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Two Preconditioners for Saddle Point Problems with Penalty Term

Axel Klawonn

1 Introduction

Many problems in the engineering sciences lead to saddle point problems. Important examples are the Stokes equations of fluid dynamics, modeling the flow of an incompressible viscous fluid, and mixed formulations of problems from linear elasticity, e.g. for almost incompressible materials, plates, and beams; cf. Braess [Bra92] or Brezzi and Fortin [BF96]. These problems can be analyzed in the framework of saddle point problems with a penalty term.

In this article, we focus on the construction of preconditioned iterative method for certain saddle point problems with a penalty term. We present a block-diagonal and a block-triangular preconditioner in combination with appropriate Krylov space methods. This yields preconditioned iterative methods which have convergence rates that are bounded independently of the penalty and the discretization parameters. Details and proofs of the results can be found in [Kla97, Kla95a, Kla95b, Kla98]. Here, we only consider symmetric saddle point problems. For related work on the non-symmetric case, see Elman and Silvester [ES96], where the Oseen operator which is obtained by applying a Picard iteration to the Navier-Stokes equations is analyzed. Earlier work on block-diagonal preconditioners can be found in Rusten and Winther [RW92] and Silvester and Wathen [SW94]. A number of methods have been proposed for solving saddle point problems. For a list of references, see e.g. [Kla97, Kla98].

2 Saddle Point Problems with a Penalty Term

In this section, we give a brief overview over saddle point problems with a penalty term; cf. Braess [Bra92].

Let $(V, \|\cdot\|_V)$ and $(M, \|\cdot\|_M)$ be two Hilbert spaces, let M_c be a dense subspace of M, and let $a(\cdot,\cdot) : V \times V \to R$, $b(\cdot,\cdot) : V \times M \to R$, $c(\cdot,\cdot) : M_c \times M_c \to R$, be three bilinear forms. Additionally, we introduce V_0, a subspace of V, given by $V_0 := \{v \in V : b(v,q) = 0 \; \forall q \in M\}$. We assume that $a(\cdot,\cdot)$ is symmetric V_0-elliptic

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and that $c(\cdot,\cdot)$ is symmetric M_c-positive semi-definite. Moreover, we assume $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to be bounded. We consider the following problem:
Find $(u,p) \in V \times M_c$, such that

$$\begin{array}{lclcll} a(u,v) & + & b(v,p) & = & < f,v > & \forall v \in V \\ b(u,q) & - & t^2 c(p,q) & = & < g,q > & \forall q \in M_c \quad t \in [0,1]. \end{array} \tag{1}$$

We denote by $X := V \times M_c$ the product space and by

$$\mathcal{A}(x,y) := a(u,v) + b(u,q) + b(v,p) - t^2 c(p,q),$$

$x = (u,p) \in X,\ y = (v,q) \in X$, the bilinear form of problem (1) on X. With the additional definition $\mathcal{F}(y) :=< f,v > + < g,q >$, we obtain an equivalent formulation of problem (1)

$$\mathcal{A}(x,y) = \mathcal{F}(y) \quad \forall y \in X. \tag{2}$$

We equip X with a new norm. We assume that we have an additional norm on M_c, i.e. $|||\cdot|||_M$, and introduce the new norm on X by

$$|||x||| := \|u\|_V + |||q|||_M \text{ for } x = (u,p) \in X.$$

If the bilinear form $c(\cdot,\cdot)$ is continuous on $M \times M$, we define $|||p|||_M := \|p\|_M$. Otherwise, $|||p|||_M$ is defined by $\|p\|_M + t|p|_c$, where $|p|_c := \sqrt{c(p,p)}$ is a semi-norm on M_c.

Theorem 1 *Let the following three assumptions be satisfied:*

1. *The continuous bilinear form $a(\cdot,\cdot)$ is symmetric and V_0-elliptic, i.e.*
$$\exists \alpha_0 > 0, \text{ such that } a(v,v) \geq \alpha_0 \|v\|_V^2 \quad \forall v \in V_0,$$
2. *The continuous bilinear form $b(\cdot,\cdot)$ fulfills an inf-sup condition, i.e.*
$$\exists \beta_0 > 0, \text{ such that } \inf_{q \in M} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_M} \geq \beta_0,$$
3. *The bilinear form $c(\cdot,\cdot)$ is symmetric and M_c-positive semi-definite, i.e.*
$$c(q,q) \geq 0 \quad \forall q \in M_c.$$

Then, $\mathcal{A}(\cdot,\cdot)$ defines an isometric isomorphism $\mathcal{A} : X \to X'$ if in addition one of the following conditions is satisfied:

1) The bilinear form $c(\cdot,\cdot)$ is continuous on $M_c \times M_c$.
2) The bilinear form $a(\cdot,\cdot)$ is $V-$elliptic.

Under these assumptions $\mathcal{A}^{-1}$ is uniformly bounded for $t \in [0,1]$.

For a proof of this theorem, we refer the reader to Braess [Bra92], Section III.4.

All these results are also valid for suitable finite element spaces; see Braess [Bra92] or Brezzi and Fortin [BF96]. We then require, additionally, that the constants in Theorem 1 are independent of h. The continuity assumptions turn into uniform boundedness with respect to h; see, e.g., Braess [Bra92].

Discretizing (2) by mixed finite elements, we obtain a linear system of algebraic equations,

$$\mathcal{A}x = \mathcal{F},$$

where

$$\mathcal{A} := \begin{pmatrix} A & B^t \\ B & -t^2 C \end{pmatrix} \in R^{n+m} \times R^{n+m}, \quad \mathcal{F} := \begin{pmatrix} f \\ g \end{pmatrix} \in R^{n+m}.$$

3 The Preconditioners

In this section, we present two different preconditioners for saddle point problems. The first is based on a block-diagonal structure, the second is a block-triangular preconditioner. Both approaches yield optimal convergence rates. To construct the preconditioners, we consider the discrete problem, using vectors and matrices, instead of functions and operators. Let us point out that it could have as well been presented in an abstract Hilbert space setting. With a slight abuse of notation, we also use, for simplicity, the same notation for the norms in both settings.

The Block-Diagonal Preconditioner

The block-diagonal preconditioner has the form

$$\hat{\mathcal{B}} := \begin{pmatrix} \hat{A} & O \\ O & \hat{C} \end{pmatrix} \in R^{n+m} \times R^{n+m}. \tag{3}$$

Here $\hat{A}$ and $\hat{C}$ satisfy certain ellipticity conditions, i.e. there exist positive constants a_0, a_1 and c_0, c_1, such that

$$\begin{aligned} a_0^2 \, \|u\|_V^2 &\le u^t \hat{A} u \le a_1^2 \, \|u\|_V^2 \quad && \forall u \in R^n, \\ c_0^2 \, |||p|||_M^2 &\le p^t \hat{C} p \le c_1^2 \, |||p|||_M^2 \quad && \forall p \in R^m. \end{aligned}$$

We assume the constants a_0, a_1, c_0, and c_1 to be independent of the critical parameters.

We use the block-diagonal preconditioner in combination with the conjugate residual method. To give a convergence estimate, it is our goal to give an upper bound for the spectral condition number of the preconditioned system $\kappa(\hat{\mathcal{B}}^{-1}\mathcal{A}) := \rho(\hat{\mathcal{B}}^{-1}\mathcal{A})\rho((\hat{\mathcal{B}}^{-1}\mathcal{A})^{-1})$, where $\rho(\cdot)$ is the spectral radius; cf. Hackbusch [Hac94], Theorem 9.5.13.

In the next theorem, we show that the spectral condition number can be uniformly bounded with respect to the penalty and discretization parameters; see Klawonn [Kla95a, Kla95b] for a detailed proof.

Theorem 2 *The condition number of $\hat{\mathcal{B}}^{-1}\mathcal{A}$ is bounded independently of the discretization and the penalty parameters, i.e.*

$$\kappa(\hat{\mathcal{B}}^{-1}\mathcal{A}) \le \frac{C_1}{C_0}.$$

Here, C_0, C_1 are positive constants independent of the penalty and the discretization parameters.

Table 1 Iteration counts for exact solvers as preconditioners for A and C, $\nu = 0.3$.

Grid	20×10	40×20	60×30	80×40	100×50	120×60	140×70
CR	17	19	19	21	21	21	21
GMRES	9	10	10	10	10	10	10
BI-CGSTAB	5	5	6	6	6	6	6

The Block-Triangular Preconditioner

In this section, we consider a block-triangular preconditioner. We note that the results presented in this subsection are, in contrast to the previous subsection, restricted to saddle point problems where the blocks A and C are positive definite. This class of problems contains, e.g., Stokes equations or mixed formulations of linear elasticity but excludes certain beam and plate problems.

The preconditioned system is either of the form $\mathcal{A}\hat{\mathcal{B}}^{-1}$ or $\hat{\mathcal{B}}^{-1}\mathcal{A}$ where $\hat{\mathcal{B}}$ is the block–triangular preconditioner.

We use the following notation

$$\hat{\mathcal{B}}_U := \begin{pmatrix} \hat{A} & B^t \\ O & -\hat{C} \end{pmatrix} \in R^{n+m} \times R^{n+m}, \quad \hat{\mathcal{B}}_L := \begin{pmatrix} \hat{A} & O \\ B & -\hat{C} \end{pmatrix} \in R^{n+m} \times R^{n+m},$$

Here $\hat{A}$ and $\hat{C}$ are positive definite. We make the following assumptions on $\mathcal{A}$ and $\hat{\mathcal{B}}$: The matrix $\hat{A}$ is a good preconditioner for A, i.e.

$$\exists a_0, a_1 > 0 \quad a_0^2 u^t \hat{A} u \leq u^t A u \leq a_1^2 u^t \hat{A} u \quad \forall u \in R^n. \tag{4}$$

The constants a_0, a_1 should preferably be close to each other and be independent of the discretization parameters but there are also other interesting cases.

We also require that $\hat{C}$ is a good preconditioner for C, i.e.

$$\exists c_0, c_1 > 0 \quad c_0^2 \, p^t \hat{C} p \leq p^t C p \leq c_1^2 \, p^t \hat{C} p \quad \forall p \in R^m. \tag{5}$$

Under the additional assumption that

$$1 < a_0 \leq a_1, \tag{6}$$

which can always be achieved by an appropriate scaling, we can show that the spectrum of $\mathcal{A}\hat{\mathcal{B}}^{-1}$ stays bounded independently of the discretization and the penalty parameters.

Introduce the notation,

$$\mathcal{H} \quad := \quad \begin{pmatrix} A - \hat{A} & O \\ O & \hat{C} \end{pmatrix}.$$

From (6), we see that $\mathcal{H}$ is positive definite. Since $A, \hat{A}, C$ are symmetric, it defines an inner product.

In the next theorem, we see that GMRES, using an inner product which is spectrally equivalent to the one defined by $\mathcal{H}^{-1}$, applied to the preconditioned system $\mathcal{A}\hat{\mathcal{B}}_U^{-1}$ yields an optimal convergence rate. The proof uses that $\mathcal{A}\hat{\mathcal{B}}_U^{-1}$ is symmetric positive definite in the $\mathcal{H}^{-1}$–inner product. For details, see Klawonn [Kla95b, Kla98].

Table 2 Iteration counts for a two-level multigrid preconditioner with a standard V-cycle defining $\hat{A}$, and $\hat{C} = C$, and $\nu = 0.3$.

Grid	20×10	40×20	60×30	80×40	100×50	120×60	140×70
CR	20	23	24	26	26	26	26
GMRES	12	12	13	13	13	13	14
BI-CGSTAB	7	7	7	7	7	7	7

Table 3 Iteration counts for a two-level multigrid preconditioner with a standard V-cycle defining $\hat{A}$ and a one-level symmetric multiplicative overlapping Schwarz method with the minimal overlap of one node defining $\hat{C}$, and $\nu = 0.3$.

Grid	20×10	40×20	60×30	80×40	100×50	120×60	140×70
CR	20	23	24	26	26	26	26
GMRES	12	12	13	13	13	13	14
BI-CGSTAB	7	7	7	7	7	7	7

Theorem 3 *Let $\hat{\mathcal{H}}$ be a positive definite matrix, such that $\bar{C}_0^2 \quad \mathcal{H}^{-1} \leq \hat{\mathcal{H}}^{-1} \leq \bar{C}_1^2 \quad \mathcal{H}^{-1}$, where $\bar{C}_0, \bar{C}_1$ are positive constants independent of the discretization and penalty parameters. Then,*

$$\|r_n\|_{\hat{\mathcal{H}}^{-1}} \leq \frac{\bar{C}_1}{\bar{C}_0} \; 2 \; \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^n \quad \|r_0\|_{\hat{\mathcal{H}}^{-1}},$$

where r_n is the n-th residual of GMRES, $r_0 = b - \mathcal{A}\hat{\mathcal{B}}^{-1}x_0$ and $\kappa := \kappa(\mathcal{A}\hat{\mathcal{B}}^{-1}) \leq \frac{C_1}{C_0}$ is the condition number of $\mathcal{A}\hat{\mathcal{B}}^{-1}$ in the $\mathcal{H}^{-1}-$inner product.

We note that our convergence estimate only depends on the square root of the condition number of the preconditioned problem. Except for a leading factor, this estimate matches the standard estimate for the conjugate gradient method applied to positive definite symmetric problems.

There also exist convergence estimates for GMRES using the L_2-metric. Since these bounds are not uniform for the meshsize h, we refer to the more detailed discussion in Klawonn [Kla95b, Kla98]. We would like to point out that these theoretical non-uniform bounds are not sharp since the convergence rates obtained from the numerical experiments are uniformly bounded ; cf. Section 4.

4 Numerical Examples

In this section, we apply our block–preconditioners to the mixed formulation of planar, linear elasticity, cf., e.g., Brezzi and Fortin [BF96] or Klawonn [Kla97]. For simplicity,

Table 4 Iteration counts for exact solvers as preconditioners for A and C on a 80×40 grid.

ν	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
CR	21	21	23	25	25	25	25	25
GMRES	10	11	12	12	12	12	12	12
BI-CGSTAB	6	7	7	7	7	7	7	7

Table 5 Iteration counts for a two-level multigrid method with a standard V-cycle defining $\hat{A}$ and $\hat{C} = C$ on a 80×40 grid.

ν	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
CR	26	29	33	33	33	33	33	33
GMRES	13	14	15	15	15	15	15	15
BI-CGSTAB	7	7	7	7	7	7	7	7

we work with the following formulation

$$\begin{aligned} \mu\,(\nabla u, \nabla v)_0 + (\mathrm{div} v, p)_0 &= \; < f, v > \quad \forall v \in V := (H^1_\Gamma(\Omega))^2, \\ (\mathrm{div} u, q)_0 - \frac{1}{\lambda+\mu}\,(p,q)_0 &= \; 0 \qquad\qquad \forall q \in M := L_2(\Omega), \end{aligned}$$

with $H^1_\Gamma(\Omega) := \{v \in H^1(\Omega) : v_{|\Gamma_0} = 0\}$. Γ_0 is the part of the boundary where Dirichlet conditions are imposed and λ, μ are the Lamé parameters. All results shown are for mixed boundary conditions with $\Gamma_0 := \{x = (x_1, x_2) \in \partial\Omega : x_1 < -0.8\}$ and the region $[-1,1] \times [-1,1]$. We note that our model is mathematically equivalent to the full elasticity problem only in the case of Dirichlet conditions on the whole boundary.

For growing λ, the considered material becomes more incompressible. Instead of using the Lamé constants λ and μ, we can also work with Young's elasticity modulus E and the Poisson ratio ν. These parameters are related to each other as follows

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{7}$$

The relation between the penalty parameter t and the Poisson ratio ν is given by $t := (1+\nu)(1-2\nu)/(E\nu)$. Without loss of generality, we set $E = 1$. We discretize by a $Q_1(h) - Q_1(2h)$ macro-element, i.e. we use piecewise bilinear polynomials on quadrilaterals on a grid with mesh size h for the displacements u and piecewise bilinear polynomials on quadrilaterals with mesh size $2h$ for the Lagrange multiplier p. For a proof that the inf-sup condition of B holds for this element; see Girault and Raviart [GR86] or Brezzi and Fortin [BF96].

All computations were carried out on a SUN SPARC 10 workstation using the numerical software package PETSc 1.0; cf. Balay, Gropp, Curfman McInnes, and Smith [BGMS96]. The initial guess is 0, and the stopping criterion is $\|r_k\|_2/\|r_0\|_2 < 10^{-5}$, where r_k is the k-th residual.

Table 6 Iteration counts for an exact solver as $\hat{A}$ and a one-level symmetric multiplicative overlapping Schwarz method with the minimal overlap of one node as $\hat{C}$ on a 80×40 grid.

ν	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
CR	21	21	23	25	25	25	25	25
GMRES	10	11	12	12	12	12	12	12
BI-CGSTAB	6	7	7	7	7	7	7	7

Table 7 Iteration counts for a two-level multigrid method with a standard V-cycle as $\hat{A}$ and a one-level symmetric multiplicative overlapping Schwarz method with the minimal overlap of one node as $\hat{C}$ on a 80×40 grid.

ν	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
CR	26	29	33	33	33	33	33	33
GMRES	13	14	15	15	15	15	15	15
BI-CGSTAB	7	7	7	7	7	7	7	7

We give numerical results for different Krylov space methods. We use the conjugate residual method (CR) in combination with the block–diagonal preconditioner and GMRES and BI-CGSTAB with the block-triangular one. We report on results with a version of GMRES without restarts but we also ran a version with restart every 10 iterations. The number of iterations for this latter version was always just one or two larger than without restart. We use right-oriented preconditioning with $\hat{\mathcal{B}}_U^{-1}$ for GMRES and we only use the L_2- rather than the $\hat{\mathcal{H}}^{-1}-$metric. Experiments were also carried out with a left-oriented preconditioner $\hat{\mathcal{B}}_L^{-1}$ and BI-CGSTAB. This latter method has the advantage of being based on a short term recurrence but it is not covered by our theory. As is shown in our experiments, there is no appreciable difference in the number of matrix-vector products of the different methods and the numerical results suggest that the number of iterations is bounded independently of the critical parameters h and t. Although we would like to point out that GMRES requires more inner products and more storage than BI-CGSTAB.

We note that in all of our experiments the block-diagonal preconditioner almost always needs about twice as many matrix-vector products than the block-triangular preconditioner. Whereas the latter is only slightly more expensive when used with a short term recurrence method.

To see how the Krylov space methods behave under the best of circumstances, we first conducted some experiments using exact solvers, i.e. $\hat{A} = A$ and $\hat{C} = C$; see Tables 1 and 4.

In another series of experiments, we use different preconditioners for A and C. We present results with a two-level multigrid preconditioner with a V-cycle including one pre- and one post-smoothing symmetric Gauss-Seidel step defining $\hat{A}$, and a one-level symmetric multiplicative overlapping Schwarz method with the minimal overlap of one node as $\hat{C}$; see Tables 2, 3, 5, 6, 7.

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