u_2^{(i)} = 0$ on $\partial\Omega_2 \setminus \Gamma_2$ and $g_2^{(i)} u_2^{(i)} + \frac{\partial u_2^{(i)}}{\partial n_2} = 0$ on Γ_2, where $g_2^{(i)}$ is an estimate of the true boundary function on Γ_2 approximated by $u_1^{(i)}$:

$$g_2^{(i)} u_1^{(i)} + \frac{\partial u_1^{(i)}}{\partial n_2} = 0 \text{ on } \Gamma_2.$$

Here, n_i denotes the unit outward normal on Ω_i. Define $u_1^{(i+1)}$ as the solution of the eigenvalue problem on Ω_1 with boundary conditions $u_1^{(i+1)} = 0$ on $\partial\Omega_1 \setminus \Gamma_1$ and $g_1^{(i)} u_1^{(i+1)} + \frac{\partial u_1^{(i+1)}}{\partial n_1} = 0$ on Γ_1, where

$$g_1^{(i)} u_2^{(i)} + \frac{\partial u_2^{(i)}}{\partial n_1} = 0 \text{ on } \Gamma_1.$$

Note that $g_1^{(i)}$ is an estimate of the true boundary function on Γ_1 approximated by $u_2^{(i)}$. The above sequences are defined once we specify the initial iterate $u_1^{(1)}$. Note that we introduced the sequence of boundary functions $g^{(i)}$ for explanation purpose only. The actual boundary conditions can be simplified to

$$u_2^{(i)} \frac{\partial u_1^{(i+1)}}{\partial n_1} - \frac{\partial u_2^{(i)}}{\partial n_1} u_1^{(i+1)} = 0 \text{ on } \Gamma_1$$

and

$$u_1^{(i)} \frac{\partial u_2^{(i)}}{\partial n_2} - u_2^{(i)} \frac{\partial u_1^{(i)}}{\partial n_2} = 0 \text{ on } \Gamma_2.$$

We have not been able to show convergence of the above sequences. The method does converge in the few numerical experiments that we have tried. Local convergence and the exact rate of convergence for the one–dimensional problem is shown in [Lui96a]:

Theorem 3 *The Schwarz alternating method for the one–dimensional eigenvalue problem converges if the initial guess is sufficiently close to the true solution.*

For $0 < a < b < \pi$, let the subdomains covering $[0, \pi]$ be $(0, b)$ and (a, π). The sequences of eigenvalue problems reduce to

$$-u_1^{(i)''} = \lambda_1^{(i)} u_1^{(i)} \text{ on } (0, b), \quad u_1^{(i)}(0) = 0, \ u_1^{(i)} u_2^{(i-1)'} - u_2^{(i-1)} u_1^{(i)'} \big|_{x=b} = 0$$

and

$$-u_2^{(i)''} = \lambda_2^{(i)} u_2^{(i)} \text{ on } (a, \pi), \quad u_2^{(i)} u_1^{(i)'} - u_1^{(i)} u_2^{(i)'} \big|_{x=a} = 0, \ u_2^{(i)}(\pi) = 0$$

for $i = 1, 2, \ldots$. The sequences are defined after prescribing the 'initial condition' $u_2^{(0)}$. The exact solutions are:

$$u_1^{(i)}(x) = \sin(\alpha_i x), \quad u_2^{(i)}(x) = \sin(\beta_i(\pi - x)), \ i = 1, 2, \ldots$$

where the constants α_i and β_i are determined by the interior boundary conditions. After some algebra, we find that these constants are the smallest positive roots of the equations

$$\beta_{i-1} \cot\left(\beta_{i-1}(\pi - b)\right) + \alpha_i \cot(\alpha_i b) = 0 \tag{2.1}$$

and

$$\alpha_i \cot(\alpha_i a) + \beta_i \cot\left(\beta_i(\pi - a)\right) = 0, \ i = 1, 2, \ldots. \tag{2.2}$$

Once the value of β_0 has been specified, these sequences are well–defined. The proof of convergence reduces to showing that both the sequences α_i and β_i converge to one. The rate of convergence can be measured by

$$r_i = \left| \frac{\alpha_i - 1}{\alpha_{i-1} - 1} \right| \quad \text{and} \quad s_i = \left| \frac{\beta_i - 1}{\beta_{i-1} - 1} \right| .$$

It can be shown that

$$\lim_{i\to\infty} r_i = \lim_{i\to\infty} s_i = \frac{\cot(\pi - b) - \frac{\pi - b}{\sin^2(\pi - b)}}{\cot(\pi - a) - \frac{\pi - a}{\sin^2(\pi - a)}} \, \frac{\cot a - \frac{a}{\sin^2 a}}{\cot b - \frac{b}{\sin^2 b}} \equiv r.$$

For $0 < a < b < \pi$, it can be shown that $0 < r < 1$ and thus the sequences α_i and β_i converge to 1 at rate r asymptotically.

3 Schur Algorithms

The first domain decomposition algorithm for eigenvalue problems was derived by Kron [Kro63] and it is a Schur–type algorithm. Most of the papers listed earlier also belong to this category. For simplicity, let the domain consist of two non–overlapping subdomains Ω_1, Ω_2 with interface Γ separating them. Assume that the discrete eigenvalue problem can be written in the form

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} . \tag{3.3}$$

Here, λ is some eigenvalue, u_i is the vector of unknowns in $\Omega_i, i = 1, 2$ and u_3 is the vector of unknowns along the interface. The matrices A_{ii} are assumed to be symmetric.

Formally, we may solve for u_1 and u_2 in terms of u_3. Substituting the results into the third equation in (3.3), we obtain

$$S(\lambda) u_3 \equiv \left[(A_{33} - \lambda) - A_{13}^T (A_{11} - \lambda)^{-1} A_{13} - A_{23}^T (A_{22} - \lambda)^{-1} A_{23} \right] u_3 = 0.$$

The matrix S is of dimension equal to the number of unknowns on the interface Γ and is thus much smaller than the size of the original matrix. Under some mild conditions ([Lui96b]), the eigenvalues of the original global matrix are precisely the values of λ at which $S(\lambda)$ has a zero eigenvalue. One way of accomplishing this is to find a root of the nonlinear equation $f(\lambda) \equiv \det S(\lambda) = 0$. It can be shown that f has poles at the union of the set of eigenvalues of A_{11} and of A_{22}. If an initial guess is not very close to the desired eigenvalue, it is quite possible that an iterate of Newton's or secant method may jump to a different interval bounded by different poles of f.

For instance, we consider the case of finding the first (smallest) eigenvalue λ_1 of the global matrix with an upper bound γ_1 which is a simple pole of f. For simplicity, assume that λ_1 is the only eigenvalue less than γ_1. Because γ_1 is a pole of f, a natural method is to find the zero of the de-singularized function $g(\lambda) = (\lambda - \gamma_1) f(\lambda)$ using Newton's, secant or Muller's method safeguarded by bisection. See also [AGG88].

Once a zero eigenvalue of $S(\lambda)$ has been found, one inverse iteration, for instance, may be used to determine its corresponding eigenvector u_3. The other components u_1 and u_2 of an eigenvector of the global matrix can subsequently be found from subdomain solves. See [Lui96b] for some theoretical results concerning the relationships among poles and zeroes of the eigenvalues of S and the eigenvalues of the original matrix. In [LG96], the smallest eigenvalue was computed using inverse iteration and employing preconditioned Krylov space methods. An open problem here is how to compute interior eigenvalues without the explicit formation of the matrix S. The explicit formation of S permits us to use direct methods to compute its inertia which in turn permits us to compute any specified eigenvalue ([Seh89]). Currently, there is no known fast method to determine the inertia of a matrix using only the knowledge of the action of the matrix on a vector. Without knowing the inertia, it does not seem possible to have an algorithm which guarantees that a prescribed eigenvalue is found.

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