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Generalized Neumann-Neumann Preconditioners for Iterative Substructuring

Patrick Le Tallec and Marina Vidrascu

1 Introduction

In iterative substructuring, the parallel solution of a complex structural problem is achieved by splitting the original domain of computation in smaller nonoverlapping simpler subdomains, and by reducing the initial problem to an interface system with matrix

$$\mathbf{S} = \sum_i \mathbf{R}_i^t \mathbf{S}_i \mathbf{R}_i, \quad \mathbf{S}_i = \bar{\mathbf{K}}_i - \mathbf{B}_i^t (\mathring{\mathbf{K}}_i)^{-1} \mathbf{B}_i$$

to be solved by a parallel preconditioned conjugate gradient method. Many variants of this approach have been proposed and investigated in the recent literature, all associated to different choices of preconditioners. It turns out, in fact, that the interface problem requires specific preconditioners which take advantage of its particular structure. Such preconditioners must have nice parallel properties, must be able to handle arbitrary elliptic operators and discretization grids, and their performance must be insensitive to the discretization step h and to the number of subdomains. Many such preconditioners have appeared in the literature, following the early work of Bramble, Pasciak and Schatz [BPS86] ([CM94], [CMW93], [DW92], [Man90], [Wid88]). For three dimensional elasticity, efficient results have been obtained using either wire-basket algorithms such as proposed in Smith [Smi92] or Neumann-Neumann preconditioners ([DLV91], [LeT94]).

This last choice uses as preconditioner the following weighted sum of inverses [MB93] :

$$\mathbf{M}^{-1} = \mathbf{P} + (\mathbf{I} - \mathbf{P}) \left(\sum_i \mathbf{D}_i \mathbf{S}_i^{-1} \mathbf{D}_i^t \right) (\mathbf{I} - \mathbf{P}),$$

with $\mathbf{P}$ a coarse projection operator, and $\mathbf{D}_i$ a local partition of unity to be adapted to coefficients heterogeneities. This preconditioner is very general and can be applied to linear or nonlinear three dimensional elasticity problems using either matching or

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

non matching grids [LSV94], to nonlinear plates or shells problems [LMVed], or to incompressible flow problems [LP96].

It turns out that all these situations can be described and analyzed by a unique abstract framework. Indeed, the Neumann-Neummann algorithm is a standard additive Schwarz algorithm based on an interface space decomposition of the type

$$\mathbf{V} = \mathbf{V}_0 + \sum_i (\mathbf{I} - \mathbf{P})\mathbf{D}_i \mathbf{V}_i.$$

The purpose of this paper is to explain how to efficiently relate the Neumann-Neumann algorithm to the more classical additive Schwarz framework. The previously known convergence results of J. Mandel or of the authors are then easily recovered. More important, this framework leads to several extensions of the algorithm for situations involving inexact domain solvers or nonconforming 3D mesh refinements. The efficiency of these different extensions will be illustrated by the results of several real life numerical experiments.

2 Model Problem and Basic Algorithm

Let us consider a second order elliptic problem with vector unknown $u(x) \in \mathbb{R}^3$ set on a given domain Ω of $\mathbb{R}^3$ with variational formulation

$$\int_\Omega \Big(a(x) \cdot \nabla u(x)\Big) \cdot \nabla v(x) dx \; = \int_\Omega f^\Omega \cdot v dx + \int_{\partial\Omega_N} f^\Gamma \cdot v da, \quad \forall v \in H(\Omega). \tag{2.1}$$

Here, $H(\Omega)$ denotes the space of admissible (finite element) solutions, $\partial\Omega_N$ the part of the boundary where Neumann boundary conditions are imposed and $\partial\Omega_D$ the part where Dirichlet boundary conditions are imposed.

Iterative substructuring techniques use non overlapping domain partitions which split the original domain into small disjoint subdomains and reduce the original problem to an interface problem solved by an iterative conjugate gradient method.

The first step is thus to split the domain into small local non overlapping subdomains

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i,$$

with interfaces

$$\Gamma_i = \partial\Omega_i \backslash \partial\Omega, \tag{2.2}$$

$$\Gamma = \cup_i \Gamma_i. \tag{2.3}$$

The second step is to construct the interface problem. Let $\mathbf{K}_i$ denote the stiffness matrix of the subdomain Ω_i

$$(\mathbf{K}_i)_{lm} = \int_{\Omega_i} \Big(a(x) \cdot \nabla \phi_l(x)\Big) \cdot \nabla \phi_m(x) dx$$

and

$$(F_i)_l = \int_{\Omega_i} f^{\Omega}(x) \cdot \phi_l(x) dx + \int_{\partial\Omega_N \cap \partial\Omega_i} f^{\Gamma}(x) \cdot \phi_l(x) da$$

the corresponding right hand side. These matrices and right hand sides can obviously be computed independently on each subdomain. For each subdomain, the degrees of freedom are then decomposed into internal degrees of freedom $\mathring{X}_i$ associated to nodes which are strictly inside the subdomain Ω_i, or on the external boundary and interface degrees of freedom $\bar{X}_i$ associated to nodes lying on the interface between two or more neighboring subdomains. With this partition, the subdomain stiffness matrix and right hand side take the form

$$\mathbf{K}_i = \begin{bmatrix} \mathring{\mathbf{K}}_i & \mathbf{B}_i \\ \mathbf{B}_i^t & \bar{\mathbf{K}}_i \end{bmatrix}, \qquad F_i = \begin{bmatrix} \mathring{F}_i \\ \bar{F}_i \end{bmatrix}. \tag{2.4}$$

Let us finally denote by $\bar{X} = \bigcup_i \bar{X}_i$ the entire set of interface degrees of freedom, and by $\bar{X}_i = \mathbf{R}_i \bar{X}$ the restriction of $\bar{X}$ on the boundary of Ω_i. Under this notation, after addition of the local contributions of all subdomains to the global stiffness matrix and right hand side, the linear system describing the global equilibrium of the domain Ω takes the block structured form

$$\begin{bmatrix} \mathring{\mathbf{K}}_1 & 0 & \cdot & 0 & \mathbf{B}_1\mathbf{R}_1 \\ 0 & \mathring{\mathbf{K}}_2 & \cdot & 0 & \mathbf{B}_2\mathbf{R}_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \mathring{\mathbf{K}}_N & \mathbf{B}_N\mathbf{R}_N \\ \mathbf{R}_1^t\mathbf{B}_1^t & \mathbf{R}_2^t\mathbf{B}_2^t & \cdot & \mathbf{R}_N^t\mathbf{B}_N^t & \sum_i \mathbf{R}_i^t\bar{\mathbf{K}}_i\mathbf{R}_i \end{bmatrix} \begin{pmatrix} \mathring{X}_1 \\ \mathring{X}_2 \\ \cdot \\ \mathring{X}_N \\ \bar{X} \end{pmatrix} = \begin{pmatrix} \mathring{F}_1 \\ \mathring{F}_2 \\ \cdot \\ \mathring{F}_N \\ \sum_i \mathbf{R}_i^t \bar{F}_i \end{pmatrix}.$$

This system is ideally solved by block Gaussian elimination of the internal degrees of freedom $\mathring{X}_i$, yielding

$$\mathring{X}_i = (\mathring{\mathbf{K}}_i)^{-1}(\mathring{F}_i - \mathbf{B}_i \bar{X}_i). \tag{2.5}$$

In mathematical terms, this elimination amounts to the parallel solution of local equilibrium problems set on subdomains Ω_i with fixed Dirichlet boundary conditions.

After elimination we obtain the reduced interface system

$$\sum_i \mathbf{R}_i^t \left(\bar{\mathbf{K}}_i \mathbf{R}_i \bar{X} + \mathbf{B}_i^t (\mathring{\mathbf{K}}_i)^{-1} \left[\mathring{F}_i - \mathbf{B}_i \mathbf{R}_i \bar{X} \right] \right) = \sum_i \mathbf{R}_i^t \bar{F}_i.$$

Introducing the so-called local Schur complement matrix

$$\mathbf{S}_i := \bar{\mathbf{K}}_i - \mathbf{B}_i^t (\mathring{\mathbf{K}}_i)^{-1} \mathbf{B}_i, \tag{2.6}$$

this interface system takes the final form :

$$(\sum_i \mathbf{R}_i^t \mathbf{S}_i \mathbf{R}_i) \bar{X} = \sum_i \mathbf{R}_i^t (\bar{F}_i - \mathbf{B}_i^t (\mathring{\mathbf{K}}_i)^{-1} \mathring{F}_i). \tag{2.7}$$

Problem (2.7) is equivalent to our original equilibrium problem, but is only written in terms of the interface unknowns $\bar{X}$.

The main idea of modern domain decomposition methods is to solve problem (2.7) by an iterative preconditioned conjugate gradient algorithm. These iterative techniques never require the explicit calculation of matrix $\mathbf{S}$ since they form the matrix vector product $\mathbf{S}\bar{X}$ by solving auxiliary Dirichlet problems (2.5) on the local subdomains [LV96]. The main issue conditioning the success and parallel efficiency of such techniques is then the choice of the preconditioner $\mathbf{M}$. This preconditioner must be easy to implement in a parallel environment and must lead to a scalable algorithm when the number of processors increases. Additive Schwarz methods give a very general and efficient way of constructing such preconditioners.

3 Abstract Additive Schwarz Method

Let us consider the solution of the abstract variational problem

$$a(u,v) = \langle f, v\rangle, \forall v \in \mathbf{V}, u \in \mathbf{V}, \tag{3.8}$$

where $\mathbf{V}$ is a given Hilbert space with duality product $\langle .,. \rangle$, and a an elliptic continuous symmetric bilinear form defined on $\mathbf{V}$.

We suppose that the space $\mathbf{V}$ can be decomposed into the sum

$$\mathbf{V} = I_0\mathbf{V}_0 + I_1\mathbf{V}_1 + I_2\mathbf{V}_2 + \ldots + I_N\mathbf{V}_N, \tag{3.9}$$

where I_i is a given continuous linear extension map from the local space $\mathbf{V}_i$ to the global space $\mathbf{V}$. On each subspace $\mathbf{V}_i$, we introduce a symmetric elliptic bilinear form $b_i(.,.)$. We denote by $A : \mathbf{V} \to \mathbf{V}'$ the linear operator associated to the form a

$$\langle Au, v\rangle = a(u,v), \forall u, v \in \mathbf{V},$$

and by $B_i : \mathbf{V}_i \to \mathbf{V}_i'$ the linear operator associated to the form b_i.

With this notation, the additive Schwarz method for solving our original problem (3.8) is defined as the conjugate gradient method preconditioned by the following sum of local operators

$$M^{-1} = I_0 B_0^{-1} I_0^t + \ldots + I_N B_N^{-1} I_N^t. \tag{3.10}$$

This preconditioner is quite easy to compute since its action on a given element $L \in \mathbf{V}'$ is simply equal to the sum

$$M^{-1}L = \sum_{i=0}^{N} I_i u_i,$$

where $u_i \in \mathbf{V}_i$ is the solution of the local variational problem

$$b_i(u_i, v_i) = \langle L, I_i v_i\rangle, \forall v_i \in \mathbf{V}_i.$$

As usual, the efficiency of the above preconditioned conjugate gradient method is inversely proportional to the condition number of the operator $M^{-1}A$ which we control by carefully choosing the subspaces I_iV_i [CM94].

4 Generalized Neumann-Neumann Preconditioner

We have seen earlier that the interface problem (2.7) takes the abstract form

$$(\sum_i \mathbf{R}_i^t \mathbf{S}_i \mathbf{R}_i)\bar{X} = \bar{F} \in \mathbf{V}', \tag{4.11}$$

with $\mathbf{R}_i$ the restriction from the space $\mathbf{V}$ of global interface values $\bar{X}$ to the space $\mathbf{V}_i$ of local interface values $\bar{X}_i$. Such an abstract problem can be solved in all generality by a Neumann-Neumann algorithm which preconditions the sum $\mathbf{S} = \sum \mathbf{R}_i^t \mathbf{S}_i \mathbf{R}_i$ by a two level weighted sum of the inverses $\mathbf{M}^{-1} = \sum \mathbf{D}_i (\mathbf{S}_i)^{-1} \mathbf{D}_i^t$. From a theoretical point of view, this algorithm turns out to be a particular case of the above additive Schwarz method.

For constructing such an abstract Neumann-Neumann preconditioner, we first need to *choose*

1. a partition of unity $\mathbf{D}_i : \mathbf{V}_i \to \mathbf{V}$ satisfying

$$\sum_{i=1}^N \mathbf{D}_i \mathbf{R}_i = \mathbf{Id}|_{\mathbf{V}}.$$

   For implementation reasons (flexibility and parallelism), the map $\mathbf{D}_i$ must be as local as possible. The generic choice consists in defining $\mathbf{D}_i$ on each interface degree of freedom $v(P_l)$ by : $\mathbf{D}_i v(P_l) = \frac{\rho_i}{\rho} v(P_k)$ if the l degree of freedom of $\mathbf{V}$ corresponds to the k degree of freedom of $\mathbf{V}_i$, and by $\mathbf{D}_i v(P_l) = 0$, if not. Here ρ_i is a local measure of the stiffness of subdomain Ω_i (for example an average Young modulus on Ω_i) and $\rho = \sum_{P_l \in \Omega_j} \rho_j$ is the sum of ρ_j on all subdomains Ω_j containing P_l.
2. an approximate local operator $\tilde{\mathbf{S}}_i$ such that

$$\tilde{\mathbf{S}} = \sum \mathbf{R}_\mathbf{i}^\mathbf{t} \tilde{\mathbf{S}}_i \mathbf{R}_i$$

   is spectrally equivalent to $\mathbf{S}$: $\omega_- \langle \mathbf{S}v, v\rangle \le \langle \tilde{\mathbf{S}}v, v\rangle \le \omega_+ \langle \mathbf{S}v, v\rangle, \ \ \forall v \in \mathbf{V}$.
   Up to now, Neumann-Neumann methods used the original Schur complement $\mathbf{S}_i$ as a local operator, but our most recent tests and analysis show that one can choose different local operators $\tilde{\mathbf{S}}_i$. In practice, one uses the local Schur complement of a simplified (unrefined, undeformed, homogenized..) problem. The calculation of its local inverse will then reduce to the solution of an approximate Neumann problem.
3. a $\tilde{\mathbf{S}}_i$ orthogonal decomposition of each local space $\mathbf{V}_i, i = 1, 2, \cdots, N$, into

$$\mathbf{V}_i = \mathbf{V}_i^0 \oplus \mathbf{Z}_i.$$

   Above, the local coarse space $\mathbf{Z}_i$ contains all potential local singularities, that is functions v_i whose extensions $\mathbf{D}_i v_i$ are of very large (usually H dependent) energy. In particular, the space $\mathbf{Z}_i$ must be such that

$$Ker \tilde{\mathbf{S}}_i \subset \mathbf{Z}_i \subset \mathbf{V}_i.$$

For elasticity problems, $\mathbf{Z}_i$ is usually taken as the space of local rigid body motions. For plate and shell problems, a better choice is to choose [LMVed] $\mathbf{Z}_i$ as the orthogonal to the space

$$\mathbf{V}_i^0 = \{v \in \mathbf{V}_i, v = 0 \text{ at cross points}\}.$$

We then define the generalized Neumann-Neumann domain decomposition technique as the additive Schwarz algorithm solving $\mathbf{S}$ on the space $\mathbf{V}$ of interface restrictions of elements of $H(\Omega)$, with

1. coarse space $\mathbf{V}_0 = \sum_{i=1}^N \mathbf{D}_i\mathbf{Z}_i \subset \mathbf{V}$, endowed with the scalar product $\tilde{\mathbf{S}}$,
2. local spaces $\mathbf{V}_i^0, i = 1, 2, \cdots, N$ endowed with the scalar product $\mathbf{B_i} = \tilde{\mathbf{S}}_i$,
3. extensions $\mathbf{I}_i = (I - \mathbf{P})\mathbf{D}_i$, with $\mathbf{P}$ the $\tilde{\mathbf{S}}$ orthogonal projection of $\mathbf{V}$ onto $\mathbf{V}_0$. The above choice of extension map is in fact the key point of the Neumann-Neummann algorithm.

By construction, this Neumann-Neummann algorithm corresponds to the preconditioning operator

$$\mathbf{M}^{-1} = \tilde{\mathbf{S}}_0^{-1} + \sum_i (I - \mathbf{P})\mathbf{D}_i\tilde{\mathbf{S}}_i^{-1}\mathbf{D}_i^t(I - \mathbf{P})^t,$$

in which we recognize a direct generalization of the expression initially proposed in [MB93].

To see how this abstract algorithm can be numerically implemented, we detail below the application of the operator $\mathbf{M}^{-1}$ to a given element r of $\mathbf{V}'$. From the above construction, we first need to project the residual onto the coarse space by solving the coarse problem

$$\langle \tilde{\mathbf{S}}u_0, v_0\rangle = \langle r, v_0\rangle, \forall v_0 \in \mathbf{V}_0,$$

to compute the local contributions u_i by solving in parallel the local "Neumann" problems

$$\begin{aligned} u_i \in \mathbf{V}_i^0 \quad : \qquad \langle \tilde{\mathbf{S}}_i u_i, v_i\rangle &= \langle r, (I - \mathbf{P})\mathbf{D}_i v_i\rangle, \\ &= \langle r - \tilde{\mathbf{S}}u_0, \mathbf{D}_i v_i\rangle \quad \forall v_i \in \mathbf{V}_i^0, \end{aligned}$$

to project these local contributions onto the coarse space

$$\langle \tilde{\mathbf{S}}(\textstyle\sum_i \mathbf{D}_i z_i), v_0\rangle = \langle \textstyle\sum_i \mathbf{D}_i u_i, \tilde{\mathbf{S}}v_0\rangle, \qquad \forall v_0 \in \mathbf{V}_0, z_i \in \mathbf{Z}_i,$$

and to set

$$\mathbf{M}^{-1}r = u_0 + \sum_{i=1}^N \mathbf{D}_i(u_i - z_i).$$

Using the general theory, we then have

Theorem 1 *The above abstract Neumann-Neumann preconditioner satisfies*

$$Cond(\mathbf{M}^{-1}\mathbf{S}) = \frac{\lambda_{max}(\mathbf{M}^{-1}\mathbf{S})}{\lambda_{min}(\mathbf{M}^{-1}\mathbf{S})} \leq \frac{(Ne+1)\omega_+}{\omega_-} \max_i \sup_{v_i \in \mathbf{V}_i^0} \frac{\|\mathbf{D}_i v_i\|_{\tilde{S}}^2}{\|v_i\|_{\tilde{S}_i}^2},$$

with Ne the maximum number of neighbors of a given subdomain.

Proof. The minimal eigenvalue $\lambda_{min}(\mathbf{M}^{-1}\mathbf{S})$ can be bounded from below by the well-known partition lemma classically used in the analysis of additive Schwarz methods. For this purpose, we split any $v \in \mathbf{V}$ into

$$v = \mathbf{P}v + (I - \mathbf{P})v = I_0 v_0 + v_\perp.$$

From the local decomposition of each local space $\mathbf{V}_i$, each local component $\mathbf{R}_i v_\perp$ can be decomposed into $\mathbf{R}_i v_\perp = v_i + z_i, v_i \in \mathbf{V}_i^0, z_i \in \mathbf{Z}_i$. By introducing our partition of unity $\mathbf{D}_i$, we then have

$$\begin{aligned} v_\perp &= (I - \mathbf{P}) \sum_i \mathbf{D}_i \mathbf{R}_i v_\perp \\ &= \sum_i (I - \mathbf{P}) \mathbf{D}_i (v_i + z_i) \\ &= \sum_i (I - \mathbf{P}) \mathbf{D}_i v_i, \\ &= \sum_i I_i v_i. \end{aligned}$$

By orthogonality of the local decomposition, we then verify

$$\begin{aligned} \sum_i b_i(v_i, v_i) &= \sum_i \langle \tilde{\mathbf{S}}_i v_i, v_i \rangle \le \sum_i \langle \tilde{\mathbf{S}}_i (v_i + z_i), v_i + z_i \rangle \\ &= \sum_i \langle \tilde{\mathbf{S}}_i \mathbf{R}_i v_\perp, \mathbf{R}_i v_\perp \rangle = \langle \tilde{\mathbf{S}} v_\perp, v_\perp \rangle. \end{aligned}$$

By orthogonality again, we have

$$\begin{aligned} b_0(v_0, v_0) + \sum_i b_i(v_i, v_i) &\le \langle \tilde{\mathbf{S}} v_0, v_0 \rangle + \langle \tilde{\mathbf{S}} v_\perp, v_\perp \rangle \\ &= \langle \tilde{\mathbf{S}} v, v \rangle \le \omega_+ \langle \mathbf{S} v, v \rangle. \end{aligned}$$

Thus, the partition lemma holds with $C_0 = \omega_+$, which implies that λ_{min} is bounded from below by $\frac{1}{\omega_+}$.

On the other hand, the derivation of an optimal upper bound for $\lambda_{max}(\mathbf{M}^{-1}\mathbf{S})$ requires specific orthogonality arguments which do not easily fit into the classical theory of additive Schwarz methods. Indeed, using orthogonality and the contraction properties of the projection $(I - \mathbf{P})$, we have

$$\begin{aligned} \langle \tilde{\mathbf{S}} \mathbf{M}^{-1} \tilde{\mathbf{S}} \bar{X}, \mathbf{M}^{-1} \tilde{\mathbf{S}} \bar{X} \rangle &= \langle \tilde{\mathbf{S}} (u_0 + \sum_i (I - \mathbf{P}) \mathbf{D}_i u_i), u_0 + \sum_i (I - \mathbf{P}) \mathbf{D}_i u_i \rangle \\ &= \langle \tilde{\mathbf{S}} u_0, u_0 \rangle + \langle \tilde{\mathbf{S}} \sum_i (I - \mathbf{P}) \mathbf{D}_i u_i, \sum_i (I - \mathbf{P}) \mathbf{D}_i u_i \rangle \\ &\le \langle \tilde{\mathbf{S}} u_0, u_0 \rangle + \langle \tilde{\mathbf{S}} \sum_i \mathbf{D}_i u_i, \sum_i \mathbf{D}_i u_i \rangle. \end{aligned}$$

By introducing the number of neighbors N_i of a given subdomain Ω_i

$$N_i = \text{Number of } j \neq i, \exists u_i \in \mathbf{V}_i, \exists u_j \in \mathbf{V}_j, \langle \tilde{\mathbf{S}}\mathbf{D}_i u_i, \mathbf{D}_j u_j \rangle \neq 0,$$

the continuity constant of $\mathbf{D}_i$

$$c_i = \sup_{v_i \in \mathbf{V}_i^0} \frac{\langle \tilde{\mathbf{S}}\mathbf{D}_i v_i, \mathbf{D}_i v_i \rangle}{\langle \tilde{\mathbf{S}}_\mathbf{i} v_i, v_i \rangle}$$

and using Cauchy Schwarz, we then deduce

$$\begin{aligned}
\langle \tilde{\mathbf{S}}\mathbf{M}^{-1}\tilde{\mathbf{S}}\tilde{X}, \mathbf{M}^{-1}\tilde{\mathbf{S}}\tilde{X} \rangle &\leq \langle \tilde{\mathbf{S}} u_0, u_0 \rangle + \max_i (N_i + 1) \sum_i \langle \tilde{\mathbf{S}}\mathbf{D}_i u_i, \mathbf{D}_i u_i \rangle \\
&\leq \langle \tilde{\mathbf{S}} u_0, u_0 \rangle + \max_i (N_i + 1)\ c_i \sum_i \langle \tilde{\mathbf{S}}_\mathbf{i} u_i, u_i \rangle \\
&\leq \langle \tilde{\mathbf{S}}\bar{X}, P u_0 \rangle + \max_i (N_i + 1)\ c_i \sum_i \langle \tilde{\mathbf{S}}\bar{X}, I_i u_i \rangle \\
&\leq [\max_i (N_i + 1)\ c_i] \langle \tilde{\mathbf{S}}\bar{X}, \mathbf{M}^{-1}\tilde{\mathbf{S}}\bar{X} \rangle \\
&\leq [\max_i (N_i + 1)\ c_i] \langle \tilde{\mathbf{S}}\bar{X}, \bar{X} \rangle^{\frac{1}{2}} \langle \tilde{\mathbf{S}}\mathbf{M}^{-1}\tilde{\mathbf{S}}\bar{X}, \mathbf{M}^{-1}\tilde{\mathbf{S}}\bar{X} \rangle^{\frac{1}{2}}.
\end{aligned}$$

The final result follows then by standard estimates

$$\begin{aligned}
\lambda_{max}(\mathbf{M}^{-1}\mathbf{S}) &= \max_{\bar{X}} \frac{\langle \mathbf{S}\bar{X}, \bar{X} \rangle}{\langle \mathbf{M}\bar{X}, \bar{X} \rangle} = \max_{\bar{X}} \frac{\langle \mathbf{S}\bar{X}, \bar{X} \rangle}{\langle \tilde{\mathbf{S}}\bar{X}, \bar{X} \rangle} \ \frac{\langle \tilde{\mathbf{S}}\bar{X}, \bar{X} \rangle}{\langle \mathbf{M}\bar{X}, \bar{X} \rangle} \\
&\leq \frac{1}{\omega_-} \max_{\bar{X}} \frac{\langle \tilde{\mathbf{S}}\mathbf{M}^{-1}\tilde{\mathbf{S}}\bar{X}, \mathbf{M}^{-1}\tilde{\mathbf{S}}\bar{X} \rangle^{\frac{1}{2}}}{\langle \tilde{\mathbf{S}}\bar{X}, \bar{X} \rangle^{\frac{1}{2}}} \leq \frac{1}{\omega_-} \max_i (N_i + 1)\ c_i.
\end{aligned}$$

If we particularize this abstract convergence result to specific elasticity problems with specific choices of local spaces and coarse grid operators , we recover the following quite general convergence theorem [LeT94], [LMVed] :

Theorem 2 *Using the above Neumann-Neumann preconditioner in the framework of three-dimensional linear elasticity problems or of plate problems, the condition number of the operator* $\mathbf{M}^{-1}\mathbf{S}$ *is bounded by*

$$Cond(\mathbf{M}^{-1}\mathbf{S}) \leq \frac{C}{\alpha_i^2}[1 + ln \max_i \frac{H_i}{h_i}]^2, \tag{4.12}$$

the constant C *being independent of the subdomains diameters* H_i*, discretization steps* h_i*, aspect ratios* α_i *and averaged coefficients* ρ_i*.*

The above result guarantees the scalability of the proposed algorithm with respect to the number of subdomains (H_i independence), and its robustness with respect to strongly heterogeneous elasticity coefficients (ρ_i independence). This independence with respect to coefficient jumps is due to our specific choice of weighting factors

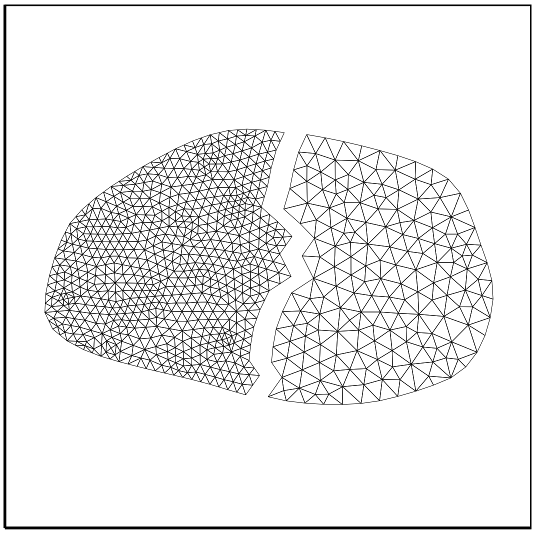
$$\mathbf{D}_i v(P_l) = \frac{\rho_i}{\rho} v(P_k).$$

But the abstract convergence result is more general because it handles situations using inexact subdomain solvers $\tilde{\mathbf{S}}_i^{-1}$ in the preconditioning step. The next paragraph illustrates this possibility.

5 Application to a Large-scale Problem with Local Refinement

Many large scale engineering problems require additional care and precision next to junctions of complex geometries. A simple way for achieving this in an industrial framework consists in first defining a global conforming finite element mesh of the whole domain Ω, to be partitioned as usual into conforming non overlapping subdomains Ω_i. In order to improve the local accuracy of the finite element solution, we then refine the finite element mesh of several subdomains (but not of all subdomains) by subdividing each original element of these subdomains into $2, 4, 8, 16, ...$ sub elements of same nature. After such local refinements the global mesh is no longer conforming : on the interface between a refined domain Ω_i and an unrefined domain Ω_j, several nodes of Ω_i will have no equivalent on Ω_j (Figure 1).

Figure 1 Nonconforming mesh refinement : refined nodes on the right domain have no counterpart on the left domain.



This lack of conformity can be handled by the so called slave node approach used in Bramble, Ewing, Parashkevov and Pasciak [BEPP92]. In this approach, all finite element displacement fields are imposed to be pointwise continuous at all subdomain interfaces. In other words, the finite element space definition is kept as

$$H(\Omega) = \Big\{ v_h : \bar{\Omega} \to \mathbb{R}^3, v_h \text{ continuous}, \;\; v_h = 0 \text{ on } \partial\Omega_D,$$

$$v_{h_{|T_l}} = v_l \circ {}'^{-1}_l, v_l \in [Q_2'(\hat{\Omega})]^3, \text{ for all elements } T_l \text{ of all subdomains } \Omega_i \Big\}.$$

With this choice, on any nonconforming interface, the values of the displacement field at any interface node P_l^i of the refined subdomain Ω_i which is not shared by the neighboring subdomain Ω_j are constrained to be equal to the value at this point of the Ω_j finite element interpolation of this field

$$u(P_l^i) = \sum_{k \in T \cap \partial\Omega_j} u(P_k^j) \phi_k^T (P_l^i).$$

Here T is the finite element of subdomain Ω_j which contains P_l^i, P_k^j are the interface nodes of this element, and $\phi_k^T(x)$ is the nodal element shape function associated to the node P_k^j. By construction, the interface nodes P_k^j of Ω_j are shared by Ω_i, and by the imposed continuity of the displacement field at the interface, the above continuity constraints can be rewritten as

$$u(P_l^i) = \sum_{k \in T \cap \partial\Omega_i} u(P_k^i) \phi_k^T (P_l^i). \tag{5.13}$$

Therefore, the additional degrees of freedom introduced by refinement on the interface $\partial\Omega_i$ are not directly related to any degree of freedom of Ω_j, but only to degrees of freedom of Ω_i. They must then be considered as internal degrees of freedom of Ω_i, that is as elements of the set $\mathring{X}_i$, and do not participate to the interface problem. In other words, the mesh refinement of the subdomain Ω_i will modify the local stiffness matrix $\mathbf{K}_i$ (new finite elements are added and the internal kinematic constraint (5.13) must be taken into account), will add elements to the set $\mathring{X}_i$ of internal unknowns, but will not modify the list and definition of the interface degrees of freedom $\bar{X}_i$. In particular, the interface problem keeps the same structure and dimension as in the unrefined case. Such situations can therefore be easily solved by our generalized Neumann-Neumann algorithm, using as interface preconditioner $\tilde{\mathbf{S}}$ the (much cheaper) interface preconditioner of the unrefined case.

We have applied this strategy to the calculation of part of a protection wall in a 3D offshore platform subjected to an external pressure. This problem can be written in the following form :

Find the displacement field $u(x)$, of a three-dimensional structure Ω, subjected to a given external loading. The external forces acting onto the body can be reduced to surface tractions f^Γ acting on the part $\partial\Omega_N$ of the boundary $\partial\Omega$. These tractions represent the external pressure or the action of icebergs on the structure. The displacement $u_i(x)$ is imposed on the remaining part $\partial\Omega_D = \partial\Omega - \partial\Omega_N$ of the boundary.

The governing equilibrium equations reduced then to the strong form :

$$
\begin{aligned}
-div(E(x)\,\varepsilon(u)) &= 0 \text{ in } \Omega, \\
E(x)\,\varepsilon(u)n &= f^{\Gamma} \text{ on } \partial\Omega_N, \\
u(x) &= 0 \text{ on } \partial\Omega_D,
\end{aligned}
$$

with E(x) the local elasticity tensor, and $\varepsilon(u)$ the linearized strain tensor

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right).$$

For isotropic materials, we have simply

$$E(x)\,\varepsilon(u) = \frac{E\nu}{(1+\nu)(1-2\nu)} div(u) Id + \frac{E}{(1+\nu)}\varepsilon(u).$$

The platform is supposed to be made of an isotropic elastic material ($E = .3710^{11}$ and $\nu = .2$) and is discretized using second order hexahedral finite elements. The domain is cut into 5 subdomains. Three calculations were performed. The first one uses 5 coarse compatible subdomains. The total mesh is rather coarse, with element aspect ratios of 5. Four subdomains are of equal size and contain each 209 elements, 1616 nodes and 4848 degrees of freedom. The fifth subdomains has 88 elements, 714 nodes (2142 d.o.f). Then this fifth subdomain is refined (Figure 2) and thus the resulting decomposition uses non matching grids. The refined domain contains 748 elements, 4181 nodes (12543 d.o.f). Finally all the subdomains were refined. In order to fit in the computer memory a decomposition in 18 subdomains was needed for the fully refined case. The number of subdomain iterations, for a precision of 10^{-6} in the conjugate gradient algorithm, is 53 for the coarse decomposition, 57 for the incompatible one and 65 for the fine one. For the coarse decomposition the calculated condition number is 408.24, compared to 457.75 for the incompatible one. In both cases, the dimension of the interface problem is 2436. As far as accuracy is concerned, the non matching grid gives results comparable to those of the totally refined case. In summary, partial refinement is as accurate as full refinement but is associated to an interface problem whose complexity and cost is as cheap as the unrefined case when solved by our generalized Neumann - Neumann technique.

Similar costs are also observed when solving nonlinear elasticity problems with frozen interface preconditioner [LV96], which validates the generalized Neumann Neumann algorithm in real life numerical examples.

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Figure 2 Non matching grid decomposition of the offshore problem.